STRATEGIC NETWORK FORMATION WITH MANY AGENTS

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Abstract. We consider a random utility model of strategic network formation, where we derive a tractable approximation to the distribution of network links using many-player asymptotics. Our framework assumes that agents have heterogeneous tastes over links, and allows for anonymous and non-anonymous interaction effects among links. The observed network is assumed to be pairwise stable, and we impose no restrictions regarding selection among multiple stable outcomes. Our main results concern convergence of the link frequency distribution from finite pairwise stable networks to the (many-player) limiting distribution. The set of possible limiting distributions is shown to have a fairly simple form and is characterized through aggregate equilibrium conditions, which may permit multiple solutions. We analyze identification of link preferences and propose a method for estimation of preference parameters. We also derive an analytical expression for agents' welfare (expected surplus) from the structure of the network.

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1. Introduction

Network models can be used to describe systems of contracts, transactions, and other formal or informal relationships between economic agents. In many economic contexts, the incentives to form new network connections exhibit strategic interdependencies across links. In models of trust and social capital, risky exchanges may be secured through transactions with third parties (see e.g. Jackson, Rodriguez-Barraquer, and Tan (2012), Ambrus, Mobius, and Szeidl (2014), Gagnon and Goyal (2016)) which may help with screening, monitoring, and enforcement of an agreement. When networks provide access to information, link formation incentives depend crucially on how a signal is transmitted through that network (see Calvó-Armengol (2004) and Calvó-Armengol and Jackson (2004)). For example, an agent may obtain more widely sourced information through a more central nodes, but may at the same time have to compete with a larger number of network neighbors for access to that information. For friendship networks, Currarini, Jackson, and Pin (2009) formulate a model

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for racial segregation with homophilous preferences, where agents choose endogenously how much effort to spend on searching for potential friends. For theoretical or empirical models of peer effects or social coordination, for example in the decision to smoke or engage in other types of risky behavior among high school students, the friendship network often has to be regarded as endogenous with respect to the relevant outcome if that activity itself plays a significant role in agents' social life, or the decision depends on unobservables that may also influence friendship formation (see e.g. Goyal and Vega-Redondo (2005), Goldsmith-Pinkham and Imbens (2012), and Badev (2016)). Strategic incentives of this type may or may not lead to formation of the most beneficial links in terms of aggregate welfare or a social planner's objective. In either case, distinguishing "strategic" externalities from "intrinsic" preferences for forming social or economic relationships are of immediate policy relevance.

This paper proposes a canonical empirical framework for models of network formation which allows to translate premises and predictions of (typically more stylized) theoretical models into testable hypotheses. Specifically, we consider a random utility model where network links are undirected and discrete, and link preferences may depend on agents' exogenous attributes and (endogenous) position in the network. To frame ideas, a parametric model could specify a net payoff to agent i from establishing a link to agent j that depends on either node's exogenous attributes x_i, x_j, the respective number of agents s_i, s_j either node is directly connected to, and an indicator t_{ij} whether i and j have another network neighbor in common. The incremental benefit to i from forming such a link could then be of the form

$$U_{ij} = x_i'\beta_1 + x_j'\beta_2 + |x_i - x_j|'\beta_3 + \gamma_1 s_i + \gamma_2 s_j + \delta t_{ij} + \eta_{ij}$$

where η_{ij} an idiosyncratic shock to i's preferences of forming a link to j. These random utilities could be regarded as continuation values or "reduced form" payoffs from economic activity on the resulting network, reflecting strategic motives of the kind discussed at the beginning of this introduction. Our framework allows for more general payoffs depending on exogenous and endogenous characteristics, and assumes that the observable network is pairwise stable (Jackson and Wolinsky (1996)), where a link ij forms if and only if the incremental benefit of that link to either node exceeds the cost of maintaining that link. Pairwise stability is the default solution concept for models strategic network formation in economics (see e.g. Jackson (2008)) and imposes only minimal requirements on agents' strategic sophistication. The main technical challenges in estimating a model of this form is that the variables s_i, s_j, t_{ij} are a function of the network graph, and therefore endogenous to the network formation model.

The main theoretical result is a tractable approximation to the resulting distribution(s) over networks, assuming that the number of nodes (agents) in the network is large. Our analysis identifies the relevant aggregate state variables that characterize equilibrium and interdependence of individual link formation decisions, and shows how to use (many-agent) limiting approximations to simplify the representation of the network in terms of these variables. We derive a sharp characterization of the set of link distributions that can be generated by pairwise stable networks. With strategic interaction effects between links, this set is in general not a singleton. Based on this limiting approximation we then propose strategies for estimation and inference regarding the model parameters. We also derive an analytical limiting expression for the expected surplus to the agent of forming the links of the pairwise stable network - in particular, when preferences depend only on exogenous node attributes, expected surplus for a given node equals the expected number of its network neighbors. That characterization of surplus can be used to analyze incentives for participating in network formation or exerting effort searching for potential friends (as in the model of Currarini, Jackson, and Pin (2009)), as well as welfare analysis for policy interventions that affect the shape of the network.

The asymptotic approximation is obtained by embedding the finite-player network corresponding to the observable data into a sequence of network formation models with an increasing number of agents. Using statistical approximation techniques, we derive the limit for the distribution of links along that sequence. The primary motivation for many-agent asymptotics in the network model is to arrive at a tractable model that does not require an explicit account for certain interdependencies that are not of first order in the limiting experiment. In particular, the limiting sequence considered has the following qualitative features: (1) each agent can choose from a large number of possible link formation opportunities, and (2) similar agents face similar choices, at least as measured by the inclusive values corresponding to link opportunity sets. (3) By construction, additional links become increasingly costly along the asymptotic sequence, so that the resulting network remains sparse. (4) The limiting distribution of links resulting from pairwise stable network formation need in general not be unique. Rather, a given realization of payoffs may support multiple pairwise stable networks that differ qualitatively both in terms of global, aggregate features, as well as locally in assigning nodes different roles under alternative equilibria. The limiting sequence does not impose any additional qualitative constraints on agents' incentives for forming network links.

Our approach incorporates some qualitative insights on many-agent limits of game-theoretic models and matching markets from Menzel (2016), Dagsvik (2000), and Menzel (2015). However the main new technical challenges in analyzing large networks cannot be addressed using the formal tools developed in these papers. Most importantly, many realistic models of strategic externalities in link formation need to allow for strong (statistical) dependence across the entire network. The limiting arguments developed in this paper (most importantly Lemma 4.1) relying on symmetric (exchangeable), rather than weak dependence are to my knowledge entirely new and may serve as a blueprint for limiting arguments in large games beyond the context of networks. Furthermore, in a network formation problem with link externalities, non-uniqueness of stable outcomes results in a non-singleton set of limiting distributions, adding conceptual difficulties in taking many-player limits. In contrast, the structure of the matching problem in Menzel (2015) was shown to imply weak dependence of matching outcomes and resulted in a unique limiting distribution.

Literature. A powerful and convenient formal framework for describing networks is the classical Erdős and Rényi (1959) random graph (RGM) model. The RGM describes links as independent binary random variables and has been extended in various ways to allow for node- and edge-level heterogeneity and has been used in applied work (see e.g. Fafchamps and Gubert (2007)). For strategic models of network formation, strategic interdependence between link formation decisions typically leads to stochastic dependence between links and can therefore in general not be represented as RGM. Also, network data often exhibit clustering and degree heterogeneity in excess of levels compatible with that model. One approach to accommodate this empirical regularity into econometric models of link formation is to allow for preferential attachment and unobserved heterogeneity in the propensity of a node to form links (see Graham (2014) and Dzemski (2014)).

As an alternative, the model may directly incorporate (endogenous) network attributes - including degree centrality or network distance - as determinants of the link probability in a generalized exponential random graph model (ERGM). Mathematical properties of ERGM are by now fairly well understood, and some fairly general results on estimation and large-sample theory are already available.[1] Here Chandrasekhar and Jackson (2016) develop a flexible approach to match not only pairwise frequencies, but also subgraph counts involving three or more nodes. Our framework differs from these papers in that we characterize the network formation process using link preferences that may depend directly on endogenous network attributes. This introduces a strategic element into the model which in some cases produces interdependencies of link formation decisions between "distant" nodes, and typically yields a multiplicity of stable network outcomes. In particular, Chandrasekhar and Jackson (2016)'s assumption that subnetworks of certain types form independently is not generally consistent with pairwise stability under preferences that exhibit strategic interdependencies between different links. Lovasz (2012) showed how to characterize a finite network graph as a sample from a continuous limiting object. However when the graph is the result of strategic decisions by the agents associated with the (finitely many) nodes, the relationship between features of the descriptive limiting "graphon" to stable, "structural" features

[1] See e.g. Frank and Strauss (1986), Wasserman and Pattison (1996), Bickel, Chen, and Levina (2011), or Snijders (2011) for a survey. Jackson and Rogers (2007) analyze characteristics of large networks of homogeneous agents that result from a sequential random meeting process where links may be added "myopically" at each step.

of an underlying population is generally not transparent or even well-defined, especially if the network formation model admits multiple stable outcomes.

Most existing approaches to structural estimation rely heavily on simulation methods - this includes Hoff, Raftery, and Handcock (2002), Christakis, Fowler, Imbens, and Kalyanaraman (2010), Mele (2012), Sheng (2014), and Leung (2016) - whereas our approach focusses on analytic characterizations of pairwise stable networks. Instead of considering the joint distribution of the adjacency matrix or larger local "neighborhoods" within the network (as considered by Sheng (2014), de Paula, Richards-Shubik, and Tamer (2014) or Graham (2012)), we argue that it is typically sufficient for estimation to consider the frequencies of links between pairs of nodes (dyads) with a given combination of exogenous attributes and endogenous network characteristics. Our analysis differs from de Paula, Richards-Shubik, and Tamer (2014) in that our limiting model is constructed as a limiting approximation to a finite network, whereas their model assumes a continuum of players. Furthermore, we model link preferences as non-anonymous in the finite network, and therefore have to characterize explicitly how subnetworks interact with the full network through link availability and strategic interaction effects with neighboring nodes. Our asymptotic approximations allow to characterize that dependence using aggregate state variables that satisfy certain equilibrium conditions in order for the network to be pairwise stable. Boucher and Mourifié (2012) give conditions for weak dependence of network links under increasing domain asymptotics, whereas our approach can be thought of as "infill" asymptotics where link frequencies between distant nodes are non-trivial under any metric on the space of node characteristics. A concurrent paper by Leung (2016) gives conditions under which strategic interaction effects remain limited to subnetworks of finite size as the number of nodes grows large. Our approach does allow for long-range dependence of arbitrary strength, and relies on symmetry and exchangeability arguments instead.

The remainder of the paper is organized as follows: we first describe the economic model, including alternative solution concepts. Section 3 defines the limiting model, and section 4 gives formal results regarding convergence to that limit. Section 5 discusses strategies for identification and estimation based on that representation, and gives an analytical characterization of agents' welfare (expected surplus) from the structure of the network. Section 6 gives an outline of the main formal steps for the convergence argument. Section 7 presents a Monte Carlo study illustrating the theoretical convergence results.

2. Model Description

The network consists of a set of n agents ("nodes" or "vertices"), which we denote with $\mathcal{N} = \{1, \ldots, n\}$. We assume that each agent is associated with a vector of exogenous attributes (types) $x_i \in \mathcal{X}$, where the type space $\mathcal{X}$ is some (continuous or discrete) subset of a Euclidean space, and the marginal distribution of types is given by the p.d.f. $w(x)$. We also use $X = [x_1, \ldots, x_n]'$ to denote the matrix containing the n nodes' exogenous attributes.

Using standard notation (see Jackson (2008)), we identify the network graph with the *adjacency matrix* $\mathbf{L}$, where the element

$$L_{ij} = \begin{cases} 1 & \text{if there is a direct link from node } i \text{ to node } j \\ 0 & \text{otherwise} \end{cases}$$

As a convention, we do not allow for any node i to be linked to itself, $L_{ii} = 0$. We assume that all links are *undirected*, so that the adjacency matrix $\mathbf{L}$ is symmetric, i.e.$L_{ij} = L_{ji}$. We also let $L - \{ij\}$ be the network resulting from deleting the edge ij from $\mathbf{L}$, that is from setting $L_{ij} = L_{ji} = 0$. Similarly, $L + \{ij\}$ denotes the network resulting from adding the edge ij to $\mathbf{L}$.

In an idealized application, the observed network data consists of $\mathbf{X}$ and $\mathbf{L}$. However, the limiting approximations do not distinguish between observable and unobservable components of x_i and can therefore also be used in settings in which relevant exogenous characteristics are unobserved. Furthermore, our results can also be applied when the researcher only observes attributes and links for a randomly selected subset of nodes according to a known sampling rule.

2.1. **Payoffs.** Player i's payoffs are of the form

$$\Pi_i(\mathbf{L}) = B_i(\mathbf{L}) - C_i(\mathbf{L})$$

where $B_i(\mathbf{L})$ denotes the gross benefit to i from the network structure, and $C_i(\mathbf{L})$ the cost of maintaining links. We will see below that identification of costs and benefits entails a location normalization of some kind. Hence, we will generally assume that the cost $C_i(\mathbf{L})$ is only a function of the number of direct links to player i, but not the identities or characteristics of the individuals that i is directly connected to under the network structure $\mathbf{L}$.

We specify the model in terms of the incremental benefit of adding a link ij to the network $\mathbf{L}$,

$$U_{ij}(\mathbf{L}) := B_i(\mathbf{L} + \{ij\}) - B_i(\mathbf{L} - \{ij\})$$

and the cost increment of adding that link,

$$MC_{ij}(\mathbf{L}) := C_i(\mathbf{L} + \{ij\}) - C_i(\mathbf{L} - \{ij\})$$

With a slight departure from common usage of those terms, we refer to $U_{ij}(\mathbf{L})$ and $MC_{ik}(\mathbf{L})$ as the *marginal benefit* and *marginal cost* (to player i), respectively, of adding the link ij to the network.

Throughout our analysis we specify the marginal benefit function as

$$U_{ij}(\mathbf{L}) = U^*_{ij}(\mathbf{L}) + \sigma\eta_{ij} \tag{2.1}$$

where $U^*_{ij}(\mathbf{L})$ is a deterministic function of attributes $x_1, \dots, x_n$ and the adjacency matrix $\mathbf{L}$, and will be referred to as the *systematic part* of the marginal benefit function. The idiosyncratic taste shifters η_{ij} are assumed to be independent of x_i and x_j and distributed according to a continuous c.d.f. $G(\cdot)$, and $\sigma > 0$ is a scale parameter. Also, marginal costs are given by

$$MC_{ij}(\mathbf{L}) := \max_{k=1,\dots,J} \sigma\eta_{i0,k} \tag{2.2}$$

where $\eta_{i0,k}$ are independent of x_i and across draws $k = 1, 2, \dots$, and the choice of the number of draws J will be discussed in section 4. In particular, we let J to grow as n increases in order for the resulting network to be sparse. In this formulation, marginal costs do not depend on the network structure, so that in the following we denote marginal cost of the link ij by MC_i without explicit reference to j or the network $\mathbf{L}$. Note that in the absence of further restrictions on the systematic parts of benefits $U^*_{ij}(\mathbf{L})$, this is only a normalization.

The main application of our asymptotic results concerns identification of - parametric or nonparametric models for the function $U^*_{ij}(L)$, where the distribution of taste shocks $G(\cdot)$ need not be specified by the researcher as long as its upper tail is assumed to satisfy the shape restriction in Assumption 4.2 below. We find below that for some relevant aspects of the model, only the sum of the systematic part of marginal utilities between the two nodes constituting an edge matters, we also define the pseudo-surplus for the edge $\{ij\}$ as

$$V^*_{ij}(\mathbf{L}) := U^*_{ij}(\mathbf{L}) + U^*_{ji}(\mathbf{L})$$

Obviously $V^*_{ij}(\mathbf{L}) = V^*_{ji}(\mathbf{L})$, so pseudo-surplus is symmetric with respect to the identities of the two nodes.

Our framework allows for various types of interaction effects on the marginal benefit function. The marginal benefit from adding the link from i to j may depend on agent i and j's exogenous attributes x_i and x_j, and the structure of the network through vector-valued statistics S_i, S_j, T_{ij} that summarize the payoff-relevant features,

$$U^*_{ij}(\mathbf{L}) \equiv U^*(x_i, x_j; S_i, S_j, T_{ij}) \tag{2.3}$$

Specifically, the marginal benefit of a link may directly depend on node i and j's exogenous attributes, x_i and x_j, respectively, as well as interaction effects between the two. $U^*_{ij}(\mathbf{L})$ may vary in x_i, e.g. because some node attributes may make i attach more value to any additional links. On the other hand, dependence on x_j allows for target nodes with certain attributes to be generally more attractive as partners. Finally, a non-zero cross-derivative between components of x_i and x_j could represent economic complementarities, or a preference for being linked to nodes with similar attributes (homophily).

In addition to preferences for exogenous attributes, the propensity of agent i to form an additional link, and the attractiveness of a link to agent j may depend on the absolute position of either node i and j in the network. To account for effects of this type, we can include node-specific network statistics of the form

$$S_i := S(\mathbf{L}, \mathbf{X}; i)$$

where we assume that the function $S(\cdot)$ is invariant to permutation of player indices.[2]

Example 2.1. ***(Degree and Composition)*** *Node specific network statistics include the network degree (number of direct neighbors),*

$$S_1(\mathbf{L}, \mathbf{X}; i) := \sum_{j \neq i} L_{ij}$$

Another statistic could measure the share of i's direct neighbors that are of a given exogenous type,

$$S_2(\mathbf{L}, \mathbf{X}; i) := \frac{\sum_{j \neq i} L_{ij} \mathbb{1}\{x_{jk} = \bar{x}_k\}}{\sum_{j \neq i} L_{ij}}$$

where the kth component of x_j may be e.g. gender or race, and $\bar{x}_k$ the value corresponding to the category in question (e.g. with respect to gender or race).

The network degree of a node plays a special role in the description of the link frequency distribution. In the remainder of the paper, we therefore partition the vector of node i's network characteristics into $s_i = (s_{1i}, s_{2i}')'$, where $s_{1i} := \sum_{j=1}^{n} L_{ij}$ denotes the network degree of node i, and s_{2i} a vector of other payoff-relevant network statistics.

Node-specific network statistics depending on node types can also be used to develop joint models of link formation other economic decisions that may be subject to peer effects on the same network:

Example 2.2. ***(Peer Effects)*** *Suppose that x_i includes a subvector $(z_i', v_i)'$, where v_i is not observed by the econometrician. Suppose that in addition to forming links, each agent chooses a discrete action $Y_i \in \{0, 1\}$, where her best response is characterized by the random payoff inequality*

$$y_i = \mathbb{1}\left\{ z_i'\gamma + \delta \frac{1}{s_{1i}} \sum_{j \neq i} L_{ij} y_j + v_i \geq 0 \right\}$$

where $s_{1j} := S_1(\mathbf{L}, \mathbf{X}; j)$ is node j's degree, and γ, δ are model coefficients. If link preference and the discrete action are determined simultaneously, we can incorporate the interaction effect into the network formation model through a node-specific statistic of the form

$$S_3(\mathbf{L}, \mathbf{X}; i) := \mathbb{1}\left\{ z_i'\gamma + \delta \frac{1}{s_{1i}} \sum_{j \neq i} L_{ij} s_{3j} + v_i \geq 0 \right\}$$

[2]Formally, we assume that for any one-to-one map $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ and $i = 1, \dots, n$ we have $S(\mathbf{L}^\pi, \mathbf{X}^\pi; \pi(i)) = S(\mathbf{L}, \mathbf{X}; i)$, where the matrices $\mathbf{X}^\pi$ and $\mathbf{L}^\pi$ are obtained from $\mathbf{X}$ and $\mathbf{L}$ by permuting the rows (rows and columns, respectively) of the matrix according to π.

where $s_{3j} := S_3(\mathbf{L}, \mathbf{X}; j)$, provided a solution to the recursive system exists.

An example for peer effects of this type would be a model for youth smoking behavior, where smokers could e.g. be more likely to form friendships with each others. Models of peer effects with endogenous friendship networks have been analyzed, among others, by Goldsmith-Pinkham and Imbens (2012) and Badev (2016). Our asymptotic results for the link frequency distribution can therefore also be used to analyze the large-network outcomes of a model of this type. However a full discussion of identification and estimation of models of peer effects with endogenous link formation is beyond the scope of this paper and will be left for future research.

Payoffs may also depend on the relative position of the node i with respect to j in the network. Specifically, the researcher may also want to include edge-specific network statistics of the form

$$T_{ij} := T(\mathbf{L}, \mathbf{X}; i, j)$$

where $T(\cdot)$ may again be vector-valued, and we assume that the function $T(\cdot)$ is invariant to permutations of player indices.[3] In the following, we also assume that the statistic is symmetric, $T_{ij} = T_{ji}$.[4] In our description of preferences regarding T_{ij} we will occasionally use t_0 to denote an arbitrarily chosen "default" value for the statistic.

Example 2.3. ***(Transitive Triads)*** *A preference for closure of transitive triads can be expressed using statistics of the form*

$$T_1(\mathbf{L}, \mathbf{X}; i, j) = \sum_{k \neq i,j} L_{ik} L_{jk}, \quad \text{or } T_2(\mathbf{L}, \mathbf{X}; i, j) = \max\{L_{ik} L_{jk} : k \neq i, j\}$$

Here, T_{1ij} counts the number of immediate neighbors that both i and j have in common, and T_{2ij} is an indicator whether i and j have any common neighbor. More generally, T_{ij} could include other measures of the distance between agents i and j in the absence of a direct link, or indicators for potential "cliques" of larger sizes.

Patterns of transitivity may emerge for example in economic models of social capital where supporting links to common neighbors may enhance the value or viability of a connection between an agent pair, see e.g. Jackson, Rodriguez-Barraquer, and Tan (2012) or Gagnon and Goyal (2016). Transitivity may also reflect a biased search process where agents may be more likely to "meet" through common neighbors.

Some of our results concern special cases in which the network statistics S_i, S_j, and T_{ij} only depend on nodes at up to a finite network distance from i and j, respectively. Specifically,

[3] That is, we assume that for any permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ and $i, j = 1, \dots, n$ we have $T(\mathbf{L}^\pi, \mathbf{X}^\pi; \pi(i), \pi(j)) = T(\mathbf{L}, \mathbf{X}; i, j)$.

[4] In order to accommodate the general case of asymmetric edge-specific statistics, it would be possible an additional argument in the marginal benefit function, and the technical results would continue to go through without substantive modifications.

we say that S_i is *a function of the network neighborhood of radius* r_S *around* i if $S(\mathbf{L},\mathbf{X};i) = S(\tilde{\mathbf{L}},\mathbf{X};i)$ for any networks $\mathbf{L},\tilde{\mathbf{L}}$ such that $\tilde{\mathbf{L}}_{kl} = L_{kl}$ whenever the network distance between i and k is less than or equal to r_S. Similarly, we say that T_{ij} *is a function of the network neighborhood of radius* r_T *around* i *and* j if $T(\mathbf{L},\mathbf{X};i,j) = T(\tilde{\mathbf{L}},\mathbf{X};i,j)$ for any networks $\mathbf{L},\tilde{\mathbf{L}}$ such that $\tilde{\mathbf{L}}_{kl} = L_{kl}$ whenever the network distance of k to i or j is less than r_T.

In contrast to node attributes x_i, x_j, the variables S_i, S_j, and T_{ij} are endogenous to the network formation process, and the characterization of the limiting model therefore must include equilibrium conditions for the joint distribution of types x_i and network statistics S_i and T_{ij}. We therefore refer to the payoff contribution of the exogenous attributes x_i, x_j as *exogenous interaction effects*, and the contribution of the endogenous network characteristics s_i, s_j, t_{ij} as *endogenous interaction effects*.

In terms of this specification, we can also rewrite the pseudo-surplus function as

$$V^*_{ij}(\mathbf{L}) = V^*(x_i, x_j; s_i, s_j, t_{ij}) := U^*(x_i, x_j; s_i, s_j, t_{ij}) + U^*(x_j, x_i; s_j, s_i, t_{ij})$$

Dependence of payoffs on network statistics s_i, s_j, t_{ij} also allows to incorporate other constraints on the shape of the network by setting payoffs for links that are not "permissible" to minus infinity. For example, in a many-to-many matching model nodes may face capacity constraints on how many partners they are allowed to match with, e.g. the stable roommate problem admits at most one link per node. Such a restriction can be formulated using the network degree s_{1i}. For coalition formation games, which produce partitions on the set of players, we can define edge-specific indicators t_{ij} that are equal to one if i and j are only connected to nodes in the same connected component of $\mathbf{L} - \{ij\}$ and zero otherwise, and only permit connections with $t_{ij} = 0$. Adapting our framework to problems of this type requires some, in certain cases only minor, modifications of our benchmark framework.[5]

2.2. **Solution Concept.** Our formal analysis assumes pairwise stability as a solution concept, which was first introduced by Jackson and Wolinsky (1996).

Definition 2.1. ***(Pairwise Stable Network)*** *The undirected graph* $\mathbf{L}$ *is a* ***pairwise stable network (PSN)*** *if for any link* ij *with* $L_{ij} = 1$,

$$\Pi_i(\mathbf{L}) \geq \Pi_i(\mathbf{L} - \{ij\}), \quad \text{and } \Pi_j(\mathbf{L}) \geq \Pi_j(\mathbf{L} - \{ij\})$$

and any link ij *with* $L_{ij} = 0$,

$$\Pi_i(\mathbf{L} + \{ij\}) < \Pi_i(\mathbf{L}), \quad \text{or } \Pi_j(\mathbf{L} + \{ij\}) < \Pi_j(\mathbf{L})$$

[5] Specifically the requirement of bounded systematic payoffs in Assumption 4.1 below has to be relaxed to allow for infinite negative payoffs, and commonly used notions of stability in matching or coalition formation models often allow for a richer set of deviations than pairwise stability, as defined in Definition 2.1. As an illustration, we discuss an extension of our results for many-to-many matching models in Appendix A.

Pairwise stability as a solution concept only requires stability against deviations in which one single link is changed at a time. For a pairwise stable network there may well be an agent who can increase her payoff by reconfiguring two or more links unilaterally. While it is possible to consider stronger notions of individually optimal choice which require stability against more complex deviations, PSN gives a set of necessary conditions which have to be met by any such refinement.

Pairwise stability also does not necessarily impose particularly high demands on participating agents' knowledge and strategic sophistication: Jackson and Watts (2002) showed that pairwise stable networks can be achieved by tâtonnement dynamics in which agents form or destroy individual connections, taking the remaining network as given and not anticipating future adjustments. This makes PSN a plausible static solution concept for a decentralized network formation process even when agents have only a limited understanding of the network as a whole, and link decisions may in fact take place over time, where the exact sequence of adjustments is not known to the researcher.

