MATH0083 Prime Numbers and their Distribution

Year:	2024–2025
Code:	MATH0083
Level:	7 (UG)
Value:	15 credits (= 7.5 ECTS credits)
Normal student group(s):	UG Year 3 and 4 Mathematics degrees
Term:	1
Assessment:	40% examination, 60% coursework
Normal Pre-requisite:	MATH0014, MATH0051, MATH0034
Lecturer:	Dr I Petrow

Course Description and Objectives

Prime numbers have fascinated humans for millennia, and one of their cardinal mysteries is their irregular distribution on the number line. One of the great theorems is mathematics is the Prime Number Theorem, which gives an asymptotic estimate for the number of primes less than a given bound. A simple yet detailed proof of the Prime Number Theorem is the capstone goal of this course.

For a detailed investigation on prime numbers, we will need to build up some tools. The main objects of study in this course are functions of number-theoretic origin, i.e. arithmetic functions. A good deal of the course will be spent building up techniques for estimating sums of arithmetic functions and showing their equivalence with Dirichlet series. The latter allows us to bring to bear powerful techniques from real and complex analysis on the study of arithmetic functions. These tools will lead us to clear proofs of Dirichlet's theorem on primes in arithmetic progressions and the Prime Number Theorem.

Recommended Texts

1. Jameson, *The Prime Number Theorem.*
2. E. Stein and R. Shakarchi, *Fourier Analysis. An introduction, Princeton Lectures in Analysis I,* (2003), Chapters 7, and 8.
3. E. Stein and R. Shalarchi, *Complex Analysis, Princeton Lectures in Analysis II,* (2003), Chapters 6, and 7.

Detailed Syllabus

The infinitude of primes: Euclid's method and Euler's method.

Sums of arithmetic functions: monotone comparison theorem, summation by parts, Euler-Maclaurin summation formula, Dirichlet convolution, von Mangoldt function, the Dirichlet convolution algebra and inverses in it, summation of Dirichlet convolutions in particular the divisor function, the Möbius function and inversion formula, Dirichlet hyperbola method, multiplicative functions, Chebyshev's theorem, p-adic valuation, Mertens's theorems.

Dirichlet Series: abscissa of absolute convergence, holomorphy of Dirichlet series, derivatives, products and reciprocals of Dirichlet series, isomorphism with Dirichlet convolution algebra, non-vanishing theorems for Dirichlet series, Euler products, isomorphism to multiplicative functions and non-vanishing theorems.

Primes in Arithmetic Progressions: characters of a finite abelian group, characters as an eigenbasis, the dual group, orthogonality relations, Dirichlet characters, Dirichlet L-functions, analytic continuation, Mertens's theorem in arithmetic progressions, non-vanishing of Dirichlet L-functions at s=1, Landau's lemma.

The Prime Number Theorem: reduction to the anti-derivative of the summatory function of the von Mangoldt function, the method of contour integration, upper bounds for $\zeta(1+it)$, the Riemann Hypothesis, non-vanishing of $\zeta(1+it)$ and lower bounds, proof of the prime number theorem by contour integration.