K3 Surfaces

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Disclaimer: These notes were originally just for me to consult while giving the talk. I've now been asked for them, so I have very rapidly added some more explanation to the first half, which previously just said things like "Introduce canonical bundle and $H^1(X, \mathcal{O}_X)$." The second half has been left as it was, so read at your own risk.

I strongly strongly recommend these notes by Svetlana Makarova: `https://gauss.math.yale.edu/~il282/Sveta_S16.pdf`. They are really excellent notes and were the main source of inspiration for my talk.

1 Definition of K3 Surfaces

1.1 Canonical Bundles

Let X be a smooth surface over a field k. For each point $p \in X$, there is a 2-dimensional vector space T_pX, called the **tangent space to X at p**, which consists of "tangent vectors" at p.

Write $\omega_{X,p}$ for the vector space of bilinear maps

$$f : T_pX \times T_pX \to k,$$

such that $f(a, b) = -f(b, a)$ for all $a, b \in T_pX$. It is easy (or at least elementary - I'm not sure how easy it is if you haven't seen it before) to show that $\omega_{X,p}$ is a one-dimensional vector space. Then the mapping

$$\omega_X : X \to \mathbf{Vect}_k, \quad p \mapsto \omega_{X,p}$$

assigns a 1-dimensional vector space to each point of X. Such an assignment is called a "line bundle" (modulo some techincal assumptions about the 1-dimensional vector spaces being "bundled" together in a well-behaved way).

This line bundle is called the **canonical bundle** of X.

1.2 First Cohomology

Again, let X be a smooth surface over k. There is a "sheaf" $\mathcal{O}_X$, which is some kind of function

$$\mathcal{O}_X : \{\text{open subsets } U \text{ of } X\} \to \{\text{abelian groups}\}$$

given by

$$\mathcal{O}_X(U) = \{\text{"nice" functions } U \to k\}.$$

In this case, "nice" generally means something like smooth or holomorphic in the differential geometry case, and regular in the algebraic geometry case.

Then $H^1(X, \mathcal{O}_X)$ denotes something called the "first cohomology" of this sheaf. If you're familiar with sheaf cohomology, then you know what this means. If you're familiar with homological algebra in general, then the point is that there is some abelian category of "sheaves", and a left exact "global sections" functor from this abelian category to the category of abelian groups. The sheaf cohomology $H^*(X, \mathcal{O}_X)$ is the then the right derived functor of this global sections functor, evaluated at $\mathcal{O}_X$.

If you're not familiar with any of this, you just need to know that $H^1(X, \mathcal{O}_X)$ is some abelian group, dependent on the surface X.

1.3 Definition of K3 Surfaces

So we have these two invariants, ω_X and $H^1(X, \mathcal{O}_X)$ of a surface X. For some reason, geometers care about these properties. Often in maths, before we try to study objects in general, we first tackle simpler special cases. For surfaces, therefore, it might make sense to consider the surfaces that are "simplest" with respect to the invariants ω_X and $H^1(X, \mathcal{O}_X)$, by assuming that these invariants are "trivial". A surface satisfying this assumption is called a **K3 surface**.

Definition 1.1. A **complex K3 surface** is a compact, connected, simply connected, complex surface X such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

In the definition above, $\mathcal{O}_X$ denotes the same sheaf we mentioned before, now viewed as the "trivial line bundle", where the vector space associated to a point $p \in X$ is the "stalk of $\mathcal{O}_X$ at p".

There is also an algebraic analogue, called an **algebraic K3 surface**, which has the advantage of being defined over any field. It's basically the same thing, but you need to be much cleverer in how you define ω_X and $H^1(X, \mathcal{O}_X)$, since fields can be really weird.

One neat relationship between algebraic and complex K3 surfaces is that any algebraic K3 surface defined over $\mathbb{C}$ is automatically a complex K3. That is,

$$\{\text{algebraic K3s over } \mathbb{C}\} \subsetneq \{\text{complex K3s}\}.$$

This is useful for coming up with examples, since it is often easier to write down the definition of an algebraic variety than of a manifold.