A major limitation of PSN as a solution concept is that, without additional restrictions on payoffs a pairwise stable network is not guaranteed to exist. While to our knowledge there are no fully general existence results, there are some relevant special cases for which existence of a PSN is not problematic.[6] Our approach builds on local stability conditions for each link in isolation, and therefore only requires that any given link satisfies the pairwise stability conditions with probability approaching 1. Hence in the context of tâtonnement dynamics, existence of a pairwise stable network will not be strictly necessary for our approach to work as long as the share of links satisfying the conditions for PSN goes to one as n grows large.

A second challenge is that pairwise stability does not predict a unique outcome for the network formation game. Neither the static nor the tâtonnement interpretation of pairwise stability in a model of decentralized network formation appear to suggest a particular rule for selecting one stable outcome over another. In their most general version our results therefore do not constrain the mechanism for selecting among multiple pairwise stable matchings, but give sharp bounds on the distribution of network outcomes.

It is also interesting to contrast our use of an essentially static solution concept to the approaches in Christakis, Fowler, Imbens, and Kalyanaraman (2010) and Mele (2012) who consider link distributions resulting from myopic random revisions of past link formation decisions, where agents are not assumed to be forward-looking regarding future stages of the formation game. Christakis, Fowler, Imbens, and Kalyanaraman (2010) specify an initial condition and a stochastic revision process, so that (in the absence of further shocks to the

[6]As an example, Miyauchi (2012) considers the case of non-negative link externalities, in which case pairwise stability can be represented as Nash equilibrium in a finite game with strategic complementarities. Hence, existence and achievability through a myopic tâtonnement mechanism follow from general results by Milgrom and Roberts (1990).

process) further iterations of the tâtonnement process would generate a distribution over pairwise stable outcomes or cycles with mixing weights depending on that specification. The revision process in Mele (2012) is represented by a potential function, favoring formation of links that lead to larger cardinal utility improvements, and networks generating a large "systematic" surplus. Our approach allows for any pairwise stable outcome and does not take an implicit stand on equilibrium selection.

For a revealed-preference analysis it is useful to represent the pairwise stability conditions as a discrete choice problem, where preferences are given by the random utility model described above, and the set of available "alternatives" for links arises endogenously from the equilibrium outcome. Specifically, given the network $\mathbf{L}$ we define the *link opportunity set* $W_i(\mathbf{L}) \subset \mathcal{N}$ as the set of nodes who would prefer to add a link to i,

$$W_i(\mathbf{L}) := \{j \in \mathcal{N}\backslash\{i\} : U_{ji}(\mathbf{L}) \geq MC_j(\mathbf{L})\}$$

Using this notation, we can rewrite the pairwise stability condition in terms of individually optimal choices from the opportunity sets arising from a network $\mathbf{L}$.

Lemma 2.1. *Assuming that all preferences are strict, a network L^* is pairwise stable if and only if for all $i = 1, \ldots, n$,*

$$L^*_{ij} = \begin{cases} 1 & \text{if } U_{ij}(\mathbf{L}^*) \geq MC_{ij}(\mathbf{L}^*) \\ 0 & \text{if } U_{ij}(\mathbf{L}^*) < MC_{ij}(\mathbf{L}^*) \end{cases} \tag{2.4}$$

for all $j \in W_i(\mathbf{L}^*)$, and $L^*_{ij} = 0$ for all $j \notin W_i(\mathbf{L}^*)$.

The proof for this lemma is similar to that of Lemma 2.1 in Menzel (2015) and is given in the appendix.

2.3. **Equilibrium Selection.** Since a pairwise stable network is generally not unique, we assume that the observed network is generated by an equilibrium selection mechanism which we formalize as a distribution over initial conditions for a tâtonnement process. Since that myopic best-response dynamics is not guaranteed to converge to a PSN even if one exists, we use the notion of *closed cycles* introduced by Jackson and Watts (2002) to characterize the outcome of tâtonnement.[7]

[7]For a definition of a closed cycle, we say that in a given network $\mathbf{L}^{(0)}$, the edge ij is *active* if either $U_{ij}(\mathbf{L}^{(0)}) - MC_i(\mathbf{L}^{(0)})$ or $U_{ji}(\mathbf{L}^{(0)}) - MC_j(\mathbf{L}^{(0)})$ violate the payoff inequalities in Definition 2.1. We then say that the chain of networks $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \ldots$ is an *improving path* if for each s, $\mathbf{L}^{(s)}$ is obtained from $\mathbf{L}^{(s-1)}$ after sequentially adjusting one single link that is active under $\mathbf{L}^{(s-1)}$.
A finite set of networks $\mathcal{L}^* := \{\mathbf{L}^{(1)}, \ldots, \mathbf{L}^{(s)}\}$ is a *closed cycle* if (a) it is a cycle in that for any two networks $\mathbf{L}, \mathbf{L}' \in \mathcal{L}^*$ there exists an improving path from $\mathbf{L}$ to $\mathbf{L}'$, and (b) it is closed in that there exists no networks $\mathbf{L} \in \mathcal{L}^*$ and $\tilde{\mathbf{L}} \notin \mathcal{L}^*$ with an improving path from $\mathbf{L}$ to $\tilde{\mathbf{L}}$.

For an initial state of the network corresponding to the adjacency matrix $\mathbf{L}^{(0)}$, we let $\mathbf{L}^*$ denote the pairwise stable network, or the first network in a closed cycle, reached by myopic adjustment process, where in the sth step, the network $\mathbf{L}^{(s)}$ results from the preceding adjacency matrix $\mathbf{L}^{(s-1)}$ by simultaneously replacing each link $L_{ij}^{(s)} = 1 - L_{ij}^{(s-1)}$ if the pairwise stability conditions for that link do not hold given the network $\mathbf{L}^{(s-1)}$, and leaving it unchanged otherwise. This process continues until it reaches a pairwise stable network, or a closed cycle, which is guaranteed to occur after a finite number of iterations according to Lemma 1 by Jackson and Watts (2002).

Representing a distribution over pairwise stable networks with a distribution over initial conditions is without loss of generality since any pairwise stable network is a fixed point with respect to the adjustment step, and therefore can be reached via tâtonnement. Note also that for the purposes of this paper, the tâtonnement process only serves as a purely representational device, and not an empirical description of the mechanism agents follow to coordinate on the pairwise stable network observed in the data.

Given the tâtonnement process defined above, a mechanism for selecting among pairwise stable outcomes is a distribution over initial network states $\mathbf{L}^{(0)}$, which may also depend on realized node attributes x_i, taste shifters η_{ij}, and marginal costs MC_i. Since any agent's link preferences depend on network structure only through the network attributes s_i, t_{ij} we can summarize the network state $\mathbf{L}^{(0)}$ equivalently by the resulting statistics $S(\mathbf{L}^{(0)}, X; i)$ and $T(\mathbf{L}^{(0)}, X; i, j)$.

Since our aim is to describe the set of pairwise stable networks, our asymptotic results below first consider the outcome of tâtonnement from arbitrary fixed initial states $\mathbf{L}^{(0)}$, where convergence is shown to be uniform with respect to the choice of $\mathbf{L}^{(0)}$. We can thereby obtain an asymptotic characterization of the full set of outcome distributions that are supported by some pairwise stable network. For large n, the distribution over network outcomes from any selection mechanism can therefore be approximated by a mixture over outcome distributions from that set, where the mixing weights may depend on node attributes $(x_i, MC_i)_{i=1}^n$ and taste shifters $(\eta_{ij})_{i,j=1}^n$.

Since our main representational results rule out closed cycles of networks that are not pairwise stable, the support of the selection mechanism has to be restricted to initial conditions from which tâtonnement converges to pairwise stable networks, and that restriction may generally depend on the realizations of the taste shifters $(\eta_{ij})_{i,j=1}^n$.

3. Asymptotic Representation of Network

This section presents the limiting model for the network for the leading case in which the local structure of the network is uniquely determined by payoffs in a manner to be made more precise below. A description of that limiting model for the fully general case requires additional notation and definitions and is therefore relegated to appendix A. This asymptotic approximation to the model can then be used for identification analysis, or to construct likelihoods and probability bounds for parametric estimation. We derive analytic characterizations for various specifications of the payoffs in (2.3) in Section 6.

3.1. **Link Frequency Distribution.** Our limiting results will be stated in terms of the **link frequency distribution,** which we define as

$$F_n(x_1, x_2; s_1, s_2, t_{12}) := \frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i} P\left(L_{ij} = 1, x_i \leq x_1, x_j \leq x_2, s_i \leq s_1, s_j \leq s_2, t_{ij} \leq t_{12}\right)$$

Note that our analysis will focus on the case of sparse networks (i.e. networks with a degree distribution that remains bounded in n), so that the natural normalization of $F_n(\cdot)$ is by the number of nodes rather than the number of dyads.

The link frequency distribution is not a proper probability distribution but integrates to a non-negative value (measure) equal to the average degree across nodes, which is in general different from one. Nonetheless and with an abuse of standard terminology we refer to the Radon-Nikodym derivative of F_n as the p.d.f. of the link frequency distribution and denote it with $f_n(x_1, x_2; s_1, s_2, t_{12})$. Apart from the different rate normalization, $f_n(\cdot)$ can be interpreted in analogy to the network density, i.e. the share of possible links of a given type that are being formed, for a dense graph.

Most existing approaches for describing distributions of networks are based on the distribution of the entire adjacency matrix (see e.g. Christakis, Fowler, Imbens, and Kalyanaraman (2010), Chandrasekhar and Jackson (2011), or Mele (2012)), which typically requires simulation of the entire network at a substantial computational cost. However for typical specifications of a network formation model, the link frequency distribution contains most of the relevant information for identification and estimation of the preference parameters: Specifically we can encode a sparse adjacency matrix more efficiently (i.e. requiring less memory) as a list of pairs of nodes that are connected by a direct link. Furthermore, under a parameterization of the model as in (2.3) the nodes are exchangeable in that the joint distribution of payoffs $U_{ij}(\mathbf{L})$ and exogenous attributes x_i is invariant to permutations of the node labels $1, \ldots, n$. Instead of node-identifiers, it is therefore sufficient to retain node-level information about the corresponding exogenous attributes x_i, x_j, and endogenous network characteristics s_i, s_j and t_{ij}. While in principle knowing which specific other links emanate from a common node could also be informative, we find that in the limit, stability of the link L_{ij} and any other link L_{ik} are conditionally independent events given the characteristics of nodes i, j, k. We do not attempt to make this informal claim more precise, but also note that the formal arguments developed in this paper can be used to describe the joint distribution of several (but finitely many) entries of the adjacency matrix.

3.2. **Limiting Model $\mathcal{F}_0^*$.** We next describe the limiting model $\mathcal{F}_0^*$ for the link frequency distribution $\hat{F}_n$. Formal conditions for convergence of the finite-n network to this "as if" statistical experiment are given in section 4. In general the limit of the link frequency distribution is not uniquely defined, due to multiplicity of pairwise stable networks in the finite-n model. Instead, we can give a sharp characterization of the set $\mathcal{F}_0^*$ of distributions such that any link frequency distribution $\hat{F}_n$ resulting from some pairwise stable network can be approximated by some element $F_0^* \in \mathcal{F}_0^*$. This limiting model then forms the basis of our approach to identification and estimation of link preferences discussed in Section 5. While decisions about whether to form or destroy a link are interrelated across nodes, the asymptotic approximation developed in this paper allows to characterize the link frequency distribution in terms of aggregate states at the network level, and a edge-level "best response" to those aggregate states.

Specifically, the limiting model $\mathcal{F}_0^*$ can be described in terms of pairwise stable **subnetworks** on finite network neighborhoods $\mathcal{N}_{ij}$ around a pair of nodes i, j. Such a network neighborhood $\mathcal{N}_{ij}$ is defined as the set of nodes $l \in \mathcal{N}$ such that $U_{lk}(\mathbf{L}) \geq MC_l$ and $U_{kl}(\mathbf{L}) \geq MC_k$ for either $k = i$ or $k = j$, so that the nodes k, l may be mutually acceptable under some configuration of the network $\mathbf{L}$. Under the assumptions made in Section 4, the number of nodes in any such network neighborhoods is stochastically bounded.

Furthermore, each node l is associated with its exogenous attributes x_l, and potential values for the endogenous network attributes s_l, t_{kl}. Here, the **potential values** for endogenous network attributes are given by the network statistics $S(\tilde{\mathbf{L}}, l)$ and $T(\tilde{\mathbf{L}}, k, l)$ evaluated at any network $\tilde{\mathbf{L}}$ that coincides with the pairwise stable network L^* everywhere except on edges between node pairs in $\mathcal{N}_{ij}$.

The model $\mathcal{F}_0^*$ describes the distribution generating the network neighborhoods $\mathcal{N}_{ij}$ in the many-player limit, as well as the distribution of network outcomes on these neighborhoods. That distribution can be fully described in terms of three components:

- the **reference distribution** M^* which is a cross-sectional p.d.f. of potential outcomes for the endogenous node characteristics s_l and $t_{l\cdot}$ given exogenous attributes in the relevant subnetwork,
- the **inclusive value function** $H^*(x, s)$ which gives a sufficient statistic for the link opportunity set $W_i(\mathbf{L}^*)$ of a node with characteristics $x_i = x$ and $s_i = s$ with respect to her link formation decisions. and
- the **edge-level response** $Q^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2) := Q^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2; H^*, M^*)$ which corresponds to a conditional probability of a link ij forming together with the resulting values of the endogenous network variables s_1, s_2, t_{12}.[8]

[8]Recall that in the Logit model for multinomial choice, the inclusive value serves as a sufficient statistic for the set of available alternative with respect to conditional choice probabilities, see e.g. Train (2009). The

We say that the edge-level response is *unique* if for any given realization of $\mathcal{N}_{ij}$, all pairwise stable networks on the neighborhood $\mathcal{N}_{ij}$ result in the same network outcomes $(\mathbf{L}_{ij}, s_i, s_j, t_{ij})$ for the edge ij. In general, the edge-level response need not be unique. For example, if j and k's payoff from a link to i increase with i's network degree s_{1i}, and i is available to either node, then there may be values of η_{ji}, η_{ki} such that both outcomes $L_{ij} = L_{ik} = 1$ and $L_{ij} = L_{ik} = 0$, respectively, are supported by different pairwise stable networks on $\mathcal{N}_{ij}$.

In general the **limiting model** $\mathcal{F}_0^*$ is a set of distribution functions corresponding to the edge-level response $Q^*(1, s_1, s_2, t_{12}|x_1, x_2; H^*, M^*)$ such that H^* and M^* satisfy a global equilibrium condition corresponding to fixed-point mappings Ψ_0 and Ω_0, respectively, to be characterized below. For expositional purposes, this section gives a characterization of the limiting distribution only for the special case of a unique edge-level response and anonymous endogenous interaction effects with a radius of interaction $r_S = 1$. Specifically, we do not allow for edge-specific interaction effects, i.e. we let $\mathcal{T} = \{t_0\}$. Sharp bounds on $\mathcal{F}_0^*$ for the general case are given in Appendix A. Examples with a unique edge-level response include models with payoffs that depend exclusively on exogenous attributes, as well as a many-to-many matching model with capacity constraints described in further detail in Appendix A. For this special case, the only relevant potential value of the node characteristic s_j for a node j in the network neighborhood around i is its value in the presence of a direct connection to i. Hence for the remainder of this section, we take the reference distribution M^* to be the marginal distribution of the potential value for s_j corresponding to the network $\tilde{\mathbf{L}}$ that is obtained from L^* after setting $L_{ij} = 1$.

The resulting limiting link frequency distribution has p.d.f.

$$\begin{aligned} f_0^*(x_1, x_2; s_1, s_2) \quad = \quad & \frac{s_{11} s_{12} \exp\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\}}{(1 + H^*(x_1, s_1))(1 + H^*(x_2, s_2))} \\ & \times M^*(s_1|x_1, x_2) M^*(s_2|x_2, x_1) w(x_1) w(x_2) \end{aligned} \tag{3.1}$$

and can therefore be characterized in closed form given the aggregate state variables H^*, M^*.

The inclusive value function $H^*(x_1, s_1)$ is a nonnegative function satisfying the fixed-point condition

$$H^*(x; s) = \Psi_0[H^*, M^*](x; s) \tag{3.2}$$

where the fixed-point operator Ψ_0 is defined as

$$\Psi_0[H, M](x; s) := \int \frac{s_{12} \exp\{U^*(x, x_2; s, s_2) + U^*(x_2, x; s_2, s)\}}{1 + H(x_2; s_2)} M^*(s_2|x_2, x_1) w(x_2) ds_2 dx_2$$

We show below that for any fixed reference distribution $M^*(s_1|x_1, x_2)$, the fixed point of (3.2) is generally unique.

inclusive value function takes that role in the many player limit of the network formation model, see Menzel (2015) for a more extensive discussion for the special case of one-to-one matching markets.

The reference distribution $M^*(s_1|x_1, x_2)$ must solve the equilibrium condition

$$M^*(s_1|x_1, x_2) = \Omega_0[H^*, M^*](x_1, x_2; s_1) \tag{3.3}$$

where the operator Ω_0 maps H, M to the conditional distribution for the network statistic s_i given x_i resulting from the edge-level response in the cross section.

The limiting model $\mathcal{F}_0^*$ then corresponds to the set of all distributions satisfying (3.1)-(3.3) for some inclusive value function H^* and reference distribution M^*. Note that for a given value of H^*, the solution to the fixed-point condition (3.3) may admit multiple solutions, so that the resulting link frequency distribution need not be uniquely defined even in the case of a unique edge-level response. In the case of no endogenous interaction effects, $\mathcal{S} = \{\}$, the fixed-point mapping for the degree distribution is given by

$$\Omega_0[H, M](x_1, x_2; s_{11}) := \frac{H(x_1)^{s_{11}}}{(1 + H^*(x_1))^{s_{11}+1}}$$

where according to the convention introduced earlier in section 2.1, s_{11} denotes the network degree of node 1. Note that the fixed point mapping and the resulting reference distribution do not depend on x_2, which is generally the case whenever the network statistics $S(\mathbf{L}, \mathbf{X}; i)$ do not depend on $\mathbf{X}$. In general, the fixed-point mapping Ω_0 has to be derived separately for each type of payoff-relevant network statistics, and we give additional examples in Appendix A.3.

Since the taste shifters η_{ij} are independent across nodes $j = 1, \dots, n$, link formation decisions are also conditionally independent. As a result, for a given pairwise stable network the selected reference distribution $M^*(s_1|x_1, x_2)$ coincides with the conditional distribution of s_1 given that node 1 is directly linked to a node 2 with attributes x_2. This observation is useful for estimation since that conditional distribution can be estimated directly from the cross-sectional sample of nodes $i = 1, \dots, n$, without having to solve the fixed-point condition (3.3) and explicitly addressing the multiplicity of solutions to that problem.[9] Furthermore, under the maintained assumption of bounded systematic payoffs and unbounded taste shifters, the probability of a link between nodes with attributes x_1, x_2 is nonzero, so that this conditional distribution is always well-defined.

More generally, a local subnetwork is consistent with pairwise stability on $\mathcal{N}_0$ if we can find a combination of potential outcomes for the network variables $(s_i, t_{ij} : i, j \in \mathcal{N}_0)$ that jointly satisfy the stability conditions given the realized types and payoff shocks. The fully general case which is discussed in Appendix A has to deal with the added complication that

[9]As pointed out in Menzel (2016), an approach of this form can be justified as conditional estimation or inference given a sufficient statistic for the selected equilibrium. The strategy of replacing equilibrium quantities with sample analogs in order to side-step a nested fixed-point problem has already been fruitfully applied in dynamic discrete choice Hotz and Miller (1993) and dynamic games Bajari, Benkard, and Levin (2007).

such a combination need not be unique. Nevertheless, the general representation retains the general structure discussed in this section in which the link L_{ij} together with network variables s_i, s_j, t_{ij} is determined locally in the subnetwork on $\mathcal{N}_{ij}$ which interacts with the full network only through the aggregate state variables H^* and M^*.

4. Convergence to the Limiting Distribution

This section presents the main convergence results for the link frequency distribution. We first state our formal assumptions, followed by the main results. An outline of the formal argument, including the main intermediate steps, is given in Section 6. The main result in this section is contained in Theorem 4.2 below. The results in this section refer to, and are proven for, the limiting model $\mathcal{F}_0^*$ for the general case, which is developed in Appendix A. Proposition A.1 in that appendix establishes that the general formulation for $\mathcal{F}_0^*$ indeed simplifies to the limiting model presented in the previous section.

4.1. **Formal Assumptions.** The main formal assumptions regarding the random utility model are similar to those in Menzel (2015). For one, we will maintain that the deterministic parts of random payoffs satisfy certain uniform bounds and smoothness restrictions:

Assumption 4.1. ***(Systematic Part of Payoffs)*** *(i) The systematic parts of payoffs are uniformly bounded in absolute value for some value of $t = t_0$, $|U^*(x, x', s, s', t_0)| \leq \bar{U} < \infty$. Furthermore, (ii) at all values of s, s', the function $U^*(x, x', s, s', t_0)$ is $p \geq 1$ times differentiable in x with uniformly bounded partial derivatives. (iii) The supports of the payoff-relevant network statistics, $\mathcal{S}$ and $\mathcal{T}$, and the type space $\mathcal{X}$ are compact sets.*

These conditions are fairly standard, where the uniform bound on systematic payoffs in part (i) serves primarily to simplify the formal argument, and might be replaced by bounds on other norms on the function $U^*(\cdot)$, a question we will leave for future research. Part (iii) may require a reparametrization of network statistics that can take arbitrarily large values, as e.g. the network degree of a node.

We next state our assumptions on the distribution of unobserved taste shifters. Most importantly, we impose sufficient conditions for the distribution of η_{ij} to belong to the domain of attraction of the extreme-value type I (Gumbel) distribution. Following Resnick (1987), we say that the upper tail of the distribution $G(\eta)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the c.d.f. satisfies

$$\lim_{s \to \infty} \frac{1 - G(s + a(s)v)}{1 - G(s)} = e^{-v}$$

for all $v \in \mathbb{R}$. We are furthermore going to restrict our attention to distributions for which the auxiliary function can be chosen as $a(s) := \frac{1-G(s)}{g(s)}$, where $g(s)$ denotes the density associated with the c.d.f. $G(s)$. This property is shared for most standard specifications of discrete choice models, e.g. if η_{ij} follows the extreme-value type I, normal, or Gamma distribution, see Resnick (1987). We can now state our main assumption on the distribution of the idiosyncratic part of payoffs:

Assumption 4.2. ***(Idiosyncratic Part of Payoffs)*** *η_{ij} and $\eta_{i0,k}$ are i.i.d. draws from the distribution $G(s)$, and are independent of x_i, x_j, where (i) the c.d.f. $G(s)$ is absolutely continuous with density $g(s)$, and (ii) the upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) := \frac{1-G(s)}{g(s)}$.*

We also need to specify the approximating sequence of networks. Here it is important to emphasize that the main purpose of the asymptotic analysis is a reliable approximation to the (finite) n-agent version of the network rather than a factual description how network outcomes would change if nodes were added to an existing network. Hence our approach is to embed the n-agent model into an asymptotic sequence whose limit preserves the main qualitative features of the finite-agent model.

Specifically, we design the asymptotic sequence to match the following properties of a finite network: (1) the network should remain sparse in that degree distribution does not diverge as the size of the market grows. (2) The limiting conditional link formation frequencies given node-level attributes should be non-degenerate, and depend non-trivially on the systematic parts of payoffs. Finally, (3) the limiting approximation should retain network features at a nontrivial frequency that are deemed important by the researcher, e.g. closed triangles or other forms of clustering among links.

For the first requirement, it is necessary to increase the magnitude of marginal costs MC_i as the number of available alternatives grows, whereas to balance the relative scales of the systematic and idiosyncratic parts we have to choose the scale parameter $\sigma \equiv \sigma_n$ at an appropriate rate. For the last requirement we have to scale the effect of edge-specific network attributes t_{ij} on the payoff functions at an appropriate rate, which we discuss for specific cases below.

Specifically we are going to assume the following in the context of the reference model:

Assumption 4.3. ***(Network Size)*** *(i) The number n of agents in the network grows to infinity, and (ii) the random draws for marginal costs MC_i are governed by the sequence $J = \left[n^{1/2}\right]$, where $[x]$ denotes the value of x rounded to the closest integer. (iii) The scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1}\left(1 - \frac{1}{\sqrt{n}}\right)$, and $a(s)$ is the auxiliary function specified in Assumption 4.2 (ii). Furthermore, (iv) for any values $t_1 \neq t_2 \in \mathcal{T}$, $|U(x, x', s, s', t_1) - U(x, x', s, s', t_2)|$ may increase with n, and there exists a constant $B_T < \infty$ such that for any sequence of pairwise stable networks $(\mathbf{L}_n^*)_{n\geq 2}$, $\sup_{x,x',s,s'} \left(\mathbb{E}\left[\exp\left\{2|U(x, x', s, s', T(\mathbf{L}_n^*, x, x', i, j)) - U(x, x', s, s', t_0)|\right\}\right]\right)^{1/2} \leq \exp\{B_T\}$ for n sufficiently large.*

The rate conditions for marginal costs and the scale parameter in parts (i) and (ii) are analogous to the matching case and discussed in greater detail in Menzel (2015). Specifically, the rate for J in part (ii) is chosen to ensure that the degree distribution from a pairwise stable network will be non-degenerate and asymptotically tight as n grows. The construction of the sequence σ_n in part (iii) implies a scale normalization for the systematic parts $U_{ij}^* = U_{ij}^*(\mathbf{L})$, and is chosen as to balance the relative magnitude for the respective effects of observed and unobserved taste shifters on choices as n grows large. Rates for σ_n for specific distributions of taste shifters are also given in Menzel (2015).

The requirement that the sequence of networks remains sparse is primarily needed to obtain the limiting characterization of link opportunity sets with inclusive value functions, where the some of the arguments break down for a network that is more dense than what is implied by the asymptotic sequence in Assumption 4.3 (ii). However, asymptotic (conditional) independence of subnetworks across distinct network neighborhoods does not rely on sparsity but continues to hold for dense or semi-sparse network sequences.

Part (iv) of Assumption 4.3 allows the effect of the edge-specific network effect t_{ij} on payoffs to increase with n, where the second half of the statement gives an upper bound on that rate of increase. We can illustrate the rate condition in part (iv) for the case of a preference for completion of transitive triads:

Proposition 4.1. *Let $t_{ij} = \max_{k \neq i,j} \{L_{ik} L_{jk}\}$ and consider the model $U^*(x, x', s, s', t) = U^*(x, x', s, s', 0) + \beta_T t$. Then Assumption 4.3 (iv) holds if $\exp\{|\beta_T|\} = O(n^{1/4})$.*

For the rate condition on β_T, we find below that in order to achieve a nondegenerate degree of clustering in the limit (i.e. a clustering coefficient taking values strictly between zero and one) we need to choose a sequence for β_T satisfying $\exp\{|\beta_T|\} = O(n^{1/6})$, which is strictly slower than the maximal rate permitted by this proposition, so that Assumption 4.3 is satisfied for this knife-edge case.

