Ouverture: the art of being a blow-up

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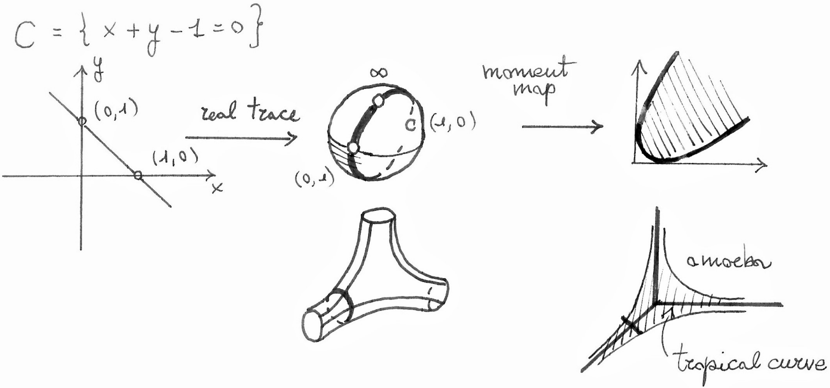
Visualization techniques

real trace

$$\mathbb{R} \hookrightarrow \mathbb{C} \to \mathbb{R}_{\geq 0} \qquad \text{“moment map”}$$
$$z \mapsto \frac{1}{2}|z|^2$$

$$\mathbb{C} \to \mathbb{R} \cup \{\infty\} \qquad \text{"tropical limit”}$$
$$z \mapsto -\log|z|.$$

Example: Drawing a curve



The challenge is to represent higher dimensional varieties...

Blow-up: Minimal Model Program

The old Italian school realized that birational equivalence is a sensible relation up to which algebraic varieties may be classified.

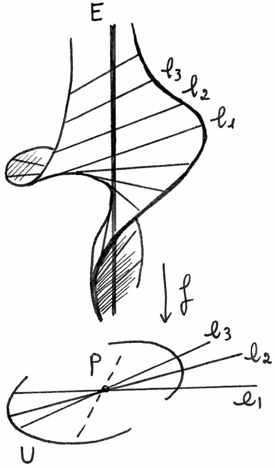
Two algebraic varieties are birational if they contain isomorphic dense Zariski-open subsets.

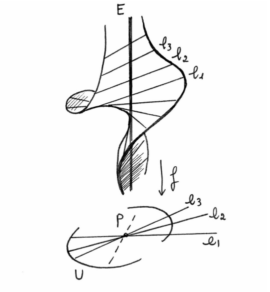
Any birational morphism between smooth surfaces can be factored as a finite sequence of blow-ups.

The Minimal Model Program in dimension two asserts that any surface can be obtained as a blow-up of a minimal surface S, i.e. K_S nef, or a Hirzebruch surface, or $\mathbb{P}^2$.

Blow-up: helicoidal picture

Let U be a small (analytic) neighbourhood of $p = (0, 0)$ in $\mathbb{C}^2$.
Denote by $f : \mathrm{Bl}_p \mathbb{C}^2 \to \mathbb{C}^2$ the blow-up of $\mathbb{C}^2$ at p.





"The infinitesimal behaviour of functions, maps, or differential forms at the point p is transformed into global phenomena on the blow-up." [Griffiths–Harris, Principle of algebraic geometry]

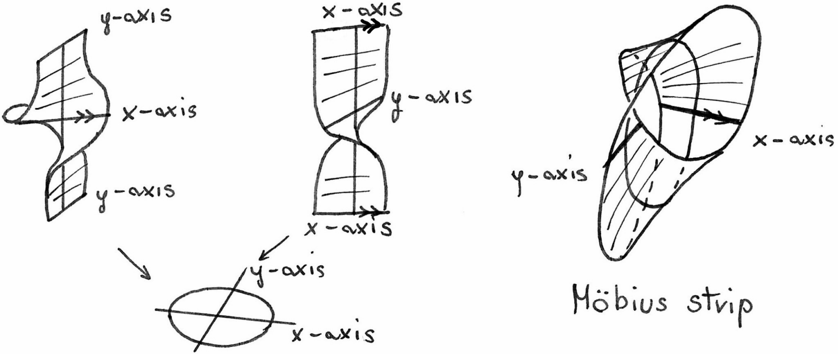
Blow-up I: separate 'infinitely near point' of x

- Separate the family of line through p.
- Replace p with the space parametrizing all tangent directions at p;
- Isomorphism away from p, i.e. the morphism $f : \mathrm{Bl}_p \mathbb{C}^2 \setminus f^{-1}(p) \to \mathbb{C}^2 \setminus \{p\}$.
- The locus where f fails to be an isomorphism, namely

