# MATH0061 Further topics in algebraic number theory

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| *Year*: | 2023–2024 |
| *Code*: | MATH0061 |
| *Level*: | 7 (UG) |
| *Normal student group(s)*: | UG Year 4 Mathematics degrees |
| *Value*: | 15 credits (= 7.5 ECTS credits) |
| *Term*: | 2 |
| *Assessment*: | 90% examination, 10% coursework |
| *Normal Pre-requisites*: | MATH0035 and MATH0022 |
| *Lecturer:* | Dr L García |

*Course Description and Objectives*

This is an advanced course in algebraic number theory. In a first course (e.g. MATH0035) one studies number fields –the fields obtained as finite extensions of the rational numbers– and learns that each number field has its own ring of integers. The failure of unique factorisation in this ring is measured by an important invariant called its class number.
The aim of this course is to delve more deeply into the arithmetic of number fields. We will study units in number fields and some of their applications to solving diophantine equations. We will also introduce $L$-functions and learn about their relation to class numbers through a fundamental theorem called the analytic class number formula. Finally, we will explore the relations between class groups and extensions of number fields with abelian Galois group, leading to the important subject of class field theory.
All the above topics will be introduced and studied with an emphasis on examples and explicit computations. At the end of the course you will be able to do many non-trivial computations in number fields. The theory we will develop is the basis for one of the most exciting current projects in all of mathematics: the Langlands program.

*Recommended Texts*

(i) D. A. Marcus, *Number Fields*, Springer Universitext, 2018.

(ii) K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer GTM, 84, 1990.

(iii) P. Samuel, *Algebraic Theory of Numbers*, Dover Books on Mathematics.

(iv) D. A. Cox, *Primes of the form $x^2 + ny^2$: Fermat, Class Field Theory and Complex Multiplication*, John Wiley & Sons, 2013.

*Assessment*

The final exam and coursework will account for 90% and 10% of the final mark respectively. The coursework will be in the form of five problem sheets that will be posted biweekly on Moodle with a one-week deadline. They will be graded and returned within a week of being handed in. The CW mark will be computed as the average mark of all five problem sheets.

*Detailed Syllabus*

- Units in number fields. Dirichlet's unit theorem. Cyclotomic units. Applications to Diophantine equations.

- Relative theory of number fields: relative integral bases, relative discriminant, Dedekind's criterion in the relative case.

- The Artin symbol. Relations to factorizing polynomials modulo primes, splitting of primes in extensions. Relation to quadratic residue symbol for quadratic extensions of the rational numbers. Examples.

- Dedekind zeta function, convergence, Euler product. Factorising the Dedekind zeta function into Dirichlet L-functions for some abelian extensions. Class number formula. Connections with Dirichlet's theorem on primes in arithmetic progressions.

- Statement of Artin reciprocity law. Ray class groups. Statement of existence theorem. Simple examples of class fields. Primes of the form $x^2 + ny^2$.

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