While there are alternative sets of primitive conditions for existence of a pairwise stable network, we make the following high-level assumption on the observed network L^*:

Assumption 4.4. ***(Pairwise Stability)*** *Let $\mathcal{N}_s \subset \{1, \ldots, n\}$ be the subset of nodes for which the network L^* satisfies the payoff conditions for pairwise stability in Definition 2.1. Then for any $\varepsilon > 0$, $|\mathcal{N}_s|/n > 1 - \varepsilon$ with probability approaching 1 as n increases.*

This condition is clearly satisfied if the network L^* is pairwise stable. Primitive conditions for existence of pairwise stable networks include cases in which links are strategic complements (see Miyauchi (2012)) or substitutes (see Sheng (2014)). Alternatively, there are conditions under which a pairwise stable outcome need not exist, but the share of "mismatched" nodes becomes arbitrarily small as n increases. One example for this is the stable roommate problem analyzed by Pęski (2014).

We next give high-level conditions on the fixed point mapping Ω_0 for the reference distribution in (3.3) and (A.4) for the general case, respectively. We let $\mathcal{R}$ denote the support of the reference distribution, which is a finite Cartesian product of the sets $\mathcal{S}$ and $\mathcal{T}$. Let $\hat{\Omega}_n[H,M]$ denote the empirical analog of the mapping $\Omega_0[H,M]$ in (A.3), where we take x_i to be distributed according to its empirical distribution in the cross-section across nodes.

In the general case of set-valued edge-level responses, Ω_0 maps to a a capacity rather than a single probability distribution, where we can represent its image as the subset of elements $\tilde{M}$ of the probability simplex $\Delta\mathcal{R}$ for distributions over $\mathcal{R}$ satisfying the constraints

$$\int_S \tilde{M}(s|x_1,x_2)ds \leq \Omega_0[H,M](x_1,x_2;S) \text{ for all } S \subset \mathcal{R}$$

That subset is generally referred to as the *core* of Ω_0, see Appendix A for a formal definition.

Since the selection mechanism on the individual response may vary discontinuously in x, the set of distributions satisfying the bounds in (3.3) and (A.4), respectively, can be fairly rich, many of which do not meet any useful smoothness criteria. However for our analysis of convergence of the set of equilibrium reference distributions, it is sufficient to restrict our attention to distributions that determine the boundary of that set on the probability simplex $\Delta(\mathcal{X}\times\mathcal{R})$. Specifically, we say that the distribution $M(s_1|x_1,x_2)$ is *on the boundary of the core* of Ω_0 if for some values of x_1,x_2, there exists a set $S(x_1,x_2) \in \mathcal{R}$ such that $\int_{S(x_1,x_2)} M(s|x_1,x_2)ds = \Omega_0(x_1,x_2,S(x_1,x_2))$ with equality.

We can now formulate the main assumptions on the fixed-point mappings $\hat{\Omega}_n$ and Ω_0 for the reference distributions in the finite network and the limiting economy, respectively:

Assumption 4.5. *(i) The mapping Ω_0 is compact and upper hemi-continuous in H,M for all $x \in \mathcal{X}$ and $S \subset \mathcal{R}$, and (ii) the core of $\Omega_0[H,M]$ is nonempty, where the boundary of the core of $\Omega_0[H,M]$ is in some compact subset $\mathcal{U} \subset \Delta(\mathcal{X}\times\mathcal{R})$ for all values of H,M. (iii)* $\sup_{x,Z\subset\mathcal{R}}\left|\hat{\Omega}_n[H,M](Z) - \Omega_0[H,M](Z)\right| \to 0$ *uniformly in $H \in \mathcal{G}$ and distributions $M \in \mathcal{U}$.*

These high-level assumptions on the equilibrium mapping Ω_0 have to be verified on a case by case basis. Furthermore, as we already argued in section A.1, the core of a capacity is a convex subset of a probability simplex, which simplifies the argument for existence of a fixed point below.

Uniform convergence of $\hat{\Omega}_n$ with respect to M in part (iii) is only stated only as a high-level condition in order to keep the result as general as possible.[10] As the case of interactions

[10]Note that the boundary distributions are pointwise minima of a selection of $\{\Omega(\cdot;S) : S \subset \mathcal{R}\}$ and $\left\{\hat{\Omega}_n(\cdot;S) : S \subset \mathcal{R}\right\}$, respectively. Hence, if $\mathcal{S}$ is finite, and $\Omega_0(x_1,x_2;S)$ and $\hat{\Omega}_n(x_1,x_2;S)$ have bounded partial derivatives with respect to x_1,x_2 of order $p \geq 1$, then the boundary distributions can be represented using a finite intersection of Glivenko-Cantelli classes, which is also Glivenko-Cantelli. In that case, uniform laws of large numbers with respect to potential boundary distributions of the core can be established under otherwise standard regularity conditions, which can be used to establish uniform convergence if $\hat{\Omega}_n(x_1,x_2;S)$ depends on sample averages with respect to the sampling distribution of x_i.

through the degree distribution in section A.3 illustrates, for some cases of applied interest the mapping Ω_0 does not depend on the sampling distribution of types, in which case uniform convergence as in part (iii) trivially holds.

4.2. **Main Limiting Results.** We can now state the main formal results of this paper. One potential concern is that the limiting distribution in (3.1) may not be well defined if there exists no fixed point for the population problem (3.2) and (3.3). We find that the assumptions on the fixed-point mapping Ω_0 are sufficient to guarantee existence of an equilibrium inclusive value function and reference distribution, as stated in the following proposition.

Theorem 4.1. ***(Fixed Point Existence)*** *Suppose that Assumptions 4.1 and 4.5 (i)-(ii) hold. Then the mapping $(H, M) \rightrightarrows (\Psi_0, \Omega_0)[H, M]$ has a fixed point.*

See the appendix for a proof. Taken together, Assumption 4.4 and Theorem 4.1 ensure the respective models for the finite network as well as the limiting model are always well-defined. We can now state our main asymptotic result, which establishes convergence to the limiting model described in Sections 3 and A, respectively.

Theorem 4.2. ***(Convergence)*** *Suppose that Assumptions 4.1-4.5 hold, and let $\mathcal{F}_0^*$ be the set of distributions characterized by (3.1)-(3.3) (A.7,A.4, and A.5, respectively). Then for any pairwise or cyclically stable network there exists a distribution $F_0^*(x_1, x_2; s_1, s_2) \in \mathcal{F}_0^*$ such that the link frequency distribution*

$$\sup_{x_1,x_2,s_1,s_2,t_{12}} |\hat{F}_n(x_1, x_2; s_1, s_2, t_{12}) - F_0^*(x_1, x_2; s_1, s_2, t_{12})| = o_p(1)$$

Furthermore, convergence is uniform with respect to selection among pairwise stable networks.

We describe the main ideas behind this convergence result in Section 6, a formal proof of the theorem is given in the appendix. This limiting model gives a tractable characterization of the link distribution. By considering only the distribution of links rather than the full adjacency matrix, we do not need to characterize the structure of the full network explicitly, but the model is closed via equilibrium conditions on the aggregate state variables H^* and M^*. In contrast, the expressions in Chandrasekhar and Jackson (2011) and Mele (2012) can only be approximated by simulation over all possible networks, the number of which grows very fast as n increases.

Finally, we want to give conditions under which the characterization of the limiting model is sharp in the sense that all distributions satisfying the fixed-point conditions (3.2) and (3.3) can be achieved by a sequence of finite networks. To this end, we rely on the notion of regularity for the solutions of the fixed-point problem which correspond to standard local stability conditions in optimization theory (see e.g. chapter 9 in Luenberger (1969), or chapter 3 in Aubin and Frankowska (1990)).

To simplify notation, we define the set-valued mapping $\Upsilon_0 : (\mathcal{G} \times \mathcal{U}) \rightrightarrows (\mathcal{G} \times \mathcal{U})$, where

$$\Upsilon_0 : \left[\begin{array}{c} H \\ M \end{array} \right] \to \left[\begin{array}{c} \Psi_0[H, M] \\ \text{core } \Omega_0[H, M] \end{array} \right]$$

Using the notation $z = (H, M) \in \mathcal{Z} := \mathcal{G} \times \mathcal{U}$, the fixed-point conditions (3.2) and (3.3) can be written in the more compact form $z^* \in \Upsilon_0[z^*]$. We also define the sample fixed point mapping $\hat{\Upsilon}_n$ in a completely analogous manner. The *contingent derivative* of Υ_0 at $(z_0', y_0)' \in \text{gph } \Phi$ is defined as the set-valued mapping $D\Upsilon_0(z_0, y_0) : \mathcal{Z} \rightrightarrows \mathcal{Z}$ such that for any $u \in \mathcal{Z}$

$$v \in D\Upsilon_0(z, y)(u) \Leftrightarrow \liminf_{h \downarrow 0, u' \to u} d\left(v, \frac{\Upsilon_0(z_0 + hu') - y}{h}\right)$$

where $d(a, B)$ is taken to be the distance of a point a to a set B.[11] Note that if the correspondence Υ_0 is single-valued and differentiable, the contingent derivative is also single-valued and equal to the derivative of the function $\Upsilon_0(z)$. The contingent derivative of Υ_0 is *surjective* at z_0 if the range of $D\Upsilon_0(z_0, y_0)$ is equal to $\mathcal{Z}$.

The following theorem states that for equilibrium points that are regular in a specific sense, the characterization of the limiting model is sharp in that any solution of (3.2) and (3.3) can be achieved as the limit of a sequence of solutions to the finite-agent network.

Theorem 4.3. *Suppose that Assumptions 4.1-4.5 hold. Furthermore, suppose that for each point z^* satisfying $z^* \in \Upsilon_0[z^*]$, the contingent derivative of $\Upsilon_0[z]$ is surjective. Then for any $z_0^* := (H_0^*, M_0^*)$ solving $z^* \in$ core $\Upsilon_0[z^*]$, there exists a sequence $\hat{z}_n := (\hat{H}_n^*, \hat{M}_n^*)$ solving $\hat{z}_n \in$ core $\hat{\Upsilon}_n[\hat{z}_n]$ such that $d(\hat{z}_n, z_0^*) \xrightarrow{p} 0$.*

See the appendix for a proof.

5. Identification, Estimation, and Welfare Analysis

We next illustrate some potential uses of the limiting approximation in Theorem 4.2, including a strategy for estimating structural payoff parameters from network data and welfare analysis. In the following we assume that all payoff-relevant attributes x_i and network characteristics s_i are observed for a random sample of nodes $i = 1, \ldots, K$ included in the sample. The arguments below could be extended to different sampling protocols and certain cases in which some components of x_i are not directly observed but generated from a distribution that is known up to a parameter to be estimated. The primary focus of this section is on the case of a single-valued best-response, a more general approach to estimation will be left for future research.

[11]See Definition 5.1.1 and Proposition 5.1.4 in Aubin and Frankowska (1990)

5.1. **Identification.** We first consider identification of the payoff functions $U^*(x_1, x_2; s_1, s_2, t_{12})$, where we assume that the researcher observes either the full network $\mathcal{L}, \mathcal{X}$, or a random sample of edges, i.e. $K \leq n(n-1)/2$ pairs i, j together with the variables $L_{ij}, x_i, x_j, s_i, s_j, t_{ij}$. Note that in this case, the link frequency distribution $f(x_i, x_j; s_i, s_j, t_{ij})$ is nonparametrically identified. These arguments can be adjusted for other sampling protocols with known sampling weights.[12] In the case of knowledge of the complete network $\mathcal{L}$ and perfectly observable attributes x_i, the network statistics S_i and T_{ij} can also be computed from the available data.

5.1.1. *Nonparametric Identification with no Endogenous Interaction Effects.* In the absence of any interaction effects between links, the marginal benefit of link ij is given by

$$U_{ij} \equiv U^*(x_i, x_j) + \sigma\eta_{ij}$$

From our results in sections 3 and 4, it follows that we can fully characterize the limiting distribution of links in pairwise stable networks in terms of the pseudo-surplus function $V^*(x_1, x_2) := U^*(x_1, x_2) + U^*(x_2, x_1)$, so that the function $U^*(x_1, x_2)$ is not separately identified. Specifically, if we let s_{1i} denote the degree of node i, the density for the limiting distribution is given by

$$f_0^*(x_1, x_2; s_1, s_2) = \frac{s_{11}s_{12}\exp\{V^*(x_1, x_2)\}M^*(s_{11}|x_1, x_2)M^*(s_{12}|x_2, x_1)w(x_1)w(x_2)}{(1 + H^*(x_1))(1 + H^*(x_2))}$$

where the inclusive value function $H^*(x)$ satisfies the fixed-point condition

$$H^*(x) = \Psi_0[H^*, M^*](x) := \int_{\mathcal{X}\times\mathcal{S}} s\frac{\exp\{V^*(x, x_2)\}}{1 + H^*(x_2)}M^*(s|x_2, x)w(x_2)dsdx_2$$

and the degree distribution $M^*(s|x)$ is given by

$$M^*(s_1|x_1, x_2) = \frac{H(x_1)^{s_1}}{(1 + H^*(x_1))^{s_1+1}}$$

and does not depend on x_2. In particular, we have for any $r = 0, 1, \dots$ that

$$P(s_{1i} \geq r|x_i = x) = \sum_{s=r}^{\infty} \frac{H^*(x)^s}{(1 + H^*(x))^{s+1}} = \left(\frac{H^*(x)}{1 + H^*(x)}\right)^r$$

so that the ratio

$$\frac{P(s_{1i} = r|x_i = x)}{P(s_{1i} \geq r|x_i = x)} = \frac{1}{1 + H^*(x)}$$

[12]For example, the researcher may sample nodes at random and eliciting all links emanating from each node ("induced subgraph"), or only the links among the nodes included in the survey ("star subgraph"), see Chandrasekhar and Lewis (2011) for a discussion.

for any natural number r, including zero. Hence for any arbitrarily chosen $r = 0, 1, \ldots,$ we can write the pseudo-surplus function in terms of log differences of link frequencies,

$$\begin{aligned} V^*(x_1, x_2) &= \log \frac{f_0^*(x_1, x_2; s_1, s_2)}{s_{11} s_{12} w^*(x_1; s_1) w^*(x_2; s_2)} \\ &\quad - \log \frac{P(s_{1i} = r | x_i = x_1)}{P(s_{1i} \geq r | x_i = x_1)} - \log \frac{P(s_{1j} = r | x_j = x_2)}{P(s_{1j} \geq r | x_j = x_2)} \end{aligned}$$

where $w^*(x; s)$ is the p.d.f. of the cross-sectional distribution of x_i, s_i. Note that all quantities on the right-hand side can be estimated nonparametrically from the observed network. Hence, the pseudo-surplus function $V^*(x_1, x_2)$ is nonparametrically identified for the "pure homophily" model. Note that the identification argument is constructive and does not require knowledge of the (unobserved) inclusive value function $H^*(x)$.

5.1.2. *Identification of the Reference Distribution.* In the fully general case, the reference distribution M^* is a joint distribution of the potential values of the network statistics (s_l, t_{il}, t_{jl}) under all possible configurations r_{ijl} of the subnetwork among the nodes i, j, l. Appendix A shows that if selection from the edge-level response is independent across nodes in the general case of a multi-valued edge-level response, then the network is characterized by a unique reference distribution M^*. In that event, the reference distribution for the potential value of s_l for any specific value of r_{ijl} equals the conditional distribution of s_l given r_{ijl} in the cross-section, and is therefore nonparametrically identified for all values of r_{ijl} that occur with strictly positive probability.

Note however that M^* is the joint distribution over potential values for all values of r_{ijl}, so that this argument does not guarantee nonparametric identification of M^* from the marginal distributions for specific potential values. In the case of a unique edge-level response, the link frequency distribution depends only on the respective marginal distributions for the potential outcomes. In that event, we can use the implied nonparametric estimator for the marginals of M^* for estimation which obviates the need to solve the fixed-point problem (3.3) explicitly.

5.1.3. *Set-Valued Edge-Level Response.* In the general case of a set-valued edge-level response, there is no guarantee that the pseudo-surplus function, or parameters governing its form, are point-identified. By Theorem 4.3, the bounds on the large-network distribution implied by the model are in general sharp, and can be used to construct moment inequalities that bound an identified set of pseudo-surplus function or other payoff parameters. This step is fully analogous to the analysis of other models with set-valued predictions, see Beresteanu, Molchanov, and Molinari (2011) and Galichon and Henry (2011).

In the literature on discrete games it is known that bounds in more tightly parameterized models may shrink to a point e.g. under large-support conditions on relevant exogenous characteristics, see Tamer (2003). It may be possible to establish similar conditions for the limiting bounds derived in this paper, however a more systematic analysis of the set-valued case is beyond the scope of this paper and will be left for future research.

5.2. **Parametric Estimation.** We now turn to estimation of parametric models for link preferences, where we assume that systematic utilities are specified as

$$U^*(x_i, x_j; s_i, s_j, t_{ij}) = U^*(x_i, x_j; s_i, s_j, t_{ij}|\theta)$$

for a finite-dimensional parameter θ. We also define the resulting pseudo-surplus function

$$V^*(x_i, x_j; s_i, s_j, t_{ij}|\theta) = U^*(x_i, x_j; s_i, s_j, t_{ij}|\theta) + U^*(x_j, x_i; s_j, s_i, t_{ij}|\theta)$$

For the case of a non-unique edge-level response, we also maintain the assumption of independent equilibrium selection stated formally in Example A.4 in the appendix.

Estimation and inference for θ in the network model is complicated by the presence of multiple stable outcomes. However, while the fixed-point conditions in (3.3) and (A.4), respectively, may admit multiple solutions, as discussed before the distribution $M^*(s_1|x_1, x_2)$ resulting from the equilibrium chosen in the data can be estimated consistently from the observed network. Our approach is therefore conditional on the non-unique equilibrium distribution M^*, which we replace by a consistent estimate. This strategy for dealing with multiple equilibria is analogous to Menzel (2016)'s approach for the case of discrete action games.

The other potential difficulty is that the limiting distribution in (A.7) depends on the (unobserved) inclusive value function. Following the approach in Menzel (2015) for the case of matching markets, we suggest to treat $H^*(x; s)$ as an auxiliary parameter in maximum likelihood estimation of the surplus function $V^*(x_1, x_2; s_1, s_2|\theta)$ satisfying the fixed-point condition (A.5).

We propose maximum likelihood estimation of θ, where the log-likelihood contribution for node i,

$$\ell_i(\theta, H) := \sum_{j=1}^{n} d_{ij} \log f^*(x_i, x_j; s_i, s_j|\theta, H)$$

is obtained from the limiting model. Hence, when the researcher observes the full network with n nodes, the log-likelihood function corresponding to the limiting distribution is

$$\mathcal{L}_n(\theta, H) := \sum_{i=1}^{n} \ell_i(\theta, H)$$

We also estimate the fixed-point mapping $\Psi_0[H, M^*]$ by its sample analog

$$\hat{\Psi}_n(\cdot) := \frac{1}{n}\sum_{i=1}^{n} \psi_i(\theta, H)$$

where the node-level contributions $\psi_i(\theta, H)$ are again derived from the limiting representation. When only a random sample of nodes or edges of the network is observed according to some known sampling protocol with uniformly bounded qualification probabilities, the formulae for $L_n(\cdot)$ and $\Psi_n(\cdot)$ can be adjusted using weights.

The maximum likelihood estimator is then obtained by maximizing the log-likelihood, where $H^*(x; s)$ is treated as an auxiliary parameter that has to satisfy the sample analog of the fixed-point condition (3.2). Specifically we solve the problem

$$\max_{\theta, H} \mathcal{L}_n(\theta, H) \quad \text{s.t. } H = \hat{\Psi}_n(H) \tag{5.1}$$

The structure of this optimization problem, where a nuisance function is defined by a fixed-point problem, is very similar to that of maximum likelihood estimation of dynamic discrete choice models where the value function has to be recomputed for each candidate value of the preference parameters. Popular approaches for estimating these models are nested fixed-point algorithms (Rust (1987), Ishakov, Lee, Rust, Schjerning, and Seo (2016)) and the MPEC algorithm (Su and Judd (2012)).

To conclude our discussion of estimation, we give the expressions for the log-likelihood $\hat{L}_n$ and the fixed-point mapping $\hat{\Psi}_n$ for a few illustrative examples which form the basis for the Monte Carlo experiments in the last section of this paper. We only consider cases for which the edge-level response is unique, or we specify the equilibrium selection rule since the main purpose of these examples is to illustrate our convergence results, abstracting from potential issues with partial identification in the general case. In each case the likelihood function is derived from the corresponding limiting model $\mathcal{F}_0^*$, assuming that the researcher observes the relevant exogenous characteristics for all nodes, $x_1, \dots, x_n$, and the full adjacency matrix $\mathbf{L}$.

Example 5.1. ***(No Endogenous Interaction Effects)*** *We first consider the case of no endogenous interaction effects, with systematic marginal utility functions of the form $U^*(x_1, x_2) = U^*(x_1, x_2; \theta)$. For this case, the only relevant network variable is the network degree $s_{1i} := \sum_{j=1}^n L_{ij}$, and the inclusive value function does not depend on endogenous network characteristics, so that $H(x; s) = H(x)$ for all $s \in \mathcal{S}$.*

Then the information in the sample can be summarized by the degree sequence $s_{11}, \dots, s_{1n}$ together with the non-zero link indicators, and the limiting model implies that the log-likelihood contribution of the ith node is given by

$$\begin{aligned} \ell_i(\theta, H) &= \frac{1}{2} \sum_{j=1}^n L_{ij} \left(V^*(x_i, x_j | \theta) - \log(1 + H(x_i)) - \log(1 + H(x_j)) \right) \\ &\quad + \log s_{1i} - \log(1 + H(x_i)) \end{aligned}$$

Note that the first term of the log-likelihood only receives weight one half to avoid double-counting of non-zero link indicators as we sum the log-likelihood contributions over the nodes $i = 1, \ldots, n$. The constrained maximum likelihood estimator maximizes the network log-likelihood $\mathcal{L}_n(\theta, H) := \sum_{i=1}^n \ell_i(\theta, H)$ subject to the fixed-point condition $H(x) = \frac{1}{n}\sum_{i=1}^n \psi_i(\theta, H)$ with

$$\psi_i(\theta, H) = w_i s_{1i} \frac{\exp\{V(x, x_i|\theta)\}}{1 + H(x_i)}$$

where $w_j := \frac{\mathbf{1}\{s_{1j}>0\}}{\frac{1}{n}\sum_{k=1}^n \mathbf{1}\{s_{1k}>0\}}$. The importance weights w_j are used to obtain an unbiased estimator for the operator Ψ_0 in (3.2), noting that the reference distribution for the potential value of s_{1j} from setting $L_{ij} = 1$ is equal to the conditional distribution of s_{1j} given $s_{1j} > 0$ in the cross-section over nodes in the network.

Example 5.2. ***(Many-to-Many Matching and Capacity Constraints)*** *Next, we state the likelihood for a many-to-many matching model that assumes the same preferences as in the previous case, but allows each node to form at most $\bar{s}$ direct links, i.e. capping the network degree at $\bar{s}$. Furthermore, in accordance with classical matching models, we modify the notion of pairwise stability for networks (PSN, Definition 2.1) to allow for deviations in which a node simultaneously severs one link and forms another, see Definition A.3 (PSN2) in the appendix for a formal definition.*

The log-likelihood contribution of the ith node resulting from the limiting model is then obtained as

$$\begin{aligned}\ell_i(\theta, H) &= \frac{1}{2}\sum_{j=1}^n L_{ij}\left(V^*(x_i, x_j|\theta) - \log(1 + H(x_i)) - \log(1 + H(x_j))\right) \\ &\quad + \log s_{1i} - \mathbb{1}\{s_{1i} < \bar{s}\}\log(1 + H(x_i))\end{aligned}$$

and the fixed-point mapping for the inclusive value function is the average of contributions

$$\psi_i(\theta, H) := w_i s_{1i} \frac{\exp\{V(x, x_i|\theta)\}}{1 + H(x_i)}$$

where $w_j := \frac{\mathbf{1}\{s_{1j}>0\}}{\frac{1}{n}\sum_{k=1}^n \mathbf{1}\{s_{1k}>0\}}$ as in the previous case.

Example 5.3. ***(Strategic Complementarities in Network Degree)*** *Finally, we consider the case in which link preferences depend on the respective network degrees of nodes i and j, $s_i = \sum_{k=1}^n L_{ik} \equiv s_{1i}$ and $s_j = \sum_{k=1}^n L_{jk} \equiv s_{1j}$. For simplicity, we assume that s_i, s_j are strategic complements with L_{ij}, that is the systematic part $U^*(x_i, x_j; s_i, s_j|\theta)$ is nondecreasing in s_i and s_j.*

With preferences of this form, the edge-level response is generally not unique, and in what follows we assume that for any realization of payoffs, the observed network is selected as the maximal pairwise stable network under the partial order $L \geq L'$ if $L_{ij} \geq L'_{ij}$ for all i, j. It follows from standard arguments for monotone comparative statics (see Milgrom and

Roberts (1990)) that the maximal stable network is well-defined and can be obtained from myopic best-response dynamics starting at the complete graph $L_{ij} = 1$ for all $i \neq j$.

Under these assumptions the probability that a given network $\mathbf{L}$ is generated by this selection mechanism is equal to the probability that $\mathbf{L}$ is pairwise stable times the conditional probability that payoffs do not support any larger network $L' > L$ given that $\mathbf{L}$ is pairwise stable. After some standard calculations, we find that under $\mathcal{F}_0^*$, the probability that the values $s_0 < s_1, \dots < s_r$ for s_{1i} are jointly supported is equal to

$$p(s_0, \dots, s_r) = \frac{H(x; s_0)^{s_0} \prod_{q=1}^{r} (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}}$$

If we define

$$\pi^*(s_0; r) := \sum_{s_0 < .. < s_r} \frac{p(s_0, s_1 \dots, s_r)}{p(s_0)} = 1 - \sum_{s_0 < .. < s_r} \frac{(1 + H(x; s_0))^{s_0+1} \prod_{q=1}^{r} (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}}$$

the conditional probability that s_0 is the largest network degree for node i given that s_0 is supported by a pairwise stable network is given by

$$\pi^*(s_0) = 1 + \sum_{r=1}^{\infty} (-1)^r \pi^*(s_0; r)$$

For an implementation of the MLE in the Monte Carlo experiments in the next section, we partially vectorize computation of $\pi^*(s_0)$. Specifically, if $H(x; s)$ only changes its value at a finite number r of values for s, then $\pi^*(s_0)$ can be computed by a double loop with a total of $2r$ iterations.

The log-likelihood contribution of the ith observation can then be written as

$$\begin{aligned} \ell_i(\theta, H) &= \frac{1}{2} \sum_{j=1}^{n} L_{ij} \left(V^*(x_i, x_j; s_i, s_j | \theta) - \log(1 + H(x_i; s_i)) - \log(1 + H(x_j; s_j)) \right) \\ &\quad + \log s_{1i} - \log(1 + H(x_i; s_i)) + \log \pi^*(s_i) \end{aligned}$$

and the fixed-point condition for the inclusive value function is obtained from the sample average of

$$\psi_i(\theta, H) := w_i s_{1i} \frac{\exp\{V(x, x_i; s, s_i | \theta)\}}{1 + H(x_i; s_i)}$$

where $w_j := \frac{\mathbf{1}\{s_{1j} > 0\}}{\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{s_{1k} > 0\}}$ as in the previous case.

