MATH0021 Homological Algebra

Year:	2023–2024
Code:	MATH0021
Level:	6 (UG)
Normal student group(s):	UG: Year 3 Mathematics degrees
Value:	15 credits (= 7.5 ECTS credits)
Term:	2
Assessment:	100% examination
Normal Pre-requisites:	MATH0053
Lecturer:	Prof FEA Johnson

Course Description and Objectives

Rings are the basic mathematical entities in which we calculate. The most elementary examples are fields and the resulting theory is completely covered in the algebra courses in years 1 and 2.

In this course we study linear algebra over more general commutative rings;here the notion of 'vector space over a field' is replaced by that of 'module over a ring'. In this wider context many of the familiar aspects of linear algebra require modification. For example the 'basis theorem' for vector spaces is replaced by the notion of 'free resolution' for more general modules.

We will concentrate on a class of 'well behaved' rings, the so called Noetherian rings. Examples are the rings $\mathbf{F}[t_1, \ldots, t_n]$, $\mathbf{Z}[t_1, \ldots, t_n]$ of polynomials over a field $\mathbf{F}$ and integers $\mathbf{Z}$. As an analogue and modification of the Jordan Normal Form Theorem we shall classify modules over Principal Ideal Domains (PIDs). A special case is the classification theorem for finitely generated abelian groups.

We shall introduce the notion of the 'dimension of a ring'. This indicates how far from being a field a given ring is. Thus fields have dimension 0, $\mathbf{Z}$ has dimension 1, whilst the polynomial ring $\mathbf{Z}[t_1, \ldots, t_n]$ has dimension $n + 1$.

This course will provide a solid foundation of commutative rings and module theory, as well as help developing foundational notions helpful in other areas such as number theory, algebraic geometry, and homological algebra.

Recommended Texts

1. N. Jacobson. *Basic Algebra, in two volumes but especially volume 2*
2. S. Lang, *Algebra*
3. C. Weibel,*Introduction to homological algebra*

Detailed Syllabus

- Examples of rings; $\mathbf{Z}$, $\mathbf{F}[t]$, $\mathbf{F}[t, t^{-1}]$, $\mathbf{F}[t_1, \ldots, t_n]$, $\mathbf{Z}[t_1, \ldots, t_n]$. Basic constructions; ideals and quotients; direct products; tensor products. Noetherian rings; Hilbert's Basis Theorem.
- Modules over rings. Basic constructions; products, direct sums, quotients; free modules and projective modules; Schanuel's Lemma; finiteness conditions and Noetherian modules.

- Exact sequences and splitting criteria; extensions of modules and the group $\mathrm{Ext}^1(M,N)$; resolution by free modules ; projective modules and the condition '$\mathrm{Ext}^1(-,N) = 0$'; cohomological interpretation of $\mathrm{Ext}^1(M,N)$.
- Classification of modules over P.I.D.s. Smith Normal Form and Generalized Euclidean rings; examples $\mathbf{Z}$, $\mathbf{F}[x]$, $\mathbf{F}[x,x^{-1}]$.
- Standard cohomological calculations. Hilbert's theorem on global dimension of $R[t_1,\ldots,t_n]$.
- Milnor's construction of nontrivial projective modules; $R[t]/(t^{2n}-1)$ when $R = \mathbf{F}[x]$.