

MATH0073 Representation Theory

<i>Year:</i>	2021–2022
<i>Code:</i>	MATH0073
<i>Level:</i>	7 (UG)
<i>Normal student group(s):</i>	UG: Year 3 and 4 Mathematics degrees
<i>Value:</i>	15 credits (= 7.5 ECTS credits)
<i>Term:</i>	2
<i>Assessment:</i>	60% Final Exam, 30% Mid-term Test, 10% Coursework
<i>Normal Pre-requisites:</i>	MATH0053, (MATH0021 recommended)
<i>Lecturer:</i>	Dr D Beraldo

Course Description and Objectives

The representation theory of finite groups, which solidifies one's knowledge of group theory, is perhaps the easiest part of the general theory of symmetry. It goes back to F. Klein who considered the possibility of representing a given abstract group by a group of linear transformations (matrices) preserving the group's structure, leading mathematicians such as G. Frobenius, I. Schur, W. Burnside and H. Maschke to follow and develop the idea further. Essentially, it is a formal calculus designed to give an explicit answer to the question

“What are the different ways (homomorphisms) a finite group G can occur as a group of invertible matrices over a particular field \mathbb{F} ?”.

The link between group representations over a field \mathbb{F} and modules is obtained using the concept of a group ring $\mathbb{F}[G]$, thus an essential step is the systematic study and classification of group rings (the so-called semisimple algebras) which behave like products of matrix rings. Therefore, the story of the representation theory of a group is the theory of all $\mathbb{F}[G]$ -modules, viz modules over the group ring of G over \mathbb{F} . The ultimate goal of this course is to teach students how to construct complex representations for popular groups as well as their character tables which serve as invariants for group rings.

In addition to the obvious applications to physical symmetry, the theory leads to significant insights into the structure theory of finite groups.

Detailed Syllabus

Examples of complex and real group representations. Semisimple modules. Schur's Lemma.

Division and endomorphism rings.

Classification of finite dimensional division algebras over \mathbb{C} and \mathbb{R} .

Characterisation theorem of semisimple modules.

Characterisation of finite dimensional semisimple algebras as products of matrix algebras over division rings.

Group rings. Maschke's Theorem. Exact Sequences.

Detailed structure of finite group rings over an algebraically closed field via the Artin-Wedderburn Decomposition Theorem. In particular, if

$$\bar{k}[G] = M_{n_1}(\bar{k}) \times \dots \times M_{n_m}(\bar{k})$$

then $m =$ number of conjugacy classes and

$$|G| = n_1^2 + \dots + n_m^2.$$

Explicit calculations for groups of small order. Tensor products of algebras. Induced representations. Character theory, idempotency, nilpotency, inner products and orthogonality relations.

Elementary representation theory over \mathbb{R} .

If time permits, co-induced representations, Frobenius Reciprocity, the Eckmann-Shapiro Lemma and Burnside's Theorem on the solubility of groups of order $p^a q^b$.

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