5.3. **Set Estimation and Bounds.** In the general case of a non-unique edge-level response, the limiting model provides bounds on the link frequency distribution. In complete analogy to estimation of discrete games, probability bounds of this type can then be used to construct moment inequalities to estimate identification regions for the payoff parameters, see Tamer (2003), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2011), and Galichon and Henry (2011).

We consider a parametric specification for payoffs,

$$U(x_i, x_j; s_i, s_j, t_{ij}) = U(x_i, x_j; s_i, s_j, t_{ij}|\theta_0)$$

and will now focus on those cases in which the edge-level response is non-unique, so that the limiting model $\mathcal{F}_0^*$ consists of non-trivial set of distributions. This set can be described in terms of lower and upper bounds for probabilities of events A_{ij} in the variables L_{ij}, s_i, s_j, t_{ij} for a dyad ij in the n-player network. That is, denoting the probability of the event A_{ij} in the selected pairwise stable network with $P_n(A_{ij}|x_i, x_j)$, we can derive functions $Q_L(\cdot), Q_U(\cdot)$ from the limiting model such that

$$Q_L(A_{ij}|x_i, x_j; \theta_0, H^*) \leq \lim_n P_n(A_{ij}|x_i, x_j) \leq Q_U(A_{ij}|x_i, x_k; \theta_0, H^*) \tag{5.2}$$

As shown by Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011), a characterization of sharp bounds on $\mathcal{F}_0^*$ typically has to account for composite events A_{ij}, i.e. events of the form $A_{ij} = \{(L_{ij}, s_i, s_j, t_{ij}) \in Z\}$ for certain non-singleton sets $Z \subset \{0,1\} \times \mathcal{S}^2 \times \mathcal{T}$. As in the single-valued case, the bounds $Q_L(\cdot)$ and $Q_U(\cdot)$ depend on the aggregate states H^*, M^* where the inclusive-value function satisfies the fixed-point condition $H^* = \Psi[H^*, M^*]$, and the reference distribution $M^* \in \Omega_0[H^*, M^*]$ with the mappings $\Psi_0[\cdot], \Omega_0[\cdot]$ are defined as before. In the remainder of this section, we focus on the case in which the reference distribution M^* is nonparametrically point-identified, so that we do not need to solve the second fixed-point problem explicitly.

Given these probability bounds, the identified set for the payoff parameter θ is

$$\begin{aligned}\Theta_I \;:=\; \Big\{&\theta \in \Theta : Q_L(A^{(r)}|x_i, x_j; \theta_0, H^*) \leq \lim_n P_n(A^{(r)}|x_i, x_j) \leq Q_U(A^{(r)}|x_i, x_k; \theta_0, H^*) \text{ a.s.}\\ &\text{for each } r = 1, \dots, R \text{ and for some } H = \Psi_0[H, M^*](\theta)\Big\}\end{aligned}$$

where $A^{(r)} := \{(d_{ij}, s_i, s_j, t_{ij}) \in Z^{(r)}\}$ and $Z^{(1)}, \dots, Z^{(R)}$ denote the subsets of $\{0,1\} \times \mathcal{S}^2 \times \mathcal{T}$.

Estimation and inference regarding the identified set Θ_I can be implemented using moment functions of the form

$$\mathbf{m}(A^{(r)}|\theta, H) := \begin{pmatrix} \mathbb{1}\{A_{ij}^{(r)}\} - Q_L(A^{(r)}|x_i, x_j; \theta, H) \\ Q_U(A^{(r)}|x_i, x_j; \theta, H) - \mathbb{1}\{A_{ij}^{(r)}\} \end{pmatrix}$$

From our convergence results and the definition of the probability bounds, we then have the asymptotic conditional moment restriction

$$\lim_n \mathbb{E}[\mathbf{m}(A^{(r)}; \theta_0)|x_i, x_j] \geq 0 \text{ a.s.}$$

These conditional restrictions can then be transformed into systems of unconditional moment equalities and inequalities for set estimation and inference, see e.g. Beresteanu, Molchanov, and Molinari (2011) for a description for the case of finite discrete games. Since the bounds in (5.2) are only satisfied as $n \to \infty$, these procedures can only be consistent (asymptotically valid, respectively) under the many-player limit.

We conclude this section by giving two examples for how to derive the probability bounds $Q_L(\cdot), Q_U(\cdot)$ from the limiting model $\mathcal{F}_0^*$.

Example 5.4. ***Completion of Transitive Triads.*** *Let payoffs be*

$$U_{ij}(\mathbf{L}, \mathbf{X}) = U^*(x_i, x_j) + \beta_{T,n} t_{ij} + \sigma \eta_{ij}$$

where $t_{ij} = t(\mathbf{L}, \mathbf{X}; i, j) := \max_{k \neq i,j} L_{ik} L_{jk}$ *is an indicator of* i *and* j *having a common network neighbor. In order to obtain a potentially non-degenerate clustering coefficient in the limiting model, we assume the sequence* $\beta_{T,n} := \frac{1}{6} \log n + \beta_T \geq 0$*, which can be shown to satisfy Assumption 4.3 (c). We also let* $t_{ij}(L_1, L_2, L_3)$ *denote the potential values for* t_{ij} *given the structure of the network after fixing* $L_{ij} = L_1$*,* $L_{ik} = L_2$*,* $L_{jk} = L_3$ *for* $L_1, L_2, L_3 \in \{0, 1\}$*. In particular,* $t_{ij}(1, L_{ij}, L_{jk}) = 1$ *only if* i, j *are part of a transitive triad for* $L_{ij} = 1$*.*

We consider probability bounds for outcomes in the subgraph on the triad consisting of nodes i, j, k*, where either of the link indicators* L_{ij}, L_{ik}, L_{jk} *may be one or zero. We do not assign specific roles to the nodes* i, j, k*, so it is without loss of generality sufficient to consider events with* $L_{ij} \geq L_{ik} \geq L_{jk}$*. For orders of magnitude, under the asymptotic sequence in Assumption 4.3 we can verify that the number of triads supporting* $l = 0, 1, 2, 3$ *stable links grows at the order* a_{ln}*, where* $a_{0n} = n^3$*,* $a_{1n} = n^2$*,* $a_{2n} = n$*, and* $a_{3n} = n$*, and their respective shares at rates* a_{ln}/n^3*. In particular, the dyad* ij *is part of* $n - 3$ *distinct triads outside* ijk*, so that the probability for* $t_{ij}(1, 0, 0) = 1$ *is of the order* $n a_{3n}/n^3 = 1/n$*.*

Next we notice that the structure of the complementarity restricts multiplicity in subnetwork outcomes: Since $\beta_{T,n} \geq 0$*, we can verify that for any payoffs supporting* $L_{ij} = L_{ik} = 1$ *and* $L_{jk} = 0$*, there exists no other pairwise stable subnetwork on the triad. Also, payoffs under which a subnetwork with one or zero links is pairwise stable, may only support the complete subgraph* $L_{ij} = L_{ik} = L_{jk} = 1$ *as an additional pairwise stable subnetwork.*

To compute the probability bounds $Q_L(L_{ij}, L_{ik}, L_{jk} | x_i, x_j, x_k; \cdot)$ *and* $Q_U(L_{ij}, L_{ik}, L_{jk} | x_i, x_j, x_k; \cdot)$*, let* $L_{ij}(t)$ *denote the potential value for* L_{ij} *from setting* $T_{ij} = 1$*. Then define* $p_{ik}(t) := P(d_{ik}(t) = 1 | x_i, x_j)$ *so that from our previous results,*

$$\lim_n n p_{ik}(t) = \frac{s_{1i} s_{1k} \exp\{U^*(x_i, x_k) + U^*(x_k, x_i) + 2\beta_T t\}}{(1 + H^*(x_i))(1 + H^*(x_k))}.$$

For any event A_{ijk} *in the variables* L_{ij}, L_{ik}, L_{jk}*, the upper bound* $Q_U(A_{ijk} | x_i, x_j, x_k)$ *corresponds to the probability that an outcome in* A_{ijk} *is supported by random payoffs. Hence we* *have*

$$\begin{aligned}
Q_U(0,0,0|x_i,x_j,x_k;\theta,H) &= \lim_n (1-p_{ij}(0))(1-p_{ik}(0))(1-p_{jk}(0)) = 1 \\
Q_U(1,0,0|x_i,x_j,x_k;\theta,H) &= \lim_n np_{ij}(0)(1-p_{ik}(0))(1-p_{jk}(0)) \\
Q_U(1,1,0|x_i,x_j,x_k;\theta,H) &= \lim_n n^2 p_{ij}(0)p_{ik}(0)(1-p_{jk}(0)) \\
Q_U(1,1,1|x_i,x_j,x_k;\theta,H) &= \lim_n n^2 p_{ij}(1)p_{ik}(1)p_{jk}(1)
\end{aligned}$$

noting that, as argued before, the contribution of triads outside ijk to link probabilities is asymptotically negligible. The limits on the right-hand side are then obtained by plugging in component-wise limits for $p_{kl}(t)$ as functions of θ and H.

For the lower bounds, note first that the upper bound on the probability of transitive triads is of a smaller asymptotic order that that for a triad with zero or one link, so that

$$\begin{aligned}
Q_L(0,0,0|x_i,x_j,x_k;\theta,H) &= Q_U(0,0,0|x_i,x_j,x_k;\theta,H) \\
Q_L(1,0,0|x_i,x_j,x_k;\theta,H) &= Q_U(1,0,0|x_i,x_j,x_k;\theta,H).
\end{aligned}$$

Furthermore, for any payoffs resulting in a triad with two links, the pairwise stable subnetwork on that triad is unique for $t_{ij}(1,1,0) = t_{ik}(1,1,0) = t_{jk}(1,1,0) = 0$, so that $Q_L(1,1,0|x_i,x_j,x_k;\theta,H) = Q_U(1,1,0|x_i,x_j,x_k;\theta,H)$. Finally, the transitive triad is the unique pairwise stable subnetwork if and only if all three links are dominant,

$$Q_L(1,1,1|x_i,x_j,x_k;\theta,H) = \lim_n n^2 p_{ij}(0)p_{ik}(0)p_{jk}(0) = 0$$

We can also verify that bounds for composite events of distinct values of L_{ij}, L_{ik}, L_{jk} do not impose additional restrictions on the limiting distribution.

We can then form moment inequality conditions by comparing these bounds to appropriately normalized subgraph counts for triads in the observed network,

$$\begin{aligned}
\hat{\mathbf{m}}_n^{(1)}(\theta,H) &:= \frac{1}{n^2}\sum_{ijk}\left(\mathbb{1}\{L_{ij}=1, L_{ik}=0, L_{jk}=0\} - Q_L(1,0,0|x_i,x_j,x_k;\theta,H)\right)\psi(x_i,x_j,x_k) \\
\hat{\mathbf{m}}_n^{(2)}(\theta,H) &:= \frac{1}{n}\sum_{ijk}\left(\mathbb{1}\{L_{ij}=L_{ik}=1, L_{jk}=0\} - Q_L(1,1,0|x_i,x_j,x_k;\theta,H)\right)\psi(x_i,x_j,x_k) \\
\hat{\mathbf{m}}_n^{(3)}(\theta,H) &:= \frac{1}{n}\sum_{ijk}\left(Q_U(1,1,1|x_i,x_j,x_k;\theta,H) - \mathbb{1}\{L_{ij}=L_{ik}=L_{jk}=1\}\right)\psi(x_i,x_j,x_k)
\end{aligned}$$

for a vector-valued function $\psi(x_1,x_2,x_3) \geq 0$ that only takes non-negative values. By the law of iterated expectations, the limit of the expectations for $\hat{\mathbf{m}}_n^{(k)}(\theta_0,H_0^*)$ is equal to zero for $k=1,2$, and greater or equal to zero for $k=3$, so that we can use the resulting moment equalities and inequalities for testing and estimation.

Example 5.5. ***(Strategic Complementarities in Network Degree)*** *Consider the payoffs from Example 5.3 with payoffs $U_{ij}(\mathbf{L})$ depending on $s_i := \sum_{j=1}^n L_{ij}$ and $s_j := \sum_{i=1}^n L_{ji}$. We now show how to construct probability bounds for dyad-level outcomes in (L_{ij}, s_i, s_j) which do not assume a particular selection mechanism.*

Similar to the discussion for the case of a specific selection mechanism, let

$$p(s_1, \dots, s_r|x) := \frac{H(x; s_1)^{s_1} \prod_{q=1}^r (H(x; s_q) - H(x; s_{q-1}))^{(s_q - s_{q-1})}}{(1 + H(x; s_r))^{r+1}}$$

for any $s_1 < \dots < s_r$, and define

$$\tau^*(\bar{s}; r|x) := \sum_{s_1 < .. < \bar{s} < .. < s_r} \frac{p(s_1, \dots, \bar{s}, \dots, s_r|x)}{p(\bar{s}|x)}$$

where the summation is over any ordered tuple of r values for s_{1i}, one component of which equals $\bar{s}$. Then the conditional probability that $\bar{s}$ is the unique pairwise stable value of s_i given that it is supported by a pairwise stable subnetwork is

$$\pi^*(\bar{s}|x) = 1 + \sum_{r=1}^\infty (-1)^r \tau^*(\bar{s}; r|x)$$

Since the sharp upper bound for the probability of the outcome L_{ij}, s_i, s_j corresponds to the probability that these values are supported by some pairwise stable subnetwork, we obtain

$$\begin{aligned} Q_U(L_{ij} = 1, s_i, s_j | x_i, x_j; \theta, H) &:= \lim_n nP(L_{ij} = 1, s_i, s_j \text{ supported } |x_i, x_j) \\ &= \frac{s_i s_j \exp\{V(x_i, x_j; s_i, s_j)\} H(x_i; s_i)^{s_i} H(x_j; s_j)^{s_j}}{(1 + H(x_i; s_i))^{s_i+1} (1 + H(x_j; s_j))^{s_j}} \end{aligned}$$

Sharp lower bounds for specific values of these network outcomes correspond to the event that no other values of L_{ij}, s_i, s_j are supported by payoffs, and can be obtained by multiplying the upper bound with the conditional probability that the given pairwise stable outcome is unique. Specifically, we let

$$Q_U(L_{ij} = 1, s_i, s_j | x_i, x_j; \theta, H) := Q_U(L_{ij} = 1, s_i, s_j | x_i, x_j; \theta, H) \pi^*(s_i|x_i) \pi^*(s_j|x_j)$$

These bounds for singleton events are not sharp, but following Beresteanu, Molchanov, and Molinari (2011) and Galichon and Henry (2011), we can obtain additional constraints by considering composite events consisting of several distinct values of these network variables.

5.4. **Welfare and Surplus.** Our limiting framework also yields a straightforward analytic approximation to expected surplus from being connected to the network. Surplus calculations of this type are necessary e.g. to characterize ex ante incentives to participate in the network and exert search effort (as e.g. in the setting described by Currarini, Jackson, and Pin (2009)), or to evaluate welfare consequences of policies affecting the composition or structure of the network.

Focusing on the case of no edge-specific endogenous interaction effects, let $U_{ij}(s) := U^*(x_i, x_j; s, s_j(\mathbf{L}^*)) + \sigma\eta_{ij}$ and let $U_{i;r}(s)$ denote the rth (largest) order statistic of the sample $\{U_{ij}(s) : j \in W_i(\mathbf{L}^*)\}$. Then if the sequence $s_1, \dots, s_{s_{1i}} = s_i$ of values for s_i results from successively adding links corresponding to the $1, \dots, r$th order statistics, agent is surplus can be obtained by integrating the marginal utilities,

$$\Pi_i(\mathbf{L}^*) = \sum_{r=1}^{s_{1i}} (U_{i;t}(s_r) - MC_i) \tag{5.3}$$

Note that if marginal link utilities are indeed derived from a benefit function $B_i(\mathbf{L})$ as in Section 2.1, the expression for $\Pi_i(\mathbf{L}^*)$ does not depend on the particular choice of such a sequence $s_1, \dots, s_r$.

For the Logit model it is known that the expected value of the first order statistic of such a sample is equal to a function of the inclusive value (see e.g. Train (2009)). We first show an analogous result for any other finite order statistic as $W_i(\mathbf{L})$ grows in size, and then derive limiting expressions for the expected net surplus in (5.3). In order to characterize the expectation of $\Pi_i(\mathbf{L}^*)$, we also let $A_i(r; s)$ denote the event that payoffs support $s_i = s$ and network degree $s_{1i} = r$. We can then derive the following limiting expressions for expected link surplus:

Proposition 5.1. *Suppose that the assumptions of Theorem 4.2 hold. Then for any $r' \geq r$,*

$$\begin{aligned}
\lim_n \mathbb{E}[U_{i;r}|A_i(r'; s)] - \frac{1}{2}\log n &= \log(1 + H^*(x; s)) + \gamma - \sum_{q=1}^{r-1} \frac{1}{q} \\
\lim_n \mathbb{E}[MC_i|A_i(r'; s)] - \frac{1}{2}\log n &= \log(1 + H^*(x; s)) + \gamma - \sum_{q=1}^{r'} \frac{1}{q}
\end{aligned}$$

where $H^*(x; s)$ is the inclusive value function and $\gamma \approx 0.5772$ denotes the Euler-Mascheroni constant.

See the appendix for the derivation of these expressions. Given this result, we can compute the expected surplus from being connected in the network. Conditional on $A_i(s_{1i}, s)$, we have

$$\begin{aligned}
\lim_n \mathbb{E}[\Pi_i(\mathbf{L}^*)|A_i(s_{1i}, s)] &= \lim_n \sum_{r=1}^{s_{1i}} (\mathbb{E}[U_{i;r}|A_i(s_{1i}; s)] - \mathbb{E}[MC_i|A_i(s_{1i}; s)]) \\
&= \sum_{r=1}^{s_{1i}} (\log(1 + H^*(x; s_r)) - \log(1 + H^*(x; s_{s_{1i}}))) + \sum_{r=1}^{s_{1i}} \left(\sum_{q=1}^{s_{1i}} \frac{1}{q} - \sum_{q=1}^{r-1} \frac{1}{q} \right) \\
&= \sum_{r=1}^{s_{1i}} (\log(1 + H^*(x; s_r)) - \log(1 + H^*(x; s_{s_{1i}}))) + s_{1i}
\end{aligned} \tag{5.4}$$

Hence, for the case where link preferences in exogenous attributes alone, it follows by the law of iterated expectations that

$$\lim_n \mathbb{E}[\Pi_i(\mathbf{L}^*)] = \mathbb{E}[s_{1i}] = H^*(x)$$

where in the case of a non-unique edge-level response, the expectation is taken given the equilibrium selection rule. On the other hand, with preferences depending on network degree, the sequence $s_1, \dots, s_{s_{1i}}$ becomes $1, 2, \dots, s_{1i}$, so that

$$\lim_n \mathbb{E}[\Pi_i(\mathbf{L}^*)] = \mathbb{E}\left[\sum_{r=1}^{s_{1i}} (\log(1 + H^*(x; r)) - \log(1 + H^*(x; s_{1i}))) + s_{1i}\right]$$

where the expectation with respect to s_{1i} also depends on the selection rule.

6. Outline of the Limiting Argument

This section gives an outline of the formal argument behind Theorem 4.2. One of the challenges in characterizing the exact model for the finite-player network is that the set of available link opportunities to each of the n nodes are unobserved and endogenous to that node's own choices. Furthermore, the pairwise stability conditions depend on the potential values for the relevant network attributes for each node (edge, respectively) under all possible configurations of the network. Clearly, the corresponding latent state space grows in dimension with the size of the network and contains both discrete and continuous components. Our argument relies on the inclusive value function and the reference distribution as asymptotically sufficient aggregate state variables to represent that state space, partly using insights from Menzel (2015).

There are three additional aspects that complicate the formal argument: For one, strategic externalities across links may lead to long-range dependence of link decisions and result in simultaneity problems that do not exist in models without strategic interdependencies. Here we rely on a novel argument based on symmetric dependence which does not require any type of ergodicity or weak dependence. Furthermore, network formation allows for several, rather than just one direct connection to each node, so that not only the maximum, but other extremal order statistics of marginal benefits are relevant for link formation decisions. Finally, PSN allows for multiple edge-level responses to a given set of link opportunities, so that even in the limit the link frequency distribution need not be unique.

Dependence. In a first step, we show that dependence between idiosyncratic taste shifters η_{ij} and endogenous network attributes of nodes in the link opportunity set $W_i(\mathbf{L}^*)$, as defined in section 2, vanishes as n grows large: In pairwise stable networks, establishing a link ij may affect subsequent decisions by other nodes that are available to i or j, which may in turn affect link choices by other agents that need not be directly linked to i or j. Such a chain of adjustments may eventually link back to either i or j's link opportunity sets. Preference cycles of this type may generally lead to dependence between taste shocks η_{ij} and link opportunity sets $W_i(\mathbf{L}^*)$ in finite network formation games, but our first technical result below establishes that dependence becomes negligible as n increases:

In the following we let $\mathbf{d}_{ji}$ be a vector of dummy variables indicating availability of node j to i for each configuration of the relevant overlap defined in Appendix A, and also let $\mathbf{s}_{ji}, \mathbf{t}_{ji}$ denote the potential values of the payoff-relevant endogenous network attributes for the dyad (i, j) corresponding to an arbitrarily chosen initial network $L^{(0)}$. Note that for the leading case of a unique edge-level response, the potential values corresponding to a network $\mathbf{L}$ are given by $\mathbf{s}_{ji} = S(\mathbf{L} + \{ij\}, X; j)$, and $\mathbf{t}_{ji} = T(\mathbf{L} + \{ij\}, X; j, i)$, respectively. We also let L^* denote the pairwise stable network, or first element of a closed cycle reached by the tâtonnement process described in section 2.3 starting from the initial condition $L^{(0)}$, and denote the potential values of the network variables under L^* with $\mathbf{d}^*_{ji}, \mathbf{s}^*_{ji}, \mathbf{t}^*_{ji}$. We also stack the potential values and attributes into vectors $z_{ij} := (x_i, x_j, \mathbf{d}'_{ji}, \mathbf{s}'_{ji}, \mathbf{t}'_{ji})$ and $z^*_{ij} := (x_i, x_j, (\mathbf{d}^*_{ji})', (\mathbf{s}^*_{ji})', (\mathbf{t}^*_{ji})')$, respectively.

Availability of a node j to i in a given network is fully determined by the potential values of $\mathbf{s}_j, \mathbf{t}_{ij}$, and the taste shocks η_{ji} and MC_j. To analyze dependence between node i's taste shifters and link opportunities, it is therefore sufficient to focus on the joint distribution of MC_i, $(\eta_{ij})_{j=1}^n$ and z^*_{ij} for $j = 1, \dots, n$.

In the following, we denote the conditional c.d.f. of $(\eta_{i1}, \dots, \eta_{in}, MC_i)$ given $\mathbf{z}^*_i := (z^*_{i1}, \dots, z^*_{in})$ with

$$G^*_n(\eta_1, \dots, \eta_n, \eta_0 | \mathbf{z}^*_i) := P(\eta_{i1} \le \eta_1, \dots, \eta_{in} \le \eta_n, MC_i \le \eta_0 | z^*_{i1}, \dots, z^*_{in})$$

with the associated p.d.f. $g^*_n(\eta_1, \dots, \eta_n, \eta_0 | \mathbf{z}^*)$, and the unconditional distribution functions with $G_n(\eta_1, \dots, \eta_n, \eta_0)$ and $g_n(\eta_1, \dots, \eta_n, \eta_0)$, respectively. The following lemma summarizes the main finding regarding dependence between taste shifters η_i and link opportunities represented by $W_i(\mathbf{L}^*)$.

Lemma 6.1. *Suppose Assumptions 4.1, 4.2, 4.3 (i), and 4.4 hold. Then, for any pairwise stable network,*

$$\lim_{n \to \infty} \frac{g^*_n(\eta_1, \dots, \eta_n, \eta_0 | \mathbf{z}^*)}{g_n(\eta_1, \dots, \eta_n, \eta_0)} = 1$$

almost surely, where convergence is uniform with respect to the initial condition $L^{(0)}$. The analogous conclusion holds for the joint distribution over any fixed finite subset of nodes $N_0 \subset \{1, \dots, n\}$, where the conditioning set excludes the dyad-level outcomes z^*_{ij} for any pair $i, j \in N_0$.

The proof of this result is given in the appendix. Note that the conclusion is analogous to that in Lemma 3.2 in Menzel (2015) for the problem of two-sided matchings. However the structure of the problem differs in one crucial aspect: the structure of the matching problem in Menzel (2015) was shown to rule out preference cycles that lead to interdependence among a non-negligible share of agents in the market. However, with general link externalities the length of preference cycles of this type is no longer bounded, and local perturbations of a given pairwise stable network may trigger cascades of subsequent adjustments that percolate through the entire network. Hence, the argument in Menzel (2015) cannot be easily extended to the network formation problem except for very restrictive cases in which link externalities are known to be sufficiently small.[13]

The proof of Lemma 6.1 considers the outcomes of tâtonnement starting from arbitrary initial conditions, where we use symmetry arguments to show that for any node pair the conditional distribution of z_{ij}^* given the initial condition z_{ij} does not depend on the taste shocks at the dyad level. This finding can then be combined with row-wise exchangeability of the distribution $g_n^*(\cdot)$ to establish the main conclusion of the Lemma. In particular, our argument does not rely on sparsity of the network, weak dependence or ergodicity in the link formation process. Instead, the conclusion of the Lemma can be interpreted as resulting from conditional independence of dyad-level outcomes given the (potentially stochastic) reference distribution associated with the selected pairwise stable network.

Conditional Choice Probabilities. The second step takes the limit of the conditional probability that agent i is willing to form a link to agent j given x_i, z_j, and her other links. As in the case of the matching model (Lemma 3.1 in Menzel (2015)), we find that given our specification the number of links "accepted" by agent i is substantially smaller than the number of "proposals" $j \in W_i(\mathbf{L}^*)$, so that the conditional probability of proposing or accepting a link depends only on the upper tail of $G(\cdot)$, the distribution for the taste shifters η_{ij}. The assumption that $G(\cdot)$ has tails of type I can then be used to establish that conditional choice probabilities can be approximated by those implied by the Logit model with taste shifters generated by an extreme-value type-I distribution. Here a complication arises from the fact that all links (and availability to j) are determined simultaneously, so that it is necessary to consider joint probabilities of the form

$$\Phi(i, j_1, \dots, j_r | \mathbf{z}_i^*) = P\left(U_{ij_1}, \dots, U_{ij_r} \geq MC_i > U_{ij'} \text{ all other } j' \in W_i(\mathbf{L}^*) | \mathbf{z}_i^*\right)$$

Notice that marginal benefits U_{ij} depends on S_i and T_{ij}, so that $\Phi(i, j_1, \dots, j_r | \mathbf{z}_i^*)$ cannot be directly interpreted as a conditional choice probability, but equals the probability that the configuration $L_{ij_1} = \cdots = L_{ij_r} = 1$ and $L_{ij'} = 0$ for all other $j' \in W_i$ satisfies the pairwise stability conditions regarding player i's payoffs. Such a configuration is not necessarily

[13]A sufficient condition for preference cycles in networks to have finite length is given by Assumption 4 ("subcriticality") in Leung (2016).

unique, but externalities among links emanating from i may support several stable outcomes for a given realization of random payoffs.

We find that under our assumptions, we can approximate the conditional probability $\Phi(i, j_1, \dots, j_r|\mathbf{z}_i^*)$ with its analog under the assumption of independent extreme-value type-I taste shifters. Let $s^*_{i,+j}, s^*_{j,+i}, t^*_{ij,+ij}$ denote the potential values for s_i, s_j, and t_{ij}, respectively, for the network $\tilde{\mathbf{L}}$ that is obtained from L^* after setting $L_{ij} = L_{ij_s} = 1$ for all $s = 1, \dots, r$ and $L_{ik} = 0$ for all $k \neq j, j_1, \dots, j_r$.

Lemma 6.2. *Suppose that Assumptions 4.1-4.3 hold, and that the marginal benefit functions $U_{i1}, \dots, U_{iJ}$ are J i.i.d. draws from the model (2.1), and marginal cost MC_i is an independent draw from (2.2). Then as $J \to \infty$,*

$$\left| n^{r/2}\Phi(i, j_1, \dots, j_r|\mathbf{z}_i^*) - \frac{r! \prod_{s=1}^r \exp\{U^*(x_i, x_{j_s}; s^*_{i,+j_s}, s^*_{j_s,+i}, t^*_{ij_s,+ij_s})\}}{\left(1 + \frac{1}{J}\sum_{j=1}^J \exp\{U^*(x_i, x_j; s^*_{i,+j}, s^*_{j,+i}, t^*_{ij,+ij})\}\right)^{r+1}} \right| \to 0 \qquad (6.1)$$

for any $r = 0, 1, 2, \dots$.

This approximation allows the use of inclusive values for the link opportunity sets to reparameterize conditional choice probabilities even if the distribution of taste shifters η_{ij} is not extreme-value type-I, but belongs to its domain of attraction. We also find that we can take joint limits for any finite set of nodes, $i_1, \dots, i_s$ conditional on $\mathbf{z}^*_{i_1}, \dots, \mathbf{z}^*_{i_s}$ in an analogous fashion.

It follows from the previous two steps that we can approximate the distribution of the edge-level response using the inclusive value of agent i's link opportunity set W, which we defined as

$$I_i[W] := \frac{1}{n^{1/2}} \sum_{j \in W} \exp\{U^*(x_i, x_j; s^*_{i,+j}, s^*_{j,+i}, t^*_{ij,+ij})\}$$

Most importantly, the composition and size of the set of link opportunities affects the conditional choice probabilities only through the inclusive value, which is a scalar parameter summarizing the systematic components of payoffs for the available options, see Luce (1959), McFadden (1974), and Dagsvik (1994).

Law of Large Numbers. The third step of the argument establishes a conditional law of large numbers for the inclusive values $I_i^* := I_i[W_i(\mathbf{L}^*)]$ which are sample averages over the characteristics of agents in the link opportunity set $W_i(\mathbf{L}^*)$, where the size of the set $|W_i(\mathbf{L}^*)|$ grows at a rate $\sqrt{n}$ for any PSN.

Lemma 6.3. *Suppose Assumptions 4.1, 4.2, and 4.3 hold. Then, (a) there exists a function $\hat{H}_n^*(x, s)$ such that for any pairwise stable network, the resulting inclusive values satisfy*

$$I_i^* - \hat{H}_n^*(x_i, s_i) = o_p(1)$$

for each i drawn from a uniform distribution over $\{1, \dots, n\}$. Furthermore, (b), if the weight functions $\omega(x, x', s, s') \geq 0$ are bounded and form a Glivenko-Cantelli class in (x, s), then

$$\sup_{x \in \mathcal{X} s \in \mathcal{S}} \frac{1}{n} \sum_{j=1}^{n} \omega(x, x_j, s, s_j)(I_j^* - \hat{H}_n^*(x_j, s_j)) = o_p(1)$$

See the appendix for a proof. The result implies that up to sampling error, for all but a vanishing share of nodes, inclusive values only depend on agents' own characteristics x_i, s_i, so that we do not need to account for the node-specific link opportunity sets separately as we take limits. In the following, we refer to $\hat{H}_n^*(x, s)$ as the inclusive value function in the finite network. Note also that part (a) still allows for some nodes to have inclusive values that differ substantially from the respective value of the inclusive value function even for large n, however their share among the n nodes vanishes as the network grows.

In the two-sided matching case an analogous result could be derived relying on bounds exploiting the ordinal structure of the set of stable matchings (see Lemma B.5 in Menzel (2015)), where the inclusive value was shown to converge to the inclusive value function for each agent. For pairwise stable networks with a non-unique edge-level response, this is in general not the case so that our argument has to rely on a different strategy.

To illustrate the difficulty, suppose that there exists a stable network in which both values $s_j = \underline{s}$ and $s_j = \bar{s} \neq \underline{s}$ are supported by the edge-level response for a nontrivial share of nodes. Then switching between a network in which $s_j = \underline{s}$ to another in which $s_j = \bar{s}$ may make j more likely to be available to i, or increase the marginal benefit to i of forming a link with j. For a given realization of taste shifters η_{ji} it may then be possible to construct a pairwise stable network in which nodes j with high values of η_{ji} choose $s_j = \underline{s}$, whereas nodes with high values of η_{jk} for another node k choose $s_j = \bar{s}$. Hence, if selection of pairwise stable networks is allowed to depend on the idiosyncratic taste shifters η_{ji}, the inclusive values I_i^*, I_k^* could deviate substantially from the average for a small number of nodes. However, we find that for any pairwise stable network the share of nodes whose inclusive value differs substantially from the respective conditional average must vanish as the size of the network grows. In particular, we find that the problematic term in the characterization of the "worst-case" selection from edge-level responses can be bounded by the eigenvalue of a symmetric random matrix which is known to converge to a finite limit.

Fixed-Point Mapping for Inclusive Value Functions. Next, we derive an (approximate) fixed-point condition for the inclusive value function $H(x; s)$ resulting from the law of large numbers in the previous step.

For any conditional distribution $M(s|x)$, define the mapping

$$\hat{\Psi}_n[H,M](x;s) := \frac{1}{n}\sum_{j=1}^{n}\int \frac{s_{1j}\exp\{U^*(x,x_j;s,s_j,t_0)+U^*(x_j,x;s_j,s,t_0)\}}{1+H(x_j;s_j)}M(s_j|x_j)ds_j \tag{6.2}$$

If we let $\hat{M}_n^*(s|x)$ denote the empirical distribution of endogenous network characteristics given exogenous traits in the PSN, the next Lemma states that the inclusive value function is a fixed point of the mapping $\hat{\Psi}_n[\cdot,\hat{M}_n^*]$:

Lemma 6.4. *The inclusive value function $\hat{H}_n^*(x,s)$ resulting from a PSN has to satisfy the approximate fixed-point condition*

$$\hat{H}_n^*(x;s) = \hat{\Psi}_n[\hat{H}_n^*,\hat{M}_n^*](x;s) + o_p(1) \tag{6.3}$$

where the remainder converges in probability uniformly in the arguments x,s.

See the appendix for a proof.

Fixed-Point Existence and Uniqueness for Inclusive Value Functions. Next we can characterize the limit for $\hat{H}_n^*$. The analog of the fixed-point operator in (6.2) for the limiting model is given by

$$\Psi_0[H,M^*](x;s) := \int \frac{s_{1j,+1}^*\exp\{U^*(x,x_j;s,s_j)+U^*(x_j,x;s_j,s)\}}{1+H(x_j;s_{1j,+1}^*)}M^*(s_{1j,+1}^*|x_j,x)dx_jds_j \tag{6.4}$$

for an appropriately chosen reference distribution M^* in the set given by (A.4). Given that reference distribution, we then let $H^*(x;s)$ be a solution of the fixed-point problem

$$H^* = \Psi_0[H^*,M^*]$$

We next give conditions under which for any given reference distribution, the fixed point exists and is unique:

Proposition 6.1. *Suppose that Assumptions 4.1-4.3 hold. Then (i) for any given reference distribution $M^*(s|x)$ for which the network degree s_{1i} satisfies $\mathbb{E}[s_{1i}|x_i] < B_s < \infty$ almost surely, the mapping $\log H \mapsto \log\Psi[H]$ is a contraction mapping with contraction constant $\lambda < \frac{B_s\exp\{2\bar{U}\}}{1+B_s\exp\{2\bar{U}\}}$. Moreover, (ii) the fixed points in (3.2) are continuous functions that have bounded partial derivatives at least up to order p.*

The formal argument for this result closely parallels the proof of Theorem 3.1 in Menzel (2015) with contraction constant equal to $\frac{B_s\exp\{2\bar{U}\}}{1+B_s\exp\{2\bar{U}\}}$, a separate proof is therefore omitted.

One case of particular interest for which the contraction property holds without additional assumptions is that of no endogenous interaction effects, as shown by the following corollary:

Corollary 6.1. *Suppose Assumptions 4.1-4.3 hold, and* $U^*(x_1, x_2; s_1, s_2, t_{12}) = U^*(x_1, x_2)$*. Then the solution* $H^*(x; s) = H^*(x)$ *to the fixed point problem (3.2) is unique.*

The proof of this corollary is given in the appendix.

6.0.1. *Fixed Point Convergence.* Finally, we can characterize the cross-sectional distribution $\hat{M}_n^*(s_1|x_1, x_2)$ of potential values for s_i in the cross-section of nodes in the n-agent network with the equilibrium conditions

$$\int_S \hat{M}_n^*(s|x_1, x_2)ds \leq \hat{\Omega}_n[\hat{H}_n^*, \hat{M}_n^*](x_1, x_2; S) + o_p(1) \text{ for all } S \in \mathcal{S} \tag{6.5}$$

We can now combine the previous steps to show joint convergence for the reference distribution $\hat{M}_n^*$ and the inclusive value function $\hat{H}_n^*(x; s)$ to solutions of the population fixed-point problem (3.2) and (3.3). Specifically, Lemmata 6.2 and 6.3 imply that link opportunity sets can be parameterized with the inclusive value functions, whereas the fixed-point conditions for the inclusive value function and reference distribution converge to their respective population limits. Finally, under our assumptions convergence of the fixed-point mappings also implies convergence of the (set of) fixed points:

Lemma 6.5. *Suppose that Assumptions 4.1-4.5 hold. Then for any stable network, the inclusive value function* $\hat{H}_n^*(x; s)$ *and reference distribution* $\hat{M}_n^*(s_1|x_1, x_2)$ *satisfy the fixed-point conditions in (6.3) and (6.5). Moreover, there exist* H^*, M^* *satisfying the population fixed-point conditions in (3.2) and (3.3) such that* $\|\hat{H}_n^* - H^*\|_\infty = o_p(1)$ *and* $\|\hat{M}_n^* - M^*\|_\infty = o_p(1)$*.*

See the appendix for a proof. Finally, the state variables $\hat{H}_n^*, \hat{M}_n^*$ are asymptotically sufficient for the global structure of the network with respect to the conditional link acceptance probabilities. Hence convergence of the fixed points together with the Logit representation of the link acceptance probabilities in Lemma B.4 imply convergence of the link frequency distribution as claimed in Theorem 4.2.

7. Simulation Study

This section reports results from Monte Carlo experiments to illustrate the performance of the limiting approximations for the case of a unique best response. We focus on simulation designs with discrete types, where the only exogenous covariate $x_i \in \{0, 1\}$ (e.g. "red" nodes vs. "blue" nodes) is a Bernoulli random variable with success probability 0.4. The taste shifters η_{ij} are i.i.d. draws from an extreme-value type-I distribution.

7.1. **Convergence of the Link Frequency Distribution.** We first simulate pairwise stable networks without endogenous interaction effects ("pure homophily" case). Link preferences are given by

$$U_{ij} = \beta_0 + \beta_1 x_i + \beta_2 |x_i - x_j| + \eta_{ij}$$

	Design 1				Design 2			
	$\mathbb{E}[s_i\|x_i = x]$		$\mathbb{E}[I_i\|x_i = x]$		$\mathbb{E}[s_i\|x_i = x]$		$\mathbb{E}[I_i\|x_i = x]$	
n	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$
100	1.971	1.948	2.307	2.300	7.540	6.560	10.557	8.955
	(2.128)	(2.106)	(0.531)	(0.535)	(6.018)	(5.353)	(1.538)	(1.364)
500	2.435	2.439	2.613	2.610	11.082	9.334	13.087	10.965
	(2.740)	(2.726)	(0.413)	(0.412)	(9.945)	(8.560)	(1.334)	(1.175)
1000	2.403	2.395	2.523	2.524	11.530	9.812	13.005	10.920
	(2.745)	(2.767)	(0.344)	(0.344)	(10.725)	(9.332)	(1.153)	(1.022)
5000	2.579	2.577	2.637	2.636	13.294	11.128	14.072	11.721
	(2.989)	(2.993)	(0.241)	(0.241)	(13.088)	(11.069)	(0.854)	(0.753)
10000	2.632	2.637	2.675	2.675	13.836	11.580	14.406	12.008
	(3.055)	(3.061)	(0.206)	(0.206)	(13.815)	(11.661)	(0.738)	(0.651)
DGP	2.660	2.664	2.718	2.718	13.883	11.617	15.012	12.463

TABLE 1. Average degree (left) and average inclusive value (right).

A nonzero coefficient for β_1 allows for the propensity to form links to vary between the two types, whereas β_2 can be interpreted as a complementarity between nodes of the same type. We use two different designs in our simulation experiments which set the preference parameters equal to $(\beta_0, \beta_1, \beta_2) = (0.5, 0, 0)$ and $(1.5, 0, -0.5)\}$, respectively. All simulation results were obtained using 200 Monte Carlo draws.

To illustrate the formal results on convergence of the link frequency distribution, we compare summary statistics of the simulated distribution and their theoretical counterparts from the limiting distribution in Table 1: The first set of columns reports the conditional mean and standard deviation (in parenthesis) of the degree of a node $s_i := \sum_{j \neq i} L_{ij}$ given the covariate $x_i = 0, 1$, and the second set of columns the conditional mean and standard deviation of the inclusive value $I_i := n^{-1/2} \sum_{j \neq i} \mathbb{1}\{U_{ji} \geq MC_j\} \exp\{U^*(x_i, x_j)\}$. The DGP values in Table 1 correspond to the inclusive value function (left) and the expected degree conditional on x_i under the limiting distribution (right).

The first simulation design results in a very sparse network in which nodes have an average degree of around 2.6, whereas for the second design, the degree distribution is centered around 12-14 links per node, which may be more typical for real-world social networks. In the first design, types do not matter for agents' preferences since $\beta_1 = \beta_2 = 0$, so that, at least up to sampling and numerical errors, inclusive values and degree distributions do not differ across types $x_i = 0, 1$. For the second design, nodes with $x_i = 0$ have larger inclusive values and degree distributions than nodes with $x_i = 1$ since the complementarity β_2 is positive and the share of nodes with $x_i = 1$ was set to 0.4. This leaves nodes of the type $x_i = 0$ with a larger number of link opportunities within their own type category than nodes with $x_i = 1$.

The simulation results replicate by and large the theoretical predictions for large networks. In particular, the conditional means of I_i and s_i converge to their asymptotic counterparts, and the cross-sectional variance of I_i decreases, although at a fairly slow rate.[14] Note also that the conditional distribution of s_i given x_i remains non-degenerate in the limit.

7.2. **Parameter Estimation with no Endogenous Interaction Effects.** We next turn to estimation of the preference parameter $\beta := (\beta_0, \beta_1, \beta_2)'$. We estimate β via pseudo-maximum likelihood, using the asymptotic log-likelihood given in Example 5.1. Note also that in the absence of strategic interaction effects $\mathbb{E}[s_i|x_i = x] = H^*(x)$ in the limiting model, so that we can use any consistent nonparametric estimator for $\mathbb{E}[s_i|x_i = x]$ to obtain starting values for $H^*(x)$. We use the same design values for the parameter vector β, and results are for 200 Monte Carlo replications.

One source for small-sample bias in the likelihood results from the use of the inclusive value function $H^*(x)$ in the limiting representation for the distribution of the edge-level response when the node forms more than one link. The derivation for Lemma 3.2 suggests a (partial) bias correction in which we replace $H^*(x_i)$ with $\tilde{I}_i := H^*(x_i) - n^{-1/2} \sum_{j=1}^n L_{ij} \exp\{U^*(x_i, x_j)\}$. Since the degree distribution remains stochastically bounded as n increases, the correction term becomes negligible in a very large network. However our simulation results suggest that such a correction substantially reduces bias for networks of moderate size, especially in the second design for which the average degree is larger than 10.

The simulation results suggest that the estimators indeed converge to the population values of the parameter β, where both bias and standard deviation of the estimator decrease as n grows. However, in contrast to standard nonlinear estimators for i.i.d. samples from a fixed DGP, the bias of the MLE in our simulation results appears not to vanish at a rate faster than its standard deviation - in fact the simulation results are consistent with a root-n rate for both bias and standard error, similar to the findings for the two-sided matching model in Menzel (2015). This behavior is primarily a result of the slower convergence rate of the inclusive value functions.

7.3. **Parameter Estimation with Capacity Constraints.** For another set of simulation results we modify the previous design by adding a capacity constraint, where the degree of each node is not permitted to exceed $\bar{s} = 5$. We also impose the modified stability notion PSN2 introduced in Appendix A.3 rather than pairwise stability. This setup can be interpreted as a model of many-to-many matching where each node can be matched with at most 5 partners.

The constrained MLE maximizes the asymptotic log likelihood given in Example 5.2. Since in this design the degree of any node is capped at $\bar{s} = 5$, we omit the bias correction of

[14]Based on the argument for Lemma 6.3, we conjecture that the rate of convergence is $n^{-1/4}$.

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.442	0.027	0.006	1.116	-0.018	-0.371
	(0.203)	(0.249)	(0.120)	(0.460)	(0.804)	(0.061)
500	0.564	0.002	0.004	1.413	-0.022	-0.432
	(0.077)	(0.099)	(0.046)	(0.229)	(0.444)	(0.022)
1000	0.542	0.004	0.003	1.451	-0.024	-0.450
	(0.053)	(0.071)	(0.030)	(0.177)	(0.364)	(0.016)
5000	0.535	0.001	0.000	1.512	0.003	-0.476
	(0.027)	(0.032)	(0.013)	(0.024)	(0.031)	(0.007)
10000	0.531	-0.002	-0.000	1.521	0.004	-0.483
	(0.016)	(0.022)	(0.009)	(0.016)	(0.022)	(0.004)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

Table 2. Model without capacity constraints - mean and standard deviation (in parentheses) of MLE

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.375	0.033	0.011	1.455	0.088	-0.434
	(0.279)	(0.287)	(0.132)	(0.337)	(0.437)	(0.095)
500	0.526	-0.009	0.004	1.521	0.153	-0.471
	(0.116)	(0.115)	(0.053)	(0.185)	(0.276)	(0.037)
1000	0.491	0.013	0.002	1.517	0.079	-0.477
	(0.086)	(0.091)	(0.036)	(0.140)	(0.221)	(0.029)
5000	0.503	0.002	-0.000	1.517	0.026	-0.491
	(0.038)	(0.038)	(0.015)	(0.066)	(0.098)	(0.013)
10000	0.509	-0.003	-0.001	1.514	0.018	-0.493
	(0.024)	(0.028)	(0.011)	(0.045)	(0.062)	(0.009)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

Table 3. Model with capacity constraints - mean and standard deviation (in parentheses) of MLE

inclusive values used in the first set of results, which produces less precise (higher-variance) estimates for networks of moderate sizes. The starting values for H^* were obtained by solving the fixed-point equations with the preference parameters β held fixed at their respective starting values. The simulation results for the MLE for the preference parameter β are reported in Table 3 and are by and large comparable to those for the baseline model.

7.4. **Endogenous Interactions based on Network Degree.** For the last simulation design, we allow for complementarities in network degree, where nodes with greater degree centrality are regarded as more "attractive" link prospects. Specifically we consider link

n	$\hat\beta_0^{ML}$	$\hat\beta_1^{ML}$	$\hat\beta_2^{ML}$	$\hat\beta_3^{ML}$	$\hat\beta_0^{ML}$	$\hat\beta_1^{ML}$	$\hat\beta_2^{ML}$	$\hat\beta_3^{ML}$
200	0.820	-0.009	-0.387	0.045	0.956	-0.399	0.009	0.079
	(0.357)	(0.173)	(0.040)	(0.039)	(0.405)	(0.124)	(0.024)	(0.041)
500	0.711	-0.008	-0.424	0.071	0.965	-0.426	0.003	0.093
	(0.256)	(0.101)	(0.023)	(0.028)	(0.264)	(0.089)	(0.014)	(0.026)
1000	0.640	0.008	-0.447	0.080	0.973	-0.462	0.001	0.098
	(0.203)	(0.077)	(0.017)	(0.021)	(0.112)	(0.069)	(0.009)	(0.010)
5000	0.528	-0.002	-0.482	0.097	1.042	-0.536	-0.001	0.100
	(0.059)	(0.031)	(0.007)	(0.006)	(0.096)	(0.074)	(0.004)	(0.007)
DGP	0.500	0.000	-0.500	0.100	1.000	-0.500	0.000	0.100

TABLE 4. Model with degree externalities - mean and standard deviation (in parentheses) of MLE

preferences of the form

$$U_{ij} = \beta_0 + \beta_1 x_i + \beta_2 |x_i - x_j| + \beta_3 \min\{10, 2 * \lceil s_j/2 \rceil\} + \eta_{ij}$$

where $s_j := \sum_{k=1}^n L_{jk}$ denotes the network degree of node j, and $\lceil x \rceil$ is the value of $x \in \mathbb{R}$ rounded up to the next integer. This specification groups agents into 6 discrete categories in terms of network degree, partitioning $\mathcal{S}$ into $\{\{0\}, \{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10,\dots\}\}$. This design follows the setup in Example 5.3, where the pairwise stable network is obtained from myopic best-response dynamics starting at the full network graph, $L_{ij} = 1$ for all $i \neq j$, in order to select the largest stable network.

We assume throughout that $\beta_3 \geq 0$ and choose the other design parameters $\beta_0, \beta_1, \beta_2$ in a way that generates a degree distribution with a reasonable amount of variation across these categories. Specifically, we use two different designs in our simulation experiments which set the preference parameters equal to $(\beta_0, \beta_1, \beta_2, \beta_3) = (1, -0.5, 0, 0.1)$ and $(0.5, 0, -0.5, 0.1)\}$, respectively.[15] All simulation results were obtained using 100 Monte Carlo draws.

Simulation results are reported in Table 4. Bias and dispersion of the MLE appear to be of a comparable order of magnitude as the previous cases, where the bias on the constant β_0 is particularly large for smaller networks. Separate maximization over $(\beta_0, \beta_1, \beta_2)$ and β_3, respectively, constraining the remaining parameters to DGP values yield much more accurate partial estimates (not reported here), suggesting that for small n the likelihood may be fairly flat in the direction of some linear combination of the two parameters.

[15]A back of the envelope calculation and simulation evidence suggest that both specifications exhibit cascading adjustments to small local changes to the network and do not meet the "subcriticality" condition of Assumption 6 in Leung (2016).

8. Discussion

This paper develops an asymptotic representation of the link frequency distribution resulting from a network formation game. In this limiting approximation, interdependence of link formation decisions can be split into a "local" component at the level of a given pair of nodes which is characterized through the edge-level response, and a "global" component, which is captured entirely by the inclusive value function H^* and reference distribution M^* which serve as aggregate state variables. The same applies to multiplicity of stable outcomes, where "local" multiplicity is resolved by selecting from a multi-valued edge-level response corresponding to an individual potential link, and "global" multiplicity corresponds to selecting among multiple roots solutions for the equilibrium conditions for the inclusive value function and reference distribution.

Appendix A. General Characterization of the Limiting Model $\mathcal{F}_0^*$

This appendix gives a general characterization of the limiting model $\mathcal{F}_0^*$, allowing for multiplicity in the edge-level response. In the absence of a unique edge-level response, pairwise stability may be consistent with a family of probability distributions each of which is generated by a different random selection from multiple pairwise stable outcomes for a given value of the relevant aggregate states. In this section we first give the most general characterization of that set in the absence of auxiliary assumptions on equilibrium selection. We then show how additional restrictions on the equilibrium selection rule can simplify that representation considerably.

Following the approach in Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011), we characterize the set of distributions generated by non-unique stable outcomes using *capacities* (see Choquet (1954), Molchanov (2005)). We next introduce the main formal concepts, and then describe the capacities and equilibrium conditions that define $\mathcal{F}_0^*$. Several illustrative examples are given in Appendix A.3.

A.1. Choquet Capacities.

For an arbitrary set $\mathcal{S}$, let $2^{\mathcal{S}}$ denotes the set of all subsets of $\mathcal{S}$, and $\Delta\mathcal{S}$ the probability simplex of distributions over elements of $\mathcal{S}$. For the following definition, we say that a sequence of sets $(A_n)_{n\geq 0}$ is *increasing* (with respect to set inclusion) if $A_n \subset A_{n+1}$ for all n, and we say that the sequence is *decreasing* if $A_{n+1} \subset A_n$ for all n.

Definition A.1. ***(Choquet capacity)*** *A mapping $\bar{Q} : 2^{\mathcal{S}} \to [0,1]$ is called a* Choquet capacity *(upper probability) on the set $\mathcal{S}$ if (a) $\bar{Q}(\emptyset) = 0$, $\bar{Q}(\mathcal{S}) = 1$, (b) $\bar{Q}$ is monotone with respect to set inclusion, i.e. $\bar{Q}(S') \leq \bar{Q}(S)$ whenever $S' \subset S \subset \mathcal{S}$, and (c) for any increasing sequence of subsets $(S_n)_{n\geq 0}$ of $\mathcal{S}$, $\lim_n \bar{Q}(S_n) = \bar{Q}\left(\bigcup_{n\geq 0} S_n\right)$, whereas for any decreasing sequence of subsets $(S_n)_{n\geq 0}$, $\lim_n \bar{Q}(S_n) = \bar{Q}\left(\bigcap_{n\geq 0} S_n\right)$.*

The normalization of the values of the capacity in part (a) is not part of the usual (i.e. more general) definition of a Choquet capacity, but is assumed throughout in this paper, so that a capacity can be interpreted as representing a set of proper probability distributions, as we discuss below. The monotonicity property in part (b), and continuity from the right in part (c) generalize the corresponding properties of standard probability distributions. Note that in order to characterize the capacity fully, it is in general not sufficient to state the upper bounds for the elementary events of the form $\{s_i = s\}$, but sharp bounds may be determined in part by composite events of the form $s_i \in S$, for arbitrary subsets $S \subset \mathcal{S}$.

