Ouverture: the art of being a blow-up

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Visualization techniques

real trace \( \mathbb{R} \hookrightarrow \mathbb{C} \rightarrow \mathbb{R}_{\geq 0} \)

\[ z \mapsto \frac{1}{2}|z|^2 \]

“moment map”

\( \mathbb{C} \rightarrow \mathbb{R} \cup \{\infty\} \)

\[ z \mapsto -\log |z| \]

“tropical limit”
Example: Drawing a curve

The challenge is to represent higher dimensional varieties...
The old Italian school realized that birational equivalence is a sensible relation up to which algebraic varieties may be classified.

Two algebraic varieties are birational if they contain isomorphic dense Zariski-open subsets.

Any birational morphism between smooth surfaces can be factored as a finite sequence of blow-ups.

The Minimal Model Program in dimension two asserts that any surface can be obtained as a blow-up of a minimal surface $S$, i.e. $K_S$ nef, or a Hirzebruch surface, or $\mathbb{P}^2$. 
Let $U$ be a small (analytic) neighbourhood of $p = (0, 0)$ in $\mathbb{C}^2$. Denote by $f : \text{Bl}_p \mathbb{C}^2 \to \mathbb{C}^2$ the blow-up of $\mathbb{C}^2$ at $p$. 
“The infinitesimal behaviour of functions, maps, or differential forms at the point \( p \) is transformed into global phenomena on the blow-up.”

[Griffiths–Harris, Principle of algebraic geometry]
Blow-up I: separate ’infinitely near point’ of $x$

- Separate the family of line through $p$.
- Replace $p$ with the space parametrizing all tangent directions at $p$;
- Isomorphism away from $p$, i.e. the morphism $f : \text{Bl}_p \mathbb{C}^2 \setminus f^{-1}(p) \to \mathbb{C}^2 \setminus \{p\}$.
- The locus where $f$ fails to be an isomorphism, namely

$$E := f^{-1}(p) \sim \mathbb{P}^1_{\mathbb{C}} \approx S^2,$$

is called **exceptional divisor**.
- The real trace of $\text{Bl}_p \mathbb{C}^2$ is a Möbius strip.
Real trace of the blow-up

Möbius strip
Blow-up as an incidence variety

\[ \text{Bl}_p \mathbb{C}^2_R = \{(x, l) \in \mathbb{R}^2 \times \mathbb{P}^1_R : x \in l\}. \]
Blow-up as a surgery I

\[ \text{Bl}_p \mathbb{C}^2 = \{(x, l) \in \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}} : x \in l\} \].

Blow-up II: total space of a disk bundle

Cut out a 4-dim ball and replace it with the total space of a disk bundle.

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Denote by $N$ the preimage of the small open set $U$ via $f$.

Is $N$ a trivial disk bundle, i.e. $N \simeq S^2 \times D^2$?

**No.** Otherwise,

$$S^3 \simeq \partial N \simeq \partial(S^2 \times D^2) \simeq S^2 \times S^1,$$

$$1 \simeq \pi_1(S^3) \neq \pi_1(S^2 \times S^1) \simeq \mathbb{Z}.$$
E as a (-1)-curve

Disk bundles over $S^2$ are identified up to diffeomorphism by the self-intersection $E^2$ of their zero section (see also Euler class).

$E^2 = \# \text{ intersection of } E \text{ with a generic smooth deformation/perturbation of it}$

$= \# E \cap s$, where $s$ is a section of the disk bundle transversal to the zero section.

Be careful! Take orientations into account!
E as a (-1)-curve

Construct a generic section of $X \to \mathbb{P}^1_{\mathbb{C}}$

- the disk bundle over $V_0 = \mathbb{P}^1_{\mathbb{C}} \setminus \infty$ is trivial;
- a smooth section over $V_0$ is simply a function $s|_{V_0} : V_0 \to D^2$;
- change chart and if $s|_{V_0}$ extends by continuity over $\infty$, it defines a global section;
- check transversality (or perturbe the section to achieve it).

$s'|_{V_0}(z) = 1 \sim s'|_{V_\infty}(w) = 1/w$ DOES NOT EXTEND

$s|_{V_0}(z) = \begin{cases} 1 & |z| \ll 1 \\ |z|^2 & |z| \gg 1 \end{cases} \sim s|_{V_\infty}(w) = \begin{cases} 1/w & |w| \gg 1 \\ \frac{1}{w} & |w| \ll 1 \end{cases}$
$E^2 = -1$

Figure: The graph of the norm of the $\mathcal{C}^\infty$-section $s|_{\nu_\infty}$. 
More about orientation: connected sum

\[ \frac{1}{2} \left( e^{i\theta} + \frac{1}{e^{i\theta}} \right) \]

Orientation Reversing
Let $M$ be a complex surface and $M'$ be the blow-up of $M$ at a point $p$.

$$M' \simeq (M \setminus U) \cup_{\partial U} s^3 \simeq \partial U_q \left( \mathbb{P}^2_C \setminus U_q \right) = M \# \mathbb{P}^2_C.$$ 

In particular, the Minimal Model Program for surfaces implies that

$$M \simeq M_{\text{min}} \# \mathbb{P}^2_C \ldots \# \mathbb{P}^2_C.$$
Moment maps and toric fan of a blow-up
“Blowing up amounts to removing the interior of a symplectic ball and collapsing the bounding sphere to the exceptional divisor by the Hopf map.” [McDuff–Salamon, Introduction to Symplectic topology]
Thank you for your attention!