2 Examples

2.1 Smooth quartic hypersuraces

Let $F \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be an irreducible homogeneous quartic polynomial, and let $X = \mathbb{V}(F) \subseteq \mathbb{P}^3$ be the associated projective variety. If X is smooth, then it is a K3 surface.

This is where I stopped adding more explanation post-talk. The rest of the notes will be A LOT terser, but they might still be useful if you want a brief overview of some important stuff.

2.2 Kummer Surfaces

Let A be an abelian surface, and let $\iota : A \to A, x \mapsto -x$ be the natural involution. Quotienting by this involution yields a surface with some double point singularities, and these can be resolved by taking blowups. The resulting surface is called a Kummer surface, and it is a K3 surface.

3 Cohomology

LES of exponential exact sequence gives $H^1(X, \mathbb{Z}) = 0$ and

$$0 \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to \mathbb{C} \to H^2(\mathcal{O}_X^\times) \to H^3(X, \mathbb{Z}) \to 0.$$

Since $\mathrm{Pic}(X)$ and $\mathbb{C}$ are torsion-free, it follows that $H^2(X, \mathbb{Z})$ is torsion-free. By torsion-shift and then Poincaré duality, we have

$$\begin{aligned} H^3(X, \mathbb{Z}) &\cong \mathrm{free}(H_3(X, \mathbb{Z})) \oplus \mathrm{torsion}(H_2(X, \mathbb{Z})) \\ &\cong \mathrm{free}(H^1(X, \mathbb{Z})) \oplus \mathrm{torsion}(H^2(X, \mathbb{Z})) \\ &= 0. \end{aligned}$$

So the only nontrivial integral cohomology of X is $H^2(X, \mathbb{Z})$. We get

i	$H^i(X, \mathbb{Z})$
0	$\mathbb{Z}$
1	0
2	$\mathbb{Z}^{22}$
3	0
4	$\mathbb{Z}$

Definition 3.1. A lattice if an fg free abelian group together with a symmetric non-degenerate bilinear form on it.

The group $H^2(X, \mathbb{Z})$ may be given the structure of a lattice by the intersection form

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}.$$

For any complex K3, we always have a lattice isomorphism

$$H^2(X, \mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} =: \Lambda,$$

and this lattice is called the **K3 lattice**. Since it is always the same, the lattice structure of $H^2(X, \mathbb{Z})$ is not very interesting. Much more interesting is the sublattice given by the image of the inclusion

$$\mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z}).$$

This is a free abelian group of rank $0 \leq \rho(X) \leq 20$. The "Picard rank" $\rho(X)$ is a very important invariant of the K3 surface X.

4 Period Map

Write $H^2(X, \mathbb{C})$ for the extension of scalars

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

By the universal coefficient theorem, this coincides with the usual notion of cohomology with coefficients in $\mathbb{C}$. There is a canonical direct sum decomposition

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

called the **Hodge decomposition of** $H^2(X, \mathbb{C})$. We have

$$\dim H^{2,0} = 1, \quad \dim H^{1,1} = 20, \quad \dim H^{0,2} = 1.$$

For any complex K3 surface X, the isomorphism $H^2(X, \mathbb{C}) \to \Lambda_{\mathbb{C}}$ takes $H^{2,0}$ to a 1-dimensional subspace of $\Lambda_{\mathbb{C}}$, satisfying certain properties.

For a certain choice of canonical isomorphisms $H^2(X, \mathbb{Z}) \to \Lambda_{\mathbb{C}}$, this gives us a map

$$\mathcal{P} : \{\text{complex } K3 \text{ surfaces}\} \to \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \text{certain nice properties}\},$$

called the **period map**. It turns out that this map is surjective, and this fact is called "surjectivity of the period map".