Choquet capacities can be used to represent sets of probability distributions, where that set is referred to as the *core* of the capacity:

Definition A.2. ***(Core)*** *The* core *of the capacity* $\bar{Q}$ *is the set of all probability distributions* $Q(s)$ *over* $\mathcal{S}$ *such that*

$$\int_S Q(s)ds \le \bar{Q}(S) \quad \text{for all subsets } S \subset \mathcal{S}$$

In that event, we also write $Q \in \text{core}(\bar{Q})$.

Hence, the core of the capacity $\bar{Q}$ is a subset of the probability simplex $\Delta\mathcal{S}$. Clearly, the core of $\bar{Q}$ is convex: if Q_1 and Q_2 are in the core, then we also have that for any $\lambda \in [0,1]$ $\int_S (\lambda Q_1(s) + (1-\lambda)Q_2(s))\, ds \le \bar{Q}(S)$ for all $S \subset \mathcal{S}$, so that $\lambda Q_1 + (1-\lambda)Q_2$ is also in the core of $\bar{Q}$.

Moreover, if $\bar{Q}$ describes a set of distributions generated by pairwise stable networks under various selection mechanisms, every distribution in the core can be attained by pairwise stable network as long as the rule for selecting from the edge-level response is unrestricted: Consider any two points in $\bar{Q}$ that are supported by selection rules corresponding to mixture weights α, α'. Then any convex combination of the two distributions can be generated by the mixture $\lambda\alpha + (1-\lambda)\alpha'$ as λ varies on the unit interval. Hence, in the absence of additional constraints on the selection mechanism we can represent the set of reference distributions consistent with pairwise stability using a capacity Ω_0 to convex subsets of the probability simplex $\Delta\mathcal{S}$. We illustrate the construction of the capacity $\bar{Q}$ describing the possible distributions of endogenous network statistics, and the associated fixed-point mapping Ω_0 with several examples in the last subsection of this appendix.

A.2. **Limiting model $\mathcal{F}_0^*$.** We now give a representation of the limiting model $\mathcal{F}_0^*$, which can be characterized in terms of subnetworks on an appropriately defined network neighborhood around a pair of nodes i, j. We start by defining the main components of our representation of $\mathcal{F}_0^*$, most importantly the edge-level response Q^*, the reference distribution M^* and the inclusive value function H^*, where the last two serve as aggregate state variables that summarize the relevant global properties of the network. We then state the equilibrium conditions determining these objects in the limit of pairwise stable networks.

A.2.1. *Main Components.* The **random network neighborhood** $\mathcal{N}_i$ around a node i is the set of nodes l such that i and l are mutually acceptable (i.e. $U_{il}(\mathbf{L}) \ge MC_i(\mathbf{L})$ and $U_{li}(\mathbf{L}) \ge MC_l(\mathbf{L})$) at least for some combination of values for the endogenous network attributes s_i, s_l, t_{il}. The network neighborhood around an edge ij is the union of the random network neighborhoods around the nodes i and j, and will be denoted by $\mathcal{N}_{ij} := \mathcal{N}_i \cup \mathcal{N}_j$.

Since the network attributes s_l, t_{il} of nodes $l \in \mathcal{N}_i$ are determined endogenously in the subnetwork on $\mathcal{N}_l$, we need to solve the model on overlapping subnetworks of a similar form. We parameterize the interdependence between adjacent subnetworks in terms of the collection of network variables L_{km}, s_k, t_{km} of nodes $k \in \mathcal{N}_{ij} \cap \mathcal{N}_l \backslash \{l\}$ and $m \in \mathcal{N}_{ij} \cap \mathcal{N}_l$. We say that the vector $\mathbf{r}_{ijl}$ containing network variables from that list is the **relevant overlap** for $\mathcal{N}_{ij}$ and $\mathcal{N}_l$ if it is a sufficient statistic for the subnetwork on $\mathcal{N}_{ij} \cap \mathcal{N}_l$ with respect to the variables s_l, t_{il}, t_{jl}. For a given network $\mathbf{L}$, we also write $\mathbf{r}_{ijl}(\mathbf{L})$ to denote the values of the network attributes in the relevant overlap.

Example A.1. *If there are only anonymous interaction effects, i.e.* $\mathcal{T} = \{t_0\}$*, then under the distribution of network neighborhoods given below in this section, distinct nodes in* $\mathcal{N}_{ij}$ *are mutually available with probability zero. Hence the relevant overlap of* $\mathcal{N}_{ij}$ *and* $\mathcal{N}_l$ *is given by* $\mathbf{r}_{ijl} = (s_i)$ *if* $l \in \mathcal{N}_j$*, and* $\mathbf{r}_{ijl} = (s_j)$ *if* $l \in \mathcal{N}_j$*.*

Example A.2. *If the only endogenous interaction effect is a preference for transitive triads, i.e.* $T_{ij} = \max_{k\in\mathcal{N}} L_{ik}L_{jk}$*, then the relevant overlap of* $\mathcal{N}_{ij}$ *and* $\mathcal{N}_l$ *is given by* $\mathbf{r}_{ijl} = (\mathbf{L}_{il}, L_{jl})$*.*

Note also that in general the number of relevant entries of $\mathbf{r}_{ijl}$ may vary with the realized structure of the network neighborhoods $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. In the following we will assume that the number of nodes in the relevant overlap is bounded at some integer $d_\cap < \infty$, and w.l.o.g. constant, potentially after introducing a placeholder for attributes that are irrelevant for a given draw of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. We also use the boldface notation $\mathbf{x}_{ijl} := (x_k)_{k\in\mathcal{N}_{ij}\cap\mathcal{N}_l} \in \mathcal{X}^{d_\cap}$ and $\mathbf{t}_{ijl} := (t_{kl})_{k\in\mathcal{N}_{ij}\cap\mathcal{N}_l} \in \mathcal{T}^{d_\cap}$ to denote the exogenous covariates (edge-specific network statistics with respect to the node l, respectively) for the nodes in the intersection of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. We also denote the range of the relevant overlap $\mathbf{r}_{ijl}$ by $\mathcal{R} \subset \{0,1\}^{d_\cap^2}\mathcal{S}^{d_\cap}\mathcal{T}^{d_\cap^2}$. In many cases it is possible to reduce the dimension of $\mathcal{R}$ to a substantially smaller number of components necessary to describe the outcome distribution, as illustrated by the previous examples.

Given the relevant overlap, the **potential values** for the endogenous network statistics are defined as

$$s_l(\mathbf{r}_{ijl}) := S(\tilde{\mathbf{L}}(\mathbf{r}_{ijl}), l) \qquad t_{kl}(\mathbf{r}_{ijl}) := T(\tilde{\mathbf{L}}(\mathbf{r}_{ijl}), k, l)$$

where $\tilde{\mathbf{L}}(\mathbf{r}_{ijl})$ is a network that coincides with L^* everywhere except on $\mathcal{N}_{ij}\cap\mathcal{N}_l$, where the network has been reconfigured to generate the specified values of the network statistics corresponding to the relevant overlap $\mathbf{r}_{ijl}$. The potential outcomes for the case in which the relevant overlap is empty, $\mathbf{r}_{ijl} = \{\}$, correspond to the values of the network statistics evaluated at L^*. While there may be more than one such network $\tilde{\mathbf{L}}$, sufficiency of the relevant overlap for s_l and $\mathbf{t}_{ijl}$ implies that either construction must result in the same potential values.

Note that for a pairwise stable network L^*, the realized links and network variables on any network neighborhood $\mathcal{N}_{ij}$ must coincide with their potential values corresponding to the relevant overlap at $\mathbf{r}_{ijl}(\mathbf{L}^*)$. For a given realization of payoffs, pairwise stability therefore amounts to a simultaneous solution of a sparse system of "structural" potential value equations for the payoff-relevant network variables, where direct interaction effects are restricted to the random network neighborhoods.

The **reference distribution** $M^*(s_l, \mathbf{t}_{ijl}; r_{ijl}|\mathbf{x}_{ijl})$ is the joint distribution of potential values of s_l and t_{kl} with components indexed by the relevant overlap $\mathbf{r}_{ijl}$ in the cross-section of nodes in $\mathcal{N}$. That is, the reference distribution is of the form

$$M^*(\bar{s}_l, \bar{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) := P(s_l(\mathbf{r}_{ijl}) = \bar{s}_l, t_{ijl} = \bar{t}_{ijl}(\mathbf{r}_{ijl})|\mathbf{x}_{ijl})$$

Hence, $M^*(s_l, \mathbf{t}_{ijl}, \mathbf{r}|\mathbf{x}_{ijl})$ becomes a distribution over the network variables s_l and $\mathbf{t}_{ijl}$ which is indexed by conditioning variables x_i, x_j, x_l and state variables $\mathbf{r}_{ijl}$. We also use the notation $M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|x_i, x_j) := \int M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) \prod_{l\neq i,j} w(x_l)dx_l$, and $M^*(s_l, \mathbf{t}_{ijl}|\mathbf{x}_{ijl}) = M^*(s_l, \mathbf{t}_{ijl}; ()|\mathbf{x}_{ijl})$ for the cross-sectional distribution of network outcomes for the pairwise stable network L^*, corresponding to a relevant overlap that is empty.

The reference distribution depends on the selection mechanism which may assign different initial values of L_{ij} to different dyads. In the absence of additional restrictions on equilibrium selection, the distribution of endogenous network characteristics may therefore be different across distinct nodes. $\mathcal{F}_0^*$ is therefore characterized by a set $\mathcal{M}^*$ of possible reference distributions, and the limiting model allows for the reference distribution at each node to correspond to a different element in $\mathcal{M}^*$.

Since the object of interest is the conditional probability of the network variables L_{ij}, s_i, s_j, t_{ij} given x_i, x_j, we can treat edges with the same values of x_i, x_j symmetrically, and integrate out other features of their network neighborhoods. Hence conditional on exogenous attributes $\mathbf{x}_{ijl}$, the reference distribution

$M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl})$ is a joint distribution over values of network attributes at the network configurations corresponding to different values of $\mathbf{r}_{ijl}$. For our analysis it is necessary to condition on x_i, x_j since we aim to obtain the conditional link-frequency distribution given the pair's exogenous attributes, whereas dependence on x_k for $k \in \mathcal{N}_{ij} \cap \mathcal{N}_l \backslash \{i,j\}$ is integrated out when we take probabilities over random network neighborhoods. The reference distribution therefore summarizes the influence of the network structure outside of a network neighborhood $\mathcal{N}_{ij}$ on the endogenous network attributes for nodes in $\mathcal{N}_{ij}$, and for a given draw of potential values we can solve for any pairwise stable configurations on that subset of nodes.

The **inclusive value function** $H^*(x_i, s_i)$ is a nonnegative function of i's attributes x_i and s_i alone. We find that in the limiting distribution, $H^*(x; s)$ serves a sufficient statistic for that agent's link opportunity set with respect to the probability that for a given combination of links the pairwise stability conditions are satisfied by agent i's random payoffs. Taken together, $\mathcal{M}^*$ and H^* serve as aggregate state variables for the network which satisfy the equilibrium (fixed point) conditions (A.4) and (A.5) if and only if they are supported by a pairwise stable network.

These three objects - i.e. the distribution of subnetworks, reference distribution, and inclusive value function - are jointly determined through equilibrium conditions developed in the remainder of this subsection. We start by specifying the distribution of the random network neighborhoods $\mathcal{N}_{ij}$ and $\mathcal{N}_l$, followed by equilibrium conditions characterizing the edge-level response, reference distribution, and inclusive value function under the limiting model $\mathcal{F}_0^*$. We can then put these individual components together to obtain the link frequency distribution associated with $\mathcal{F}_0^*$.

A.2.2. *Distribution of* $\mathcal{N}_{ij}$. In order to characterize the distribution for drawing available nodes $l \in \mathcal{N}_{ij}$, we define

$$p(x_i, x_l, s_l, t_{il}; \mathbf{r}_{ijl}) := s_{1l} \frac{\exp\{U^*(x_l, x_i; s_l, s_i, t_{il})\}}{1 + H^*(x_1; s_1)} M^*(s_l, t_{il}; \mathbf{r}_{ijl}|x_i, x_j, x_l)$$

and

$$\bar{p}(x_i, x_l) := \sup_{s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}} p(x_l, s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}).$$

Since $H^*(x; s) \geq 0$, it follows that $\bar{p}(x_1, x_2) \leq \exp\{\bar{U}\} < \infty$ whenever $\mathbb{E}[s_{1i}|x_i]$ is bounded and Assumption 4.1 holds.

Under $\mathcal{F}_0^*$, a network neighborhood $\mathcal{N}_{ij}$ is generated as follows:

- For either node $k = i, j$, the link opportunity set $\mathcal{N}_k$ is generated by a point process with Poisson intensity

$$\mu(x_l, x_k) = \bar{p}(x_k, x_l) w(x_l)$$

- Payoffs on the subnetwork $\mathcal{N}_{ij} := \{i, j\} \cup \mathcal{N}_i \cup \mathcal{N}_j$ are constructed as

$$U_{kl}(\mathbf{L}) := U^*(x_k, x_l; s_k(\mathbf{L}), s_l(\mathbf{L}), t_{kl}(\mathbf{L})) + \eta_{kl}$$

  for $k = i, j$ and $l \in \mathcal{N}_i \cup \mathcal{N}_j$, where $s_l(\mathbf{L}) := s_l(\mathbf{r}_{ijl}(\mathbf{L}))$, $t_{kl}(\mathbf{L}) \equiv t_{kl}(\mathbf{r}_{ijl}(\mathbf{L}))$, and the taste shifters η_{kl} are i.i.d. draws from an extreme-value distribution of type I.
- For any fixed values of s_l, t_{kl} each node $l \in \mathcal{N}_i \cup \mathcal{N}_j$ is available to another node $k \in \mathcal{N}_0/\{l\}$ with probability $\frac{p(x_k, x_l, s_l, t_{kl}; \mathbf{r}_{ijl}(\mathbf{L}))}{\bar{p}(x_k, x_l)} \in [0, 1]$, where availability is independent across k, l. In the absence of edge-specific interaction effects, i.e. $\mathcal{T} = \{0\}$, that probability is changed to zero for all $k, l \in \mathcal{W}_i^* \cup \mathcal{W}_j^*$.
- Marginal costs are given by $MC_k = \max_{l \in J_k} \eta_{k0,l}$ for $k = i, j$, where J_k is Poisson with intensity $\mu = 1$, and $\eta_{k0,l}$ are i.i.d. extreme-value type I, and the maximum over an empty set is taken to be minus infinity.

While this description of the distribution over network neighborhoods $\mathcal{N}_{ij}$ could in principle be used to simulate from the limiting model, probabilities for availability and link stability from this statistical model can also be obtained in closed form. The probabilities and bounds that constitute the limiting model $\mathcal{F}_0^*$ can then be fully characterized in terms of the random subnetwork on $\mathcal{N}_{ij}$.

A.2.3. *Edge-Level Response.* The **edge-level response** describes link formation for the edge ij, together with the values of the endogenous network variables s_i, s_j, t_{ij}. Formally, we let $Q^*(L_{ij}, s_i, s_j, t_{ij}|x_i, x_j)$ denote the joint distribution of the link indicator L_{ij} and the variables s_i, s_j, t_{ij} in the limiting model, conditional on x_i, x_j.

We say that L_{ij} and s_i, s_j, t_{ij} are supported by the subnetwork on $\mathcal{N}_{ij}$ if there exists a pairwise stable network L_0^* on $\mathcal{N}_0$ given the payoffs defined above. Note that the number of nodes in $\mathcal{N}_{ij}$ is random but finite. Probabilities over events in L_{ij} and s_i, s_j, t_{ij} on this subnetwork are evaluated conditional on the number of Poisson draws in $\mathcal{N}_k$ exceeding the minimum number of edges to k necessary to obtain the value s_k for the network characteristics for $k = i, j$. These probabilities are generally available in closed form given the functions $U^*(\cdot)$ and the inclusive value function $H^*(x, s)$ using results by Dagsvik (1994).

Since even at the level of the edge ij there may be multiple pairwise stable outcomes regarding s_i, s_j, t_{ij} and L_{ij}, the model admits a set of edge-level responses which will be described in terms of upper bounds on probabilities for events in these variables: For any sets $\mathcal{L} \subset \{0, 1\}$, $S_1, S_2 \subset \mathcal{S}$ and $T_{12} \subset \mathcal{T}$, we can obtain the bound

$$\begin{aligned}\bar{Q}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) \quad = \quad & P\Big(L_{ij}^* \in \mathcal{L}, S(\mathbf{L}^*, i) \in S_1, S(\mathbf{L}^*, j) \in S_2, T(\mathbf{L}^*, i, j) \in T_{12} \\ & \text{for some pairwise stable network } L_0^* \text{ on } \mathcal{N}_{ij} \Big| x_i = x_1, x_j = x_2 \Big)\end{aligned}$$

In words, the upper bound $\bar{Q}^*(\cdot|x_1, x_2)$ is the conditional probability that the link outcome $L_{ij} = l$ and some values $s \in S_i, s' \in S_j$ and $t \in T_{ij}$ are supported by some pairwise stable subnetwork on $\mathcal{N}_{ij}$.

Using the terminology introduced in section A.1, we can interpret the bound $\bar{Q}^*$ as a capacity characterizing the family of edge-level responses, where any edge-level response $Q^*(l, s_1, s_2, t_{12}|x_1, x_2)$ has to satisfy the constraints

$$\int_{\mathcal{L}} \int_{S_1} \int_{S_2} \int_{T_{12}} Q^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2) dt_{12} ds_2 ds_1 dl_{12} \le \bar{Q}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) \tag{A.1}$$

for any sets $\mathcal{L}, S_1, S_2, T_{12}$. In analogy to the approaches in Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011) for static discrete games, the set of edge-level responses can be formally characterized as the core of a capacity.

To obtain the edge-level response from the distribution of subnetworks on $\mathcal{N}_{ij}$ described above, we can first consider the probability that an "elementary" outcome corresponding to specific values of L_{kl}, s_l, t_{kl} and $\mathbf{r}_{ijl}$ is supported by the subnetwork for $k, l \in \mathcal{N}_{ij}$. For a given configuration $L_{\mathcal{N}_{ij}} := (\mathbf{L}_{kl})_{k,l \in \mathcal{N}_{ij}}$ of the subnetwork on $\mathcal{N}_{ij}$, the probability for that event is given by

$$q(\mathbf{L}_{\mathcal{N}_{ij}}, \mathcal{N}_{ij}) := \prod_{k,l \in \mathcal{N}_{ij}} M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}(\mathbf{L}_{\mathcal{N}_{ij}})|\mathbf{x}_{ijl})(q_{kl}q_{lk})^{L_{kl}}(1 - q_{kl}q_{lk})^{1-L_{kl}}$$

and zero otherwise, where $q_{kl} = p(x_k, x_l; s_k, s_l)/\bar{p}(x_k, x_l)$ if $k \notin \{i, j\}$ and $q_{kl} = p(x_k, x_l; s_k, s_l)$ if $k \in \{i, j\}$. Sharp bounds on the edge-level response $Q^*(1, s_1, s_2, t_{12}|x_1, x_2)$ can then be obtained by aggregating probabilities over all "elementary" outcomes corresponding to a given event in the network variables L_{ij}, s_i, s_j, t_{ij}.

Specifically, the upper bound $\bar{H}^*$ is given by

$$\bar{Q}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) := \mathbb{E}\left[\sum_{L_{\mathcal{N}_{ij}}} q(\mathbf{L}_{\mathcal{N}_{ij}}, \mathcal{N}_{ij})\mathbb{1}\{s_1(\mathbf{L}_{\mathcal{N}_{ij}}) \in S_1, s_2(\mathbf{L}_{\mathcal{N}_{ij}}) \in S_2, t_{12}(\mathbf{L}_{\mathcal{N}_{ij}}) \in T_{12}\}\right| x_1, x_2\Bigg] \tag{A.2}$$

where the expectation is taken with respect to the distribution of $\mathcal{N}_{ij}$.

A.2.4. *Fixed-Point Condition for the Reference Distribution.* An upper bound on the reference distribution $M^*(s_l, t_{kl}; \cdot|\cdot)$ is given by the probability that a given distribution of potential outcomes for node l is supported by some pairwise stable subnetwork on the network neighborhood $\mathcal{N}_l$. To compute this bound, we can draw a network neighborhood $\mathcal{N}_l$ for the node l with covariates $x_l = x_3$ as described in the previous steps, where we fix the covariates of the first two nodes at x_1, x_2. Note that, since nodes in the network neighborhood are realizations of a Poisson process, this gives the conditional distribution of $\mathcal{N}_l$ given the respective values of exogenous attributes for the first two nodes.

Holding the relevant overlap between the nodes fixed at $\mathbf{r}_{123}$, a subnetwork $L_{\mathcal{N}_l} := (\mathbf{L}_{ij})_{i,j\in\mathcal{N}_l}$ is supported by a pairwise stable network on $\mathcal{N}_l$ with probability

$$q(\mathbf{L}_{\mathcal{N}_l}, \mathcal{N}_l) = \prod_{i,j\in\mathcal{N}_l} M^*(s_i, s_j, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}(\mathbf{L}_{\mathcal{N}_l})|\mathbf{x}_{ijl})(q_{ij}q_{ji})^{L_{ij}}(1 - q_{ij}q_{ji})^{1-L_{ij}}$$

where $M^*(s_i, s_j, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) = M^*(s_i, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl})M^*(s_j; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}, \mathbf{t}_{ijl})$ denotes the joint distribution of potential outcomes for $s_i, s_j, \mathbf{t}_{ijl}$ given $\mathbf{x}_{ijl}$ implied by the reference distribution, and q_{ij} is defined in the analogous way as for the description of the edge-level response. Hence for any event $S_3 \subset \mathcal{S}$ and a given reference distribution M^*, the probability that a value $s_l \in S_3$ for the endogenous network attributes is supported on $\mathcal{N}_l$ after holding the overlap fixed at $\mathbf{r}_{123}$ is obtained after summing $q(\mathbf{L}_{\mathcal{N}_l}, \mathcal{N}_l)$ over all configurations of $L_{\mathcal{N}_l}$ that result in $s_l \in S_3$.

We can then define the capacity

$$\Omega_0[H, M](\mathbf{x}_{123}; \mathbf{r}_{123}, S, T) := \mathbb{E}\left[\sum_{L_{\mathcal{N}_l}} q(\mathbf{L}_{\mathcal{N}_l}, \mathcal{N}_l)\mathbb{1}\{s_l(\mathbf{L}_{\mathcal{N}_l}) \in S, \mathbf{t}_{12l}(\mathbf{L}_{\mathcal{N}_l}) \in T, \mathbf{r}(\mathbf{L}_{\mathcal{N}_l}) = \mathbf{r}_{123}\}\right| \mathbf{x}_{123}\Bigg] \tag{A.3}$$

where dependence on M, H was implicit in the definition of the probabilities $q(\mathbf{L}_{\mathcal{N}_l}, \mathcal{N}_l)$, and the expectation is taken with respect to the distribution of $\mathcal{N}_l$. Note that the exact form of Ω_0 depends on the functions $S(\cdot)$ and $T(\cdot)$ in the construction of the network characteristics. We derive the edge-level response and resulting fixed point mappings for a few special cases in the next subsection below.

The model is then closed by the equilibrium condition that the reference distribution $M^*(s_l, \mathbf{t}_{12l}; \mathbf{r}_{12l}|\mathbf{x}_{12l})$ has to be generated by some mixture over edge-level responses in the cross-section. Specifically, for any given value of H, the set $\mathcal{M}^*$ is the largest set of distributions satisfying

$$\mathcal{M}^* = \text{conv}\left(\bigcup_{M\in\mathcal{M}^*} \text{core } \Omega_0[H, M]\right) \tag{A.4}$$

where conv(A) denotes the convex hull of a set A. Note that the set $\mathcal{M}^*$ in A.4 is nonempty whenever the mapping $\Omega_0[\cdot, M]$ has a fixed point with respect to M.

To illustrate the general structure of the set $\mathcal{M}^*(H)$, consider the case of link preferences with strategic complementarities:

Example A.3. ***(Strategic Complements)*** *Suppose that the function $U^*(x_1, x_2; s_1, s_2, t_{12})$ is nondecreasing in each component of s_1, s_2, t_{12} for each value of x_1, x_2, and that the network statistics $S(\mathbf{L}, \cdot)$ and $T(\mathbf{L}, \cdot)$*

are nondecreasing in $\mathbf{L}$ *with respect to the partial ordering:* $L \geq L'$ *iff* $L_{ij} \geq L'_{ij}$ *for each* $i, j = 1, \ldots, n$*. If* $\mathcal{S}, \mathcal{T}$ *are finite sets, it can then be shown using standard arguments (see Milgrom and Roberts (1990)) that* $\Omega_0[H, M]$ *is (a) nondecreasing in* M*, where (b) its image is a lattice, and (c) its (nonempty) set of fixed points forming a lattice on the distributions over* $\mathcal{S} \times \mathcal{S} \times \mathcal{T}$*.*

For the smallest fixed point $\underline{M}^*$*, we also have that* $\underline{M}^* \leq \Omega_0[H, \underline{M}^*]$*: if there was a point* $\tilde{M} \in \Omega_0[H, \underline{M}^*]$ *such that* $\tilde{M} \leq \underline{M}^*$ *with strict inequality in at least one component, then by monotonicity and continuity of* Ω_0 *there would exist another fixed point that is strictly smaller than* $\underline{M}^*$*. Similarly, for the largest fixed point* $\overline{M}^*$ *we have* $\overline{M}^* \geq \Omega_0[H, \overline{M}^*]$*.*

Hence, $\mathcal{M}^* \subset \{M : \underline{M}^* \leq M \leq \overline{M}^*\}$*. Clearly,* $\mathcal{M}^*$ *also contains the convex hull of the set of fixed points of* $\Omega_0[H, M]$*.*

We can also use auxiliary assumptions on the equilibrium selection mechanism to narrow the set $\mathcal{M}^*$ of reference distributions. In particular, if equilibrium selection exhibits independence across dyads (a condition similar to the "no coordination" Assumption 5 in Leung (2016)), then the set $\mathcal{M}^*(H)$ is a singleton:

Example A.4. ***(Independent Selection)*** *Suppose that equilibrium selection is independent across dyads in that the links in the initial condition* $L_{ij}^{(0)}$ *are independent draws from a common distribution with conditional probability mass function* $p(\mathbf{L}_{ij}^{(0)}|x_i, x_j)$*. Then the resulting link frequency distribution can be characterized by a singleton reference distribution* $\mathcal{M}^* = \{M^*\}$ *which satisfies the fixed point condition*

$$M^* = \Omega_0[H^*, M^*]$$

As discussed in Section 3 before, in the case of a single reference distribution, $M^*(s_1|x_1, x_2)$ can be estimated directly from the observed sample as the conditional distribution of s_1 given that node 1 is directly linked to a node 2 with attributes x_2. This obviates the need of explicitly characterizing the fixed-point operator Ω_0 and solving for the - possibly non-unique - fixed points of the mapping for estimation of the payoff functions $U^*(\cdot)$.

A.2.5. *Fixed-Point Condition for the Inclusive Value Function.* For a given selection rule, we let

$$\tilde{M}^*(s|x) := \frac{1}{n} \sum_{l=1}^{n} M_l^*(s|x)$$

be the average reference distribution across nodes $i = 1, 2, \ldots,$ where $M_l^* \in \mathcal{M}^*$ denotes the reference distribution selected for node l. Note that the set $\mathcal{M}^*$ is convex for any value of H so that the mixture $\tilde{M}^* \in \mathcal{M}^*$. Under $\mathcal{F}_0^*$, the inclusive value function then satisfies the fixed-point condition

$$H^*(x; s) = \Psi_0[H^*, \tilde{M}^*](x; s) \tag{A.5}$$

for all values of x, s, where the mapping

$$\Psi_0[H, M](x; s) := \int \frac{s_{12} \exp\{U^*(x, x_2; s, s_2, t_0) + U^*(x_2, x; s_2, s, t_0)\}}{1 + H(x_2; s_2)} M(s_2|x_2) w(x_2) ds_2 dx_2$$

Note that according to the notational convention introduced earlier, the first component of s_2, $s_{12} := \sum_{j \neq 2} L_{j2}$ denotes the network degree of node 2. Using the notation introduced before, the cross-sectional distribution of network outcomes $\tilde{M}^*(s_2|x_2)$ denotes the reference distribution corresponding to an empty relevant overlap.

Hence, we can summarize the equilibrium conditions on the aggregate state variables $H^*, \mathcal{M}^*$ as follows:

$$\begin{aligned} \mathcal{M}^* &= \operatorname{conv}\left(\bigcup_{M\in\mathcal{M}^*} \text{core } \Omega_0[H,M]\right) \\ H^* &= \Psi_0[H^*,\tilde{M}^*] \quad \text{for some } \tilde{M}^* \in \mathcal{M}^* \end{aligned} \tag{A.6}$$

A.2.6. *Link Frequency Distribution.* Our characterization of the set of limiting distributions $\mathcal{F}_0^*$ consists exclusively of these three components. The p.d.f. associated with $F_0^* \in \mathcal{F}_0^*$ is of the form

$$f_0^*(x_1,x_2;s_1,s_2,t_{12}) = Q^*(1,s_1,s_2,t_{12}|x_1,x_2)w(x_1)w(x_2), \tag{A.7}$$

where the edge-level response $Q^*(\cdot|x_1,x_2)$ satisfies (A.1). Most importantly, the probability that a given link $\{ij\}$ is established depends on the structure of the larger network only through H^* and M^* in addition to "local" characteristics of the two nodes i and j. This general representation simplifies considerably for certain special cases of practical interest. We show how to derive the capacities $\bar{Q}(\cdot)$ and $\Omega_0(\cdot)$ for some special cases in Appendix A.3.

To understand how this limiting approximation simplifies the description of the network formation model, notice that verifying the pairwise stability conditions in the small subnetwork on $\mathcal{N}_0$ in the construction of the edge-level response is completely analogous to that of static game-theoretic models analyzed in Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011), and therefore amenable to the techniques developed in these two papers. The resulting bounds are asymptotically sharp as the network grows large.

A.3. **Characterization of $\mathcal{F}_0^*$ for Special Cases.** We now illustrate how to use this characterization to derive the limiting distribution for four cases with non-trivial endogenous interaction effects.

Unique Edge-Level Response. First, we briefly show how the general model nests the case of a unique edge-level response with only anonymous interaction effects with radius of interaction equal to $r_S = 1$, which was discussed in the main text. In the absence of edge-specific endogenous interaction effects, $\mathcal{N}_i \cap \mathcal{N}_l = \{i,l\}$, and since the radius of interaction is equal to 1, the relevant overlap reduces to $\mathbf{r}_{ijl} = L_{il}$ if $l \in \mathcal{N}_i$, or $\mathbf{r}_{ijl} = L_{jl}$ if $l \in \mathcal{N}_j$. Furthermore, availability of l to i only depends on the potential outcome of s_l for $L_{il} = 1$ by inspection, so that the other potential outcome is irrelevant for the construction of the link frequency distribution or the fixed point mapping Ω_0. Hence we can suppress dependence of the link frequency distribution on $\mathbf{r}_{ijl}$ and x_i and let $M^*(s_l|x_l) = M^*(s_l;1|x_l)$ denote the conditional distribution of the potential value of s_l in the presence of a direct link to i or j. Finally, uniqueness of the edge-level response implies that the fixed point mapping Ω_0 is also singleton-valued, so that the description of the limiting model for this special case in Section 3 indeed derives from the more general formulation presented in this appendix. We can summarize this finding in the following proposition:

Proposition A.1. *In the case of a unique edge-level response, the limiting model $\mathcal{F}_0^*$ is characterized by the equations (3.1)-(3.3).*

Many to Many Matching with Capacity Constraints. The network formation problem considered in this paper can be viewed as an extension of certain matching models, where we interpret a direct link between two nodes as a match between the corresponding agents. This includes marriage markets, the stable roommate problems, and the college admissions problem. One important "strategic" feature of matching models consists in capacity constraints capping the number of matching partners at some maximum degree $\bar{s}$, which could in principle be allowed to vary across individuals. Specifically, let s_i denote node i's network degree and suppose that payoffs are $U_{ij} = U^*_{ij} + \sigma\eta_{ij}$, where

$$U^*_{ij} = \begin{cases} U^*(x_i, x_j) & \text{if } s_i < \bar{s} \\ -\infty & \text{if } s_i \geq \bar{s} \end{cases}$$

In typical applications, agents may also have different "genders" (e.g. schools vs. students, firms vs. employees, etc.) where matches take place only between agents of different genders, but not within the same group. This would require some minor and straightforward adjustments to our framework, but for greater clarity we do not analyze that case explicitly in this paper. In general, additional restrictions on the set of matching opportunities will simply remove some of the payoff inequalities from the derivation of the analog to the conditional choice probability in (6.1).

For this type of problem, it is important to notice that the notion of pairwise stability in matching models (see Gale and Shapley (1962) and Roth and Sotomayor (1990)) allows for richer deviations from a status quo than PSN, the stability concept for networks. Specifically, a proposed matching is blocked by a pair if at least one agent would prefer to reject her current match (i.e. break a current link) in favor of another available matching partner (i.e. simultaneously form a link to a new available node). We can define PSN2 as stability of a network with regard to these slightly richer deviations:

Definition A.3. ***(Pairwise Stability, PSN2)*** *The undirected network* $\mathbf{L}$ *is a* ***pairwise stable network according to PSN2*** *if for any link* ij *with* $L_{ij} = 1$,

$$U_{ij}(\mathbf{L}) \geq \max\{MC_i(\mathbf{L}), U_{ik}(\mathbf{L} - \{ij\})\}, \quad \text{and } U_{ji}(\mathbf{L}) \geq \max\{MC_j(\mathbf{L}), U_{jl}(\mathbf{L} - \{ij\})\}$$

and for any link ij *with* $L_{ij} = 0$,

$$U_{ij}(\mathbf{L}) < \min\{MC_i(\mathbf{L}), U_{ik}(\mathbf{L} - \{ij\})\},, \quad \text{or } U_{ji}(\mathbf{L}) < \min\{MC_j(\mathbf{L}), U_{jl}(\mathbf{L} - \{ij\})\}$$

for any k *such that* $U_{ki}(\mathbf{L}) \geq MC_k(\mathbf{L})$ *and* l *such that* $U_{lj}(\mathbf{L}) \geq MC_l(\mathbf{L})$.

Note that for simplicity we formulate the stability conditions only in terms of marginal utilities, in analogy with the characterization of pairwise stability in Lemma 2.1. The added requirement stipulates that at the margin, each agent selects the "best" link opportunity over alternatives with lower marginal utility, thereby removing one major source of multiplicity in the edge-level response. In particular for the case of matching subject to a capacity constraint, the edge-level response under PSN2 is unique, so that we can use the same simplified notation as in the main text.

To characterize the edge-level response, player i accepts the links to $j_1, \dots, j_r$ and rejects links to $j_{r+1}, \dots, j_n$ if $U_{ij_1}, \dots, U_{ij_r} \geq MC_i > U_{ij_{r+1}}, \dots, U_{ij_n}$ when $r < \bar{s}$, and $U_{ij_1}, \dots, U_{ij_r} \geq MC_i, U_{ij_{r+1}}, \dots, U_{ij_n}$ when $r = \bar{s}$. In particular the conclusion of Lemma 6.2 holds for the corresponding probabilities. The remaining steps of the formal argument go trough without any modifications, so that we obtain the p.d.f.

$$f(x_1, x_2; s_1, s_2) = \frac{\min\{s_{11}, \bar{s}\} \min\{s_{12}, \bar{s}\} \exp\{U^*(x_1, x_2) + U^*(x_2, x_1)\} M^*(s_1|x_1) M^*(s_2|x_2) w(x_1) w(x_2)}{(1 + H^*(x_1))(1 + H^*(x_2))}$$

for the limiting link frequency distribution. The inclusive value functions $H(x)$ satisfy the fixed-point condition

$$\Psi_0[H, M](x) := \int_{\mathcal{X} \times \mathcal{S}} \min\{s, \bar{s}\} \frac{\exp\{U^*(x, x_2) + U^*(x_2, x)\}}{1 + H(x_2)} M(s|x_2) ds dx_2$$

As a minor modification relative to the case of no interaction effects, the degree distribution $M^*(s|X)$ is given by

$$M^*(s|x) = \begin{cases} \frac{H(x_1)^{s_1}}{(1+H(x_1))^{s_1+1}} & \text{for } s = 0, \dots, \bar{s} - 1 \\ \left(\frac{H(x_1)}{1+H(x_1)}\right)^{s_1+1} & \text{for } s_1 = \bar{s} \\ 0 & \text{otherwise} \end{cases}$$

Since $s_{1i} \leq \bar{s}$ with probability 1, it follows from Proposition 6.1 that the fixed point mapping for the inclusive value function $H^*(x)$ is a contraction, so that the resulting matching distribution is again unique.

Anonymous Interactions: Degree Centrality. In order to illustrate the role of the equilibrium condition (A.4), we show how to derive the edge-level response and reference distribution for the case of preferences over the degree (i.e. the number of direct links) of an agent. The degree of node i is defined as the network statistic

$$s_i = S(\mathbf{L}; x_i, i) := \sum_{j \neq i} L_{ij}$$

In terms of the latent random utility model, $S_i = s$ corresponds to the event that MC_i is the $(s+1)$st highest order statistic of the sample $\{MC_i\} \cup \{U_{ij}\}_{j \in W_i(\mathbf{L}^*)}$. Given the scalar network characteristics S_i, S_j we can consider a version of the reference model (2.3) with payoffs

$$U_{ij} \equiv U^*(x_i, x_j; s_i, s_j) + \sigma \eta_{ij}$$

To simplify the exposition, we also assume that $U^*(x_1, x_2; s_1, s_2)$ is nondecreasing in s_1, s_2. For other signs of the interaction effect, the derivations are completely analogous.

Now consider $l \in \mathcal{N}_i$. Since there are no edge-specific interaction effects in this specification, $l \notin \mathcal{N}_j$ with probability 1. Also, s_i clearly doesn't affect s_l, holding L_{il} fixed. Hence the relevant overlap between the network neighborhoods can be parameterized via $\mathbf{r}_{ijl} = L_{il}$. Furthermore, the network degree of a node l only affects the probability of link formation of a node i directly if $L_{il} = 1$, so that the potential outcome for s_l under $L_{il} = 0$ is irrelevant for the edge-level response and degree distribution. Hence it is sufficient to explicitly model the reference distribution for the potential outcome of s_l corresponding to the subnetwork state $L_{il} = 1$.

The edge-level response and the fixed-point mapping Ω_0 derive from the probabilities of elementary events in L_{ij}, s_i, s_j which can in turn be calculated from the limiting model. Specifically, we consider subsets $\tilde{S}_1 = \{s_{11}, \dots, s_{1r_1}\} \subset S_1$ and $\tilde{S}_2 = \{s_{21}, \dots, s_{2r_2}\} \subset S_2$, where we assume w.l.o.g. that $s_{11} \leq s_{12} \cdots \leq s_{1r_1}$ and $s_{21} \leq s_{22} \cdots \leq s_{2r_2}$. We then let $\bar{q}(\mathbf{L}_{12}, \tilde{S}_1, \tilde{S}_2 | x_1, x_2)$ denote the probability that any possible combination of values $s_1 \in \tilde{S}_1, s_2 \in \tilde{S}_2$ is supported by a pairwise stable network together with a direct link $L^*_{ij} = 1$.

From Lemma 6.2 and elementary calculations we then obtain

$$\begin{aligned} \bar{q}(\mathbf{L}_{12}, \tilde{S}_1, \tilde{S}_2 | x_1, x_2) \quad = \quad & \frac{s_{11} s_{21} \exp\{U^*(x_i, x_j; s_{11}, s_{21}) + U^*(x_j, x_i; s_{21}, s_{11})\}}{(1 + H(x_i; s_{1r_1}))(1 + H(x_j; s_{2r_2}))} \\ & \times \frac{H(x_i; s_{11})^{s_{11}-1} H(x_j; s_{21})^{s_{21}-1}}{(1 + H(x_i; s_{11}))^{s_{11}} (1 + H(x_j; s_{21}))^{s_{21}}} \\ & \times \left(\prod_{k=1}^{r_1 - 1} \left(\frac{H(x_i; s_{1(k+1)})}{1 + H(x_i; s_{1(k+1)})} - \frac{H(x_1; s_{1k})}{1 + H(x_i; s_{1k})} \right)^{s_{1(k+1)} - s_{1k}} \right) \\ & \times \left(\prod_{k=1}^{r_2 - 1} \left(\frac{H(x_j; s_{2(k+1)})}{1 + H(x_j; s_{2(k+1)})} - \frac{H(x_j; s_{2k})}{1 + H(x_j; s_{2k})} \right)^{s_{2(k+1)} - s_{2k}} \right) \end{aligned}$$

Similarly, the capacity $\bar{q}(\tilde{S}_1, \tilde{S}_2|x_1, x_2)$ for the event $s_1 \in S_1, s_2 \in S_2$ is given by

$$\begin{aligned}\bar{q}(\tilde{S}_1, \tilde{S}_2|x_i, x_j) &= \frac{H(x_i; s_{11})^{s_{11}} H(x_j; s_{21})^{s_{21}}}{(1+H(x_i; s_{11}))^{s_{11}}(1+H(x_j; s_{21}))^{s_{21}}} \\ &\quad \times \left(\prod_{k=1}^{r_1-1} \left(\frac{H(x_i; s_{1(k+1)})}{1+H(x_i; s_{1(k+1)})} - \frac{H(x_1; s_{1k})}{1+H(x_i; s_{1k})}\right)^{s_{1(k+1)}-s_{1k}}\right) \\ &\quad \times \left(\prod_{k=1}^{r_2-1} \left(\frac{H(x_j; s_{2(k+1)})}{1+H(x_j; s_{2(k+1)})} - \frac{H(x_j; s_{2k})}{1+H(x_j; s_{2k})}\right)^{s_{2(k+1)}-s_{2k}}\right)\end{aligned}$$

Hence, given values for $H(x; s)$, we can characterize the capacities $\bar{Q}^*$ and Ω_0 in closed form using the formulae in (A.2) and (A.3), which is in turn sufficient to describe the core of link distributions generated by pairwise stable networks. The same principle can be applied to other network attributes of node i. For example, "deeper" network characteristics that depend on a wider network neighborhood of a given node can be characterized recursively in this manner by defining $S(\mathbf{L}; x, i)$ as a function of network characteristics of i's neighbors.

Appendix B. Proofs

B.1. **Proof of Lemma 2.1.** To verify that the statement in Lemma 2.1 is indeed equivalent to the usual definition of pairwise stability, notice that if L^* is not pairwise stable, there exists two nodes i, j with $L^*_{ij} = 0$ such that $U_{ij}(\mathbf{L}^*) > MC_{ij}(\mathbf{L}^*)$ and $U_{ji}(\mathbf{L}^*) > MC_{ji}(\mathbf{L}^*)$. In particular, j is available to i under L^*, i.e. $j \in W_i(\mathbf{L}^*)$, violating (2.4). Conversely, if (2.4) does not hold for node i, then there exists $j \in N_i[L^*]$ such that $U_{ij}(\mathbf{L}^*) \geq MC_{ij}(\mathbf{L}^*)$. On the other hand, $j \in W_i(\mathbf{L}^*)$ implies that $U_{ji}(\mathbf{L}^*) \geq MC_{ji}(\mathbf{L}^*)$, where all inequalities are strict in the absence of ties. □

B.2. **Proof of Proposition 4.1.** Note first that $(\mathbb{E}\left[\exp\left\{2|U(x, x', s, s', T(\mathbf{L}^*_n, x, x', i, j)) - U(x, x', s, s', t_0)|\right\}\right])^{1/2} = P(T_{ij} = 1)^{1/2} \exp\{\beta_T\}$. Now consider the probability that i and j have a common neighbor, k. By the law of total probability, we can write

$$\begin{aligned} P(\mathbf{L}_{ik} &= L_{jk} = 1) = P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0) + P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1) \\ &\quad + P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = 0, T_{jk} = 1) + P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = 1, T_{jk} = 0) \end{aligned} \tag{B.1}$$

where

$$P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1) \leq P(\mathbf{L}_{ik} = L_{jk} = 1, L_{ij} = 1) + P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1, L_{ij} = 0)$$

It is easy to verify that under the rates assumed in the claim of this proposition, the leading terms for the right-hand side expression in (B.1) are $P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0)$ and $P(\mathbf{L}_{ik} = L_{jk} = 1, L_{ij} = 1)$, so that for n large enough we can bound

$$\begin{aligned} P(T_{ij} = 1) &\leq nP(\mathbf{L}_{ik} = L_{jk} = 1) \leq 2n(P(\mathbf{L}_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0) + P(\mathbf{L}_{ik} = L_{jk} = 1, L_{ij} = 1)) \\ &\leq 2\left(\frac{\exp\{4\bar{U}\}}{n} + \frac{\exp\{6(\bar{U} + \beta_T)\}}{n^2}\right) \end{aligned}$$

where the last line follows from the same steps as in the proof of Lemma 6.2. Hence, $P(T_{ij} = 1)\exp\{2\beta_T\} = O(1)$ if $\exp\{|\beta_T|\} = O(n^{1/4})$ □

B.3. **Proof of Theorem 4.1.** Let $\mathbf{w}_{ijl} := (s'_l, \mathbf{t}_{ijl})$ and $\mathbf{r}_{ijl}, \mathbf{x}_{ijl}$ denote the state variables for the relevant overlap, where in the following we omit the ijl subscript for notational convenience.

Note first that the conditions of Proposition 6.1 ensure that $\Psi_0[H, M]$ is a continuous, single-valued compact mapping. Next, notice that for any two distributions $M_1(\mathbf{w}; \mathbf{r}|\mathbf{x}), M_2(\mathbf{w}; \mathbf{r}|\mathbf{x})$ satisfying $\int_S M_j(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} \leq \Omega_0(\mathbf{x}; \mathbf{r}, W)$ for all core-determining sets $W \subset \mathcal{S} \times \mathcal{T}^{d_\cap}$, the convex combination $\lambda M_1 + (1-\lambda) M_2$ satisfies the same inequality constraints. Hence, the core of Ω_0 is a convex subset of the probability simplex. Furthermore, if M_3 is in the complement of the core, there exists at least one set $S \in \mathcal{S}^\circ$ such that $\int_S M_3(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} > \Omega_0(\mathbf{x}; \mathbf{r}, W) + \varepsilon$, where $\varepsilon > 0$. Then for any distribution M' with $\|M' - M_3\|_\infty \leq \varepsilon/2$, we have $\int_S M'(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} ds > \Omega_0(\mathbf{x}; \mathbf{r}, W) + \varepsilon/2$. Hence the complement of the core is open, implying that the core is also a closed subset of the relevant probability simplex with respect to the L_∞-norm. Hence, given the conditions on Ω_0 in Assumption 4.5 (i)-(ii), existence of a fixed point is a direct consequence of the Kakutani-Fan fixed point theorem for Banach spaces (Theorem 3.2.3 in Aubin and Frankowska (1990)) $\square$

B.4. **Auxiliary results for the Proof of Theorem 4.2.** We next prove the Lemmas from section 4 which are then used to establish the conclusion of Theorem 4.2.

B.5. **Proof of Lemma 6.1.** We start by introducing some notation: fix an initial state of a network with n nodes, corresponding to an arbitrarily chosen adjacency matrix $L^{(0)}$. We then let $z_{ij} := (x_i, x_j, \mathbf{d}_{ji}, \mathbf{s}_{ji}, \mathbf{t}_{ji})$ be the potential values for the payoff-relevant network statistics resulting from $L^{(0)}$, as defined in Appendix A. Also, let $(\eta_{i0j})_{j=1}^J$ and $(\eta_{ij})_{i,j=1}^n$ be arrays of i.i.d. draws from the unconditional distribution of preference shocks. In the following we denote the last J taste shocks $\eta_{i01}, \ldots, \eta_{i0J}$, corresponding to the marginal cost of establishing a link $MC_i = \sigma \max_{j=1,\ldots,J} \eta_{i0j}$, with $\eta_{in+1}, \ldots, \eta_{in+J}$ for convenience. Similarly, let z_{ij}^* denote the vector of potential values for endogenous and exogenous attributes for the dyad (i,j) under the pairwise stable network L^*, resulting from tâtonnment starting at $L^{(0)}$, as described in the main text. We also let $g_n^*(\cdot)$ denote the p.d.f. for the joint distribution of

$$\boldsymbol{\eta}_i := \left(\eta_{i1}, \ldots, \eta_{i(i-1)}, \eta_{i(i+1)}, \ldots, \eta_{in+J}\right)$$

with $\mathbf{z}_i^* := \left(z_{i1}^*, \ldots, z_{i(i-1)}^*, z_{i(i+1)}^*, \ldots, z_{i(n+J)}^*\right)$. Similarly, we denote $\mathbf{z}_i := \left(z_{i1}, \ldots, z_{i(i-1)}, z_{i(i+1)}, \ldots, z_{i(n+J)}\right)$, where we set $z_{(n+s)i}^*$ and $z_{(n+s)i}$ to some arbitrary default value for $s = 1, \ldots, J$.

Before we prove the Lemma, we have the following intermediate result:

Lemma B.1. ***(Sufficiency)*** *Let*

$$\hat{G}_{in}^{z,\eta}(z, \eta) := \frac{1}{J+n} \sum_{j=1}^{J+n} \mathbb{1}\{z_{ij} \leq z, \eta_{ij} \leq \eta\}$$

denote the empirical distribution of components of $\mathbf{z}_i, \boldsymbol{\eta}_i$. *Then*

$$(\mathbf{z}_i^* | \mathbf{z}_i, \boldsymbol{\eta}_i) \stackrel{d}{=} \left(\mathbf{z}_i^* | \mathbf{z}_i, \hat{G}_{in}^{z,\eta}\right)$$

B.6. **Proof of Lemma B.1.** Recall also that for any initial condition, the conditional distribution of $\boldsymbol{\eta}_i$ given $\mathbf{z}_i$ and $\mathbf{z}_i^*$ derives from the joint distribution of these variables implied by the network formation model. That is for a given draw of exogenous attributes and taste shifters, $z_{i1}^*, \ldots, z_{in}^*$ are the network and exogenous attributes for the pairwise stable network resulting from tâtonnement starting at the initial network L with implied attributes $z_{i1}, \ldots, z_{in}$. In particular, the conditional distribution $g_n^*(\boldsymbol{\eta}_i | \mathbf{z}_i^*)$ implicitly regards the attributes and taste shifters for all nodes other than i as random.

The proof is based on exchangeability arguments: For any permutation π of indices $\{1, \ldots, n+J\}$, let $\boldsymbol{\eta}_i^\pi = (\eta_{i\pi(1)}, \ldots, \eta_{i\pi(n+J)})$ and $\mathbf{z}_i^\pi = (z_{i\pi(1)}, \ldots, z_{i\pi(n+J)})$. Our main claim is that

$$\left(z_{ij}^* | z_{ij} = z_0, \mathbf{z}_i, \boldsymbol{\eta}_i\right) \quad \stackrel{d}{=} \quad \left(z_{ij}^* | z_{ij} = z_0, \mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi\right) \tag{B.2}$$

almost surely for any i, j and permutation π.

We establish that relation by comparing the outcome of tâtonnement given $\mathbf{z}_i$ and $\boldsymbol{\eta}_i$ to that given an arbitrary permutation $\mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi$. Here tâtonnement is again assumed to follow the process defined in section 2.3 where all other variables in the model are left unchanged. Now, some of the link opportunities are acceptable to node i given $\mathbf{z}_i, \boldsymbol{\eta}_i$ but not given $\mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi$. Hence a finite number of the adjustments that occur during the first stage of tâtonnment given $\mathbf{z}_i, \boldsymbol{\eta}_i$ do not occur given $\mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi$, and vice versa. Each of these adjustments may affect subsequent stages of the process, so that changing node i's taste shifters triggers a cascade of changes that may percolate through the entire network.

Now consider another node k, and suppose that after starting tâtonnement from the initial condition chosen above, one such chain of adjustments reaches a node p with attributes x_p and taste shocks $\eta_{pjk} := (\eta_{pj}, \eta_{pk}, MC_p)$ such that the previous stage resulted in a change to $(\mathbf{s}'_{pj}, \mathbf{t}'_{pj})'$. Whether such an adjustment to the direct links to p results in changes in the potential outcomes $(\mathbf{s}'_{ji}, \mathbf{t}'_{ij})'$ or $(\mathbf{s}'_{ki}, \mathbf{t}'_{ik})'$ is fully determined by the potential outcomes $(\mathbf{s}'_{pj}, \mathbf{t}'_{pj})', (\mathbf{s}'_{pk}, \mathbf{t}'_{pk})'$ and the taste shocks $\eta_{jp}, MC_j, \eta_{pj}, MC_p$ and $\eta_{kp}, MC_k, \eta_{pk}, MC_p$, respectively.