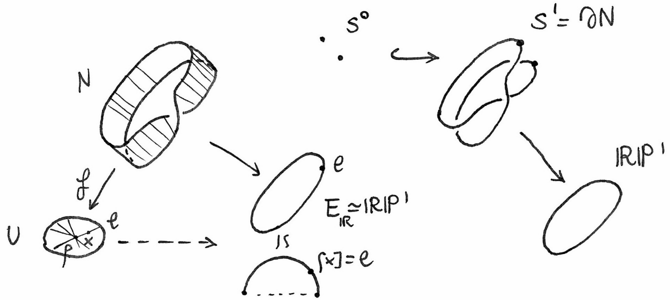
$$E := f^{-1}(p) \simeq \mathbb{P}^1_{\mathbb{C}} \simeq S^2,$$

  is called **exceptional divisor**.
- The real trace of $\mathrm{Bl}_p \mathbb{C}^2$ is a Möbius strip.

Real trace of the blow-up

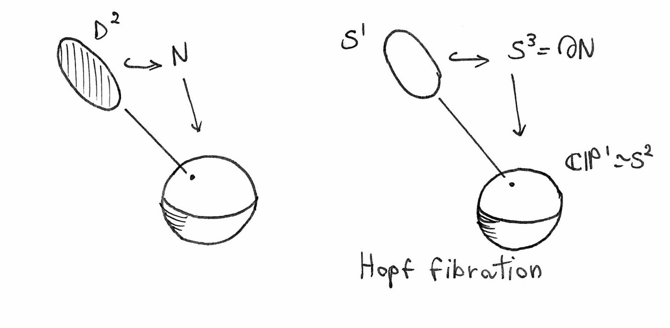


Blow-up as an incidence variety



$$\mathrm{Bl}_p \, \mathbb{C}^2_{\mathbb{R}} = \{(x, l) \in \mathbb{R}^2 \times \mathbb{P}^1_{\mathbb{R}} : x \in l\}.$$

Blow-up as a surgery I



$$\mathrm{Bl}_p \, \mathbb{C}^2_{\mathbb{C}} = \{(x, l) \in \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}} : x \in l\}.$$

Blow-up II: total space of a disk bundle

Cut out a 4-dim ball and replace it with the total space of a disk bundle.

Blow-up as the total space of a disk bundle

Denote by N the preimage of the small open set U via f.

Is N a trivial disk bundle, i.e. $N \simeq S^2 \times D^2$?

No. Otherwise,

$$S^3 \simeq \partial N \simeq \partial(S^2 \times D^2) \simeq S^2 \times S^1,$$

$$1 \simeq \pi_1(S^3) \neq \pi_1(S^2 \times S^1) \simeq \mathbb{Z}.$$

E as a (-1)-curve

Disk bundles over S^2 are identified up to diffeomorphism by the self-intersection E^2 of their zero section (see also Euler class).

$$\begin{aligned} E^2 &= \#\text{ intersection of E with a generic smooth} \\ &\qquad \text{deformation/perturbation of it} \\ &= \# E \cap s, \text{ where s is a section of the disk bundle transversal} \\ &\qquad \text{to the zero section.} \end{aligned}$$

Be careful! Take orientations into account!

E as a (-1)-curve

Construct a generic section of $X \to \mathbb{P}^1_{\mathbb{C}}$

- the disk bundle over $V_0 = \mathbb{P}^1_{\mathbb{C}} \setminus \infty$ is trivial;
- a smooth section over V_0 is simply a function

$$s|_{V_0} : V_0 \to D^2;$$

- change chart and if $s|_{V_0}$ extends by continuity over ∞, it defines a global section;
- check transversality (or perturbe the section to achieve it).

$$s'|_{V_0}(z) = 1 \quad \rightsquigarrow \quad s'|_{V_\infty}(w) = 1/w \text{ DOES NOT EXTEND}$$

$$s|_{V_0}(z) = \begin{cases} 1 & |z| \ll 1 \\ |z|^2 & |z| \gg 1 \end{cases} \quad \rightsquigarrow \quad s|_{V_\infty}(w) = \begin{cases} 1/w & |w| \gg 1 \\ \overline{w} & |w| \ll 1 \end{cases}$$

