MATH0083 Prime Numbers and their Distribution

Year:	2021–2022
Code:	MATH0083
Level:	7 (UG)
Value:	15 credits (= 7.5 ECTS credits)
Normal student group(s):	UG Year 3 and 4 Mathematics degrees
Term:	1
Assessment:	40% examination, 60% coursework
Normal Pre-requisite:	MATH0013, MATH0051, MATH0034
Lecturer:	Dr I Petrow

Course Description and Objectives

The prime numbers appear to be distributed in a very irregular way. The Prime Number Theorem gives an asymptotic expression for the number of primes less than a given number. It is unquestionably one of the great theorems of mathematics. We will provide a simple and clear exposition of the theorem and its proof.
The Prime Number Theorem is one case of a wider circle of theorems on Dirichlet series. These were introduced to prove that there are infinitely many primes in arithmetic progressions.
We will provide a thorough account of the concepts and methods needed (Dirichlet series, summation by parts, contour integration techniques).
We will meet the Riemann hypothesis (unsolved problem) and understand its relation to the distribution of prime numbers.

Recommended Texts

1. Jameson, *The Prime Number Theorem.*
2. E. Stein and R. Shakarchi, *Fourier Analysis. An introduction, Princeton Lectures in Analysis I,* (2003), Chapters 7, and 8.
3. E. Stein and R. Shalarchi, *Complex Analysis, Princeton Lectures in Analysis II,* (2003), Chapters 6, and 7.

Detailed Syllabus

Arithmetic functions, multiplicative functions, the divisor function and its averages, the Möbius function, Möbius inversion formula, Dirichlet convolution.
Elementary Prime Number Theory: The series $\sum 1/p$, the von Mangoldt function, Chebysheff's theorem, Mertens' first and second theorems.
Dirichlet series: convergence, the relation between convolution of arithmetic functions and the corresponding Dirichlet series. Euler products.
Primes in arithmetic progressions: Dirichlet characters, orthogonality of characters, Dirichlet L-functions. Analytic continuation to $\Re(s) > 0$. Non-vanishing of $L(1, \chi)$. Proof of Dirichlet's theorem.
The Prime Number Theorem: Non-vanishing of $\zeta(1 + it)$. Tauberian argument. Method of contour integration. Proof of the Prime Number Theorem.
Riemann's zeta function: Analytic continuation and functional equation. Riemann's memoir and the Riemann hypothesis. Zero-free regions, counting zeroes in the critical strip, Riemann's explicit formula. Sketch of the Prime Number Theorem with error term.