Conditional on $z_{ij} = z_{ik} = z_0$, since by assumption η_{pk} is drawn from the same distribution as η_{pi}, we have that the potential outcomes $(\mathbf{s}'_{pj}, \mathbf{t}'_{pj})' \stackrel{d}{=} (\mathbf{s}'_{pk}, \mathbf{t}'_{pk})'$. Since the tuples $(\eta_{jp}, MC_j, \eta_{pj}, MC_p)$ and $(\eta_{lp}, MC_l, \eta_{pl}, MC_p)$ of payoff shocks also follow the same distribution, the probability of a change to $(\mathbf{d}'_{ji}, \mathbf{s}'_{ji}, \mathbf{t}'_{ij})'$ is therefore equal to that of an equivalent adjustment of $(\mathbf{d}'_{ki}, \mathbf{s}'_{ki}, \mathbf{t}'_{ik})'$ at any stage of the adjustment process, and independent of η_{ij}, η_{ik} and MC_i, MC_k. It follows that

$$\left(z_{ij}^* | z_{ij} = z_0, \mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi\right) \stackrel{d}{=} \left(z_{ik}^* | z_{ik} = z_0, \mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi\right) \tag{B.3}$$

Moreover, by symmetry we have

$$\left(z_{i\pi(j)}^* | z_{i\pi(j)} = z_0, \mathbf{z}_i^\pi, \boldsymbol{\eta}_i^\pi\right) \stackrel{d}{=} \left(z_{ij}^* | z_{ij} = z_0, \mathbf{z}_i, \boldsymbol{\eta}_i\right)$$

so that, in combination with (B.3) for $k = \pi(j)$ this establishes claim (B.2). Since $\hat{G}_{in}^{z,\eta}$ is the maximal invariant for $\mathbf{z}_i, \boldsymbol{\eta}_i$ under the permutations of the indices $1, \dots, n$, we also have

$$\left(z_{ij}^* | z_{ij} = z_0, \mathbf{z}_i, \boldsymbol{\eta}_i\right) \stackrel{d}{=} \left(z_{ij}^* | z_{ij} = z_0, \hat{G}_{in}^{z,\eta}\right)$$

The same argument applies to the conditional distribution of any finite number of components $z_{ij_1}^*, \dots, z_{ij_r}^*$ given $z_{ij_1}, \dots, z_{ij_r}$ and $\boldsymbol{\eta}_i$, establishing the claim $\square$

We now conclude the proof of Lemma 6.1: By Assumption 4.1 and 4.2, the conditional probability that a value of z_i^* is the unique pairwise stable outcome given that it is supported by some pairwise stable network is bounded away from zero. In particular, $g_n^*(\mathbf{z}^* | \mathbf{z}, \hat{G}_n^{z,\eta}) > 0$. Hence it follows from the dominated convergence theorem that for any distribution of the initial condition $\mathbf{z}_i$,

$$\begin{aligned}
\frac{g_n^*(\boldsymbol{\eta} | \mathbf{z}^*, \mathbf{z}, \hat{G}_n^{z,\eta})}{g_n(\boldsymbol{\eta} | \mathbf{z}, \hat{G}_n^{z,\eta})} &= \frac{g_n^*(\boldsymbol{\eta}, \mathbf{z}^* | \mathbf{z}, \hat{G}_n^{z,\eta})}{g_n^*(\mathbf{z}^* | \mathbf{z}, \hat{G}_n^{z,\eta}) g_n(\boldsymbol{\eta} | \mathbf{z}, \hat{G}_n^{z,\eta})} \\
&= \frac{g_n^*(\mathbf{z}^* | \boldsymbol{\eta}, \mathbf{z}, \hat{G}_n^{z,\eta}) g_n(\boldsymbol{\eta} | \mathbf{z}, \hat{G}_n^{z,\eta})}{g_n^*(\mathbf{z}^* | \mathbf{z}, \hat{G}_n^{z,\eta}) g_n(\boldsymbol{\eta} | \mathbf{z}, \hat{G}_n^{z,\eta})} = \frac{g_n^*(\mathbf{z}^* | \boldsymbol{\eta}, \mathbf{z}, \hat{G}_n^{z,\eta})}{g_n^*(\mathbf{z}^* | \mathbf{z}, \hat{G}_n^{z,\eta})} = 1
\end{aligned}$$

almost surely, where the last step uses sufficiency of $\hat{G}_{in}^{z,\eta}$ from Lemma B.1.

Finally notice that the joint distribution $g_n(\boldsymbol{\eta} | \hat{G}_n^{z,\eta})$ corresponds to the experiment of drawing without replacement from $\{\eta_{i1}, \dots, \eta_{in+J-1}\}$. By the Glivenko-Cantelli theorem, the marginal c.d.f. $\hat{G}_n^\eta(\eta)$ converges almost surely to the marginal c.d.f. $G(\eta)$ specified in Assumption 4.2. It follows that the joint p.d.f. $\lim_n g_n(\boldsymbol{\eta}|\hat{G}_n^\eta)/\prod_{j=1}^{J+n-1} g(\eta_j) = 1$ almost surely, which establishes the first part of the lemma.

For the second part of the conclusion, it is sufficient to notice that the preceding argument can be extended to the joint distribution of taste shifters and network attributes for any finite number of nodes without any further adjustments to the proof. $\square$

B.6.1. *Proof of Lemma 6.2.* This result is a generalization of Lemma B.1 in Menzel (2015). We therefore refer to the proof of that result for some of the intermediate technical steps below. Define $\tilde{U}_{ij} := U^*(x_i, x_j; s_i, s_j, t_{ij})$ for $j = 1, \dots, n$, where s_i, s_j, t_{ij} denote the potential outcomes of the respective network variables after setting $L_{ij} = 1$.

By independence of $\eta_{i1}, \dots, \eta_{iN}$,

$$\begin{aligned} J^r \Phi(i, j_1, \dots, j_r|\mathbf{z}_i^*) &= J^r P(U_{ij_1} \geq MC_i, \dots, U_{ij_r} \geq MC_i, U_{ij_{r+1}} < MC_i, \dots, U_{ij_J} < MC_i|\mathbf{z}_i^*) \\ &= J^r \int \left(\prod_{q=1}^r P(U_{ij_q} \geq \sigma s) \right) \left(\prod_{q=r+1}^J P(U_{ij_q} < \sigma s|\mathbf{z}_i^*) \right) JG(s)^{J-1} g(s) ds \\ &= J^r \int \left(\prod_{q=1}^r (1 - G(s - \sigma^{-1}\tilde{U}_{ij_q})) \right) \left(\prod_{q=r+1}^J G(s - \sigma^{-1}\tilde{U}_{ij_q}) \right) JG(s)^{J-1} g(s) ds \\ &= \int \left(\prod_{q=1}^r J(1 - G(s - \sigma^{-1}\tilde{U}_{ij_q})) \right) J \frac{g(s)}{G(s)} \\ &\quad \times \exp\left\{ J \log G(s) + \frac{1}{J} \sum_{q=r+1}^J J \log G(s - \sigma^{-1}\tilde{U}_{ij_q}) \right\} ds \end{aligned}$$

Now let $b_J := G^{-1}\left(1 - \frac{1}{J}\right)$ and $a_J = a(b_J)$, where $a(\cdot)$ is the auxiliary function in Assumption 4.2 (ii). By Assumption 4.3 (iii), $\sigma = \frac{1}{a(b_J)}$, so that a change of variables $s = a_J t + b_J$ yields

$$\begin{aligned} J^r \Phi(i, j_1, \dots, j_r|\mathbf{z}_i^*) &= \int \left(\prod_{q=1}^r J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q}))) \right) J \frac{a_J g(b_J + a_J t)}{G(b_J + a_J t)} \\ &\quad \times \exp\left\{ J \log G(b_J + a_J t) + \frac{1}{J} \sum_{q=r+1}^J J \log G(b_J + a_J(t - \tilde{U}_{ij_q})) \right\} dt \end{aligned}$$

By Assumption 4.2 (ii), $J(1 - G(b_J + a_J t)) \to e^{-t}$ and

$$J a_J g(b_J + a_J t) = J a(b_J) g(b_J + a(b_J)t) = a(b_J) \frac{1 - G(b_J + a_J t)}{a(b_J + a_J t)(1 - G(b_J))} \to e^{-t}$$

where the last step uses Lemma 1.3 in Resnick (1987). Also, following steps analogous to the proof of Lemma B.1 in Menzel (2015), we can take limits and obtain

$$\begin{aligned} \prod_{q=1}^r J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q}))) &\to \exp\left\{ -rt + \sum_{q=1}^r \tilde{U}_{ij_q} \right\} \\ J \log G(b_J + a_J(t - \tilde{U}_{ij_q})) &\to -e^{-t} \exp\{\tilde{U}_{ij_q}\} \end{aligned}$$

Combining the different components, we can take the limit of the integrand in (B.4),

$$\begin{aligned} R_J(t) &:= \left(\prod_{q=1}^{r} J(1-G(b_J+a_J(t-\tilde{U}_{ij_q})))\right) J\frac{a_J g(b_J+a_J t)}{G(b_J+a_J t)} \\ &\quad \times \exp\left\{J\log G(b_J+a_J t)+\frac{1}{J}\sum_{q=r+1}^{J} J\log G(b_J+a_J(t-\tilde{U}_{ij_q}))\right\} \\ &= \exp\left\{-e^{-t}\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\{\tilde{U}_{ij_q}\}\right)-(r+1)t+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}+o(1) \end{aligned} \tag{B.4}$$

for all $t \in \mathbb{R}$. Using the same argument as in the proof of Lemma B.1 in Menzel (2015), pointwise convergence and boundedness of the integrand imply convergence of the integral by dominated convergence, so that we obtain

$$\begin{aligned} J^r\Phi(i,j_1,\dots,j_r|\mathbf{z}_i^*) &\to \int_{-\infty}^{\infty}\exp\left\{-e^{-t}\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\{\tilde{U}_{ij_q}\}\right)-(r+1)t+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}dt \\ &= \int_{-\infty}^{0}\exp\left\{s\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\{\tilde{U}_{ij_q}\}\right)+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}s^r ds \\ &= \frac{r!\exp\{\sum_{q=1}^{r}\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\left\{\tilde{U}_{ik_q}\right\}\right)^{r+1}} \end{aligned}$$

where the first step uses a change of variables $s=-e^{-t}$, and the last step can be obtained recursively via integration by parts. Furthermore, if $\frac{r}{J}\to 0$, boundedness of the systematic parts from Assumption 4.1 implies that

$$\left|\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}-\frac{1}{J}\sum_{q=r+1}^{J}\exp\left\{\tilde{U}_{ik_q}\right\}\right|\to 0$$

so that

$$J^r\Phi(i,j_1,\dots,j_r|\mathbf{z}_i^*)\to\frac{r!\prod_{q=0}^{r}\exp\{\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}\right)^{r+1}}$$

which completes the proof □

B.6.2. *Proof of Lemma 6.3.* Without loss of generality, we develop the formal argument only for the case in which the payoff-relevant network characteristic is binary, $\mathcal{S}=\{\underline{s},\bar{s}\}$, where $U^*(x,x';\underline{s},s',t)\le U^*(x,x';\bar{s},s',t)$ and $U^*(x,x';s,\underline{s},t)\le U^*(x,x';s,\bar{s},t)$ for all values of x,x',s',t. An extension to the general case follows the exact same steps but requires additional case distinctions. Also note that under Assumption 4.3 (iv), the effect of edge-specific interaction effects through $U^*(\cdot,t_{ij})-U^*(\cdot,t_0)$ on the inclusive value is negligible in the limit, so that in the following, we evaluate all systematic utilities at $t=t_0$.

Let $S_i^*\subset\mathcal{S}$ denotes the set of values for s_i supported by the edge-level response for node i, and let

$$B_0 := \{j : S_j^*=\mathcal{S}\}$$

denote the set of nodes for whom both values for s_j are supported by j's edge-level response. For each node i we also define

$$A_i := \{j : U^*(x_j,x_i;\underline{s},s_i,t_0)\ge MC_j-\sigma\eta_{ji}\}\cap B_0$$

and

$$B_i := \{j : U^*(x_j,x_i;\underline{s},s_i,t_0)< MC_j-\sigma\eta_{ji}\le U^*(x_j,x_i;\bar{s},s_i,t_0)\}\cap B_0$$

be the set of nodes with a non-unique edge-level response that are available to i for any value of s_j. As a notational convention, $i \notin A_i \cup B_i$. Note that by Assumption 4.3 and Lemma 6.2, $P(j \in A_i), P(j \in B_i) = O(n^{-1/2})$.

Define $a_{ij} := \mathbb{1}\{j \in A_i\} \exp\{U^*(x_i, x_j; s_i, \underline{s}, t_0)\}$, $b_{ij} := \mathbb{1}\{j \in B_i\} \exp\{U^*(x_i, x_j; s_i, \bar{s}, t_0)\}$, and $c_{ij} := \mathbb{1}\{j \in A_i\}(\exp\{U^*(x_i, x_j; s_i, \bar{s}, t_0)\} - \exp\{U^*(x_i, x_j; s_i, \underline{s}, t_0)\})$, and $g_{ij} := b_{ij} + c_{ij}$. Note that given x_i, x_j, (b_{ij}, c_{ij}) are conditionally independent across i, j. We also let $\Delta_i a_{ij} := a_{ij} - \mathbb{E}[a_{ij}|x_i = x, s \in S_i^*]$ and $\Delta_i g_{ij} := g_{ij} - \mathbb{E}[g_{ij}|x_i = x, s \in S_i^*]$.

We now introduce the allocation parameter $\alpha_j \in [0, 1]$ corresponding to the probability with which node j is assigned to choose the edge-level response $s_j = \bar{s}$, so that $s_j = \underline{s}$ will be chosen with probability $1 - \alpha_j$. In particular, for a given choice of $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n)'$, the inclusive value for agent i is given by

$$I_i[\boldsymbol{\alpha}] = n^{-1/2} \sum_{j=1}^{n} (a_{ij} + \alpha_j g_{ij}),$$

and the inclusive value function

$$\hat{H}_n^*(x, s; \boldsymbol{\alpha}) := n^{-1/2} \sum_{j=1}^{n} (\mathbb{E}[a_{ij}|x_i = x, s \in S_i^*] + \alpha_j \mathbb{E}[g_{ij}|x_i = x, s \in S_i])$$

Hence, we can write

$$\begin{aligned} I_i[\boldsymbol{\alpha}] - \hat{H}_n^*(x, s; \boldsymbol{\alpha}) &= n^{-1/2} \sum_{j=1}^{n} (a_{ij} - \mathbb{E}[a_{ij}|x_i = x, s \in S_i^*] + \alpha_j (g_{ij} - \mathbb{E}[g_{ij}|x_i = x, s \in S_i])) \\ &= n^{-1/2} \sum_{j=1}^{n} (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \end{aligned}$$

We can now measure the average dispersion of I_i about its conditional mean by

$$\hat{V}_n[\boldsymbol{\alpha}] := \frac{1}{n} \sum_{i=1}^{n} (I_i[\alpha] - \hat{H}_n^*(x_i, s_i; \alpha))^2$$

for a given value of $\boldsymbol{\alpha}$. To find an upper bound for a given realization of payoffs, we can solve the problem

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}] \text{ subject to } \alpha_1, \ldots, \alpha_n \in [0, 1]. \tag{B.5}$$

This upper bound is generally not sharp since for some nodes j only either value of s_j may be supported by the edge-level response. Multiplying out the square, we obtain

$$\begin{aligned} \hat{V}_n[\boldsymbol{\alpha}] &= \frac{1}{n} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \right)^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \right)^2 + 2 \left(n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \right) \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right) + \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right)^2 \end{aligned}$$

where by a LLN, $n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \to 0$ (see also Lemma B.5 in Menzel (2015) for a detailed proof), so that

$$\begin{aligned} \max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] &= \frac{1}{n} \max_{\boldsymbol{\alpha}} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right)^2 + o_p(1) \\ &= \frac{1}{n^2} \max_{\boldsymbol{\alpha}} \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \sum_{i=1}^{n} \Delta_i g_{ij} \Delta_i g_{ik} + o_p(1) \end{aligned}$$

where in the last step we multiplied out the square and changed the order of summation.

Now, for $j \neq k$,

$$\text{Var}(\Delta_i g_{ij}\Delta_i g_{ik}) = \mathbb{E}[(\Delta_i g_{ij})^2(\Delta_i g_{ik})^2] - \left(\mathbb{E}[\Delta_i g_{ij}\Delta_i g_{ik}]\right)^2 = O(n^{-1}) - O(n^{-2})$$

and

$$\text{Var}(\Delta_i g_{ij}^2) = \mathbb{E}[(\Delta_i g_{ij})^4] - \left(\mathbb{E}[\Delta_i g_{ij}^2]\right)^2 = O(n^{-1/2}) - O(n^{-1})$$

Hence, we can use a CLT to conclude that for any $j \neq k$

$$Z_{jk,n} := \sum_{i=1}^n \Delta_i g_{ij}\Delta_i g_{ik} = O_p(1), \text{ and } Z_{jj,n} := n^{-1/4}\sum_{i=1}^n \Delta_i g_{ij}^2 = O_p(1)$$

where Assumption 4.1 implies that the asymptotic variances of $Z_{jk,n}$ and $Z_{jj,n}$ are bounded. Furthermore, $\mathbb{E}[Z_{jk}] = 0$ for $j \neq k$, and $Z_{jk,n}$ are independent across $1 \leq j \leq k \leq n$.

Next, we can bound the sum corresponding to the "diagonal" elements $Z_{jj,n}$ by

$$\frac{1}{n^2}\sum_{j=1}^n \alpha_j^2 \sum_{i=1}^n \Delta_i g_{ij}^2 \leq \frac{1}{n^2}\max_{\boldsymbol{\alpha}} \sum_{j=1}^n \alpha_j^2 n^{1/4} Z_{jj,n} = n^{-7/4}\sum_{j=1}^n Z_{jj,n} = O_p(n^{-3/4})$$

noting that $Z_{jj,n} \geq 0$ a.s., so that the maximum in the second expression is attained at $\alpha_1 = \cdots = \alpha_n = 1$. In the following, we let $\mathbf{Z}_n$ be the symmetric matrix whose (j,k)th element is $Z_{jk,n}$ for $j \neq k$, and where we set Z_{jj} equal to zero.

Given these definitions, we can express the maximum in matrix notation and bound

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] = \frac{1}{n}\max_{\boldsymbol{\alpha}} \frac{1}{n}\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha} + o_p(1) \leq \frac{1}{n}\max_{\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\alpha}} + o_p(1) \equiv n^{-1/2}\lambda_{max}(n^{-1/2}\mathbf{Z}_n) + o_p(1)$$

where $\lambda_{max}(\mathbf{A}$ denotes the largest eigenvalue of a symmetric matrix $\mathbf{A}$. For the second step, notice that $|\alpha_j|^2 \leq 1$ for each j, so that the scalar product $\boldsymbol{\alpha}'\boldsymbol{\alpha} \leq n$ for each permissible $\boldsymbol{\alpha}$.

Also, $\mathbf{Z}_n$ is a symmetric matrix with entries which, conditional on $x_1, \ldots, x_n$, are independent although in general not identically distributed, bounded, mean zero random variables. Furthermore, if we pre- and post-multiply the matrix $\mathbf{Z}_n$ with the diagonal matrix $H := \text{diag}(1/\sigma_i)$, where $\sigma_i^2 := \frac{1}{n}\sum_{j\neq i}\text{Var}(\Delta_i g_{ij}^2|x_i)$, then the entries also have constant variance. It therefore follows from Theorem 2 of Füredi and Komlós (1981) that the maximal eigenvalue of $n^{-1/2}H\mathbf{Z}_nH$ is bounded from above by a finite constant with probability approaching 1, so that

$$\mathbb{E}\left[\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}]\right] = O(n^{-1/2}) \tag{B.6}$$

which converges to zero.

Now let $\tilde{j}$ be a uniform random draw from the set $\{1, \ldots, n\}$. Then we can use Chebyshev's Inequality to show that for an arbitrary selection from the edge-level responses, we can bound

$$\begin{aligned} P\left((I_{\tilde{j}} - \hat{H}_n^*(x_{\tilde{j}}, s_{\tilde{j}}))^2 > \varepsilon^2\right) &= \frac{1}{n}\sum_{i=1}^n P\left((I_i - \hat{H}_n^*(x_i, s_i))^2 > \varepsilon^2\right) \\ &\leq \frac{1}{\varepsilon^2}\frac{1}{n}\sum_{i=1}^n \mathbb{E}[(I_i - \hat{H}_n^*(x_i, s_i))^2] \\ &\leq \frac{1}{\varepsilon^2}\mathbb{E}\left[\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}]\right] = o(1) \end{aligned}$$

where the right-hand side bound is uniform across all possible selections from pairwise stable networks and converges to zero by (B.6). This establishes convergence that is pointwise in x, s but uniform in all selections from the best response.

This establishes claim (a) of the Lemma for the case in which $\mathcal{S}$ has only two elements. An generalization to the case in which $\mathcal{S}$ has $K < \infty$ elements follows the exact same steps but requires additional case distinctions and an allocation parameter α in the K-dimensional probability simplex. Finally, for the general case in which $\mathcal{S}$ may have infinitely many elements, note that by Assumption 4.1, the systematic part of payoffs only varies over a bounded interval $[-\bar{U}, \bar{U}]$ as s_1, s_2 vary. Furthermore, $U(x_1, x_2; s_1, s_2; t_0)$ is Lipschitz in s_2, so that by compactness of $\mathcal{S}$, we can cover the set of functions $\{\exp\{U(x_1, x_2; s_1, s; t_0)\} : s \in \mathcal{S}\}$ with a finite number K of L_2-norm brackets of width $\varepsilon/2$, using standard arguments (see e.g. Example 19.7 in van der Vaart (1998)). Identifying the kth bracket with an element $\exp\{U(x_1, x_2; s_1, s^{(k)}; t_0)\}$, for any $s \in \mathcal{S}$ we can therefore find $s^{(k)} \in \{s^{(1)}, \dots, s^{(K)}\} \subset \mathcal{S}$ such that

$$\int |\exp\{U(x_1, x_2; s_1, s; t_0)\} - \exp\{U(x_1, x_2; s_1, s^{(k)}; t_0)\}| w(x_1) w(x_2) dx_1 dx_2 < \varepsilon$$

for each $s_1 \in \mathcal{S}$. A simple calculation then shows that the difference between the analogs of the worst-case bounds in B.5 for the discrete set $s^{(1)}, \dots, s^{(K)} \subset \mathcal{S}$ and the full set $\mathcal{S}$ is less than ε, which can be made arbitrarily small.

For claim (b), note however that the argument for point-wise convergence in part 1 still goes through after multiplying the contribution of node i with bounded weights $\omega(x_i; s_i)$. Uniformity with respect to $\omega(\cdot)$ then follows from the GC condition and using arguments that are analogous as for part (b) of Lemma B.5 in Menzel (2015). For the case of $2 < |\mathcal{S}| < \infty$, the argument is identical except that allocation parameter α_j is now $(|\mathcal{S}| - 1)$-dimensional which increases the bounding constant by a finite multiple $\square$

B.6.3. *Size of Opportunity Sets.* The next auxiliary result concerns the rate at which the number of available potential spouses increases for each individual in the market. For a given PSN L^*, we let

$$J_i^* := J_i[L^*] := \sum_{j=1}^{n} \mathbb{1}\{U_{ji}(\mathbf{L}^*) \geq MC_j\}$$

denote the size of the link opportunity set available to agent i. Similarly, we let

$$K_i^* = \sum_{j=1}^{n} \mathbb{1}\{U_{ij}(\mathbf{L}^*) \geq MC_i\}$$

so that K_i^* is the number of nodes to whom i is available.

Lemma B.2 below establishes that in our setup, the number of available potential matches grows at a root-n rate as the size of the market grows.

Lemma B.2. *Suppose Assumptions 4.1-4.3 hold. Then for any pairwise stable network,*

$$\begin{aligned} \exp\{-\bar{U} - B_T\} \leq \quad & n^{-1/2} J_i^* \quad \leq \exp\{\bar{U} + B_T\} \\ \exp\{-\bar{U} - B_T\} \leq \quad & n^{-1/2} K_i^* \quad \leq \exp\{\bar{U} + B_T\} \end{aligned}$$

for each $i = 1, \dots, n$ with probability approaching 1 as $n \to \infty$.

Proof of Lemma B.2: Notice that in the absence of interaction effects across links, D_{ji} does not depend on the number of "proposals" that can be reciprocated, but only the magnitude of MC_i. Furthermore, by Assumption 4.1, the systematic parts of payoffs are uniformly bounded for all values of s_i, s_j. Hence the proof closely parallels the argument for the matching case. We therefore only demonstrate that externalities across links do not alter that conclusion, for the remaining technical steps we refer the reader to the proof of Lemma B.2 in Menzel (2015), which is the analogous result for the two-sided matching problem.

Fix $i, j \leq n$, and let $\tilde{U}_{ij} := U^*(x_i, x_j, s_i, s_j, T^*_{ij})$, where $T^*_{ij} := T(\mathbf{L}^*_n, x_i, x_j, i, j)$. By Assumption 4.1, $|U^*(x_i, x_j, s_i, s_j, t_0)| \leq \bar{U}$. Also, by Assumption 4.3 (iv) and Jensen's Inequality, we have

$$\mathbb{E}\left[\exp\left\{|U^*(x_i, x_j, s_i, s_j, T^*_{ij}) - U^*(x_i, x_j, s_i, s_j, t_0)|\right\}\right] \leq \exp\{B_T\}$$

for n sufficiently large, so that, using the Law of iterated expectations and the triangle inequality, $\mathbb{E}\left[\exp\left\{|\tilde{U}_{ij}|\right\}\right] \leq \exp\{\bar{U} + B_T\}$ for n large enough.

Hence, following a similar series of steps as in the proof of Lemma 6.2, the marginal probability

$$\begin{aligned} JP(U_{ij} \geq MC_i) &= J\int_{-\infty}^{\infty} G^J(\tilde{U}_{ij} + s)g(s)ds \\ &\leq \mathbb{E}\left[J\int_{-\infty}^{\infty} G^J(\bar{U} + |U^*(x_i, x_j, s_i, s_j, T^*_{ij}) - U^*(x_i, x_j, s_i, s_j, t_0)| + s)g(s)ds\right] \\ &\to \exp\{\bar{U} + B_T\} \end{aligned}$$

Similarly, we find that

$$JP(U_{ij} \geq MC_i) \geq J\int_{-\infty}^{\infty} G^J(-\bar{U} + s)g(s)ds \to \exp\{-\bar{U} - B_T\}$$

Since $K^*_i := \sum_{j=1}^n \mathbb{1}\{U_{ij} \geq MC_i\}$, we can bound the expectation,

$$\exp\{-\bar{U} - B_T\} \leq n^{-1/2}\mathbb{E}[K^*_i] \leq \exp\{\bar{U} + B_T\}$$

as n grows large. Similarly, $J^*_i := \sum_{j=1}^n \mathbb{1}\{U_{ji} \geq MC_j\}$ so that for n sufficiently large,

$$\exp\{-\bar{U} - B_T\} \leq n^{-1/2}\mathbb{E}[J^*_i] \leq \exp\{\bar{U} + B_T\}$$

These bounds are uniform for $i = 1, 2, \ldots$. Given these rates for the expectation of the upper and lower bounds for J^*_i and K^*_i, the conclusion of this lemma follows the same sequence of steps as in the proof of Lemma B.2 in Menzel (2015) □

B.6.4. *Proof of Lemma 6.4.* Aggregating over $j \neq i$, we obtain

$$\begin{aligned} \hat{H}^*_n(x_i; s_i) &= n^{-1/2}\sum_{j\neq i} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\}P(D_{ji} = 1|W_j(\mathbf{L}^*)) \\ &= n^{-1/2}\sum_{i\in W_j(\mathbf{L}^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\}P(D_{ji} = 1|W_j(\mathbf{L}^*)) \\ &\quad + n^{-1/2}\sum_{i\notin W_j(\mathbf{L}^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\}P(D_{ji} = 1|