$E^2 = -1$

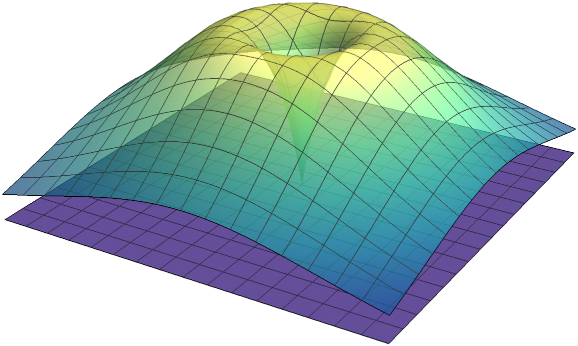
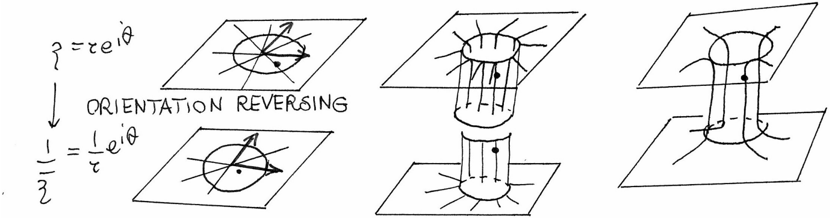
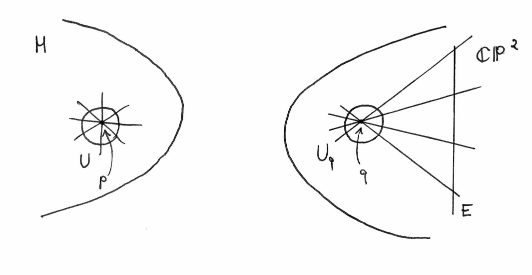


Figure : The graph of the norm of the $\mathcal{C}^\infty$-section $s|_{V_\infty}$.

More about orientation: connected sum



Blow-up as a surgery II



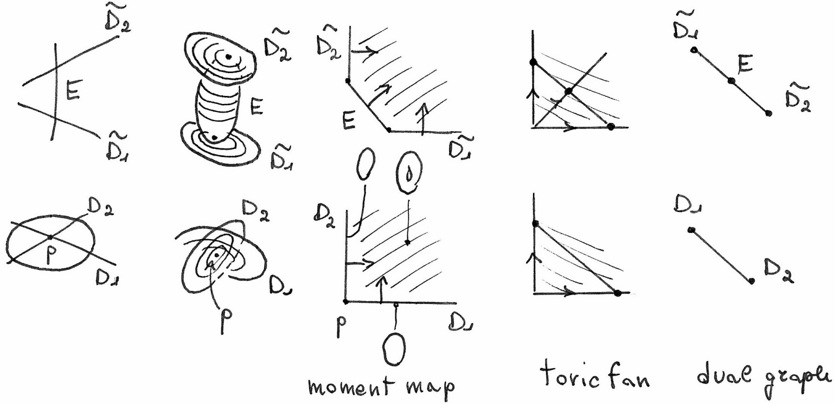
Let M be a complex surface and M' be the blow-up of M at a point p

$$M' \simeq (M \setminus U) \cup_{\partial U \simeq S^3 \simeq \partial U_q} (\overline{\mathbb{P}}^2_{\mathbb{C}} \setminus U_q) = M \# \overline{\mathbb{P}}^2_{\mathbb{C}}.$$

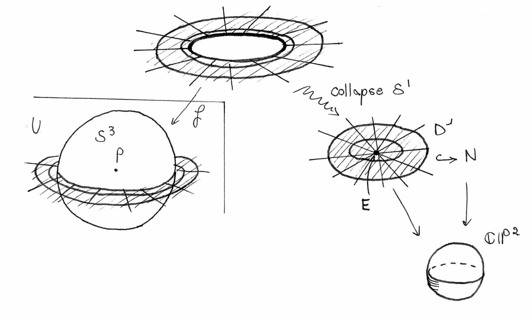
In particular, the Minimal Model Program for surfaces implies that

$$M \simeq M_{\min} \# \overline{\mathbb{P}}^2_{\mathbb{C}} \dots \# \overline{\mathbb{P}}^2_{\mathbb{C}}.$$

Moment maps and toric fan of a blow-up



Symplectic blow-up



Blow-up III: symplectic cut

"Blowing up amounts to removing the interior of a symplectic ball and collapsing the bounding sphere to the exceptional divisor by the Hopf map." [McDuff–Salamon, Introduction to Symplectic topology]

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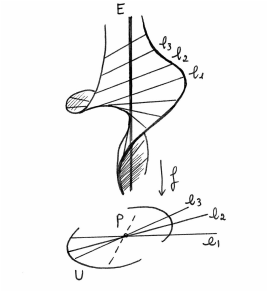

Thank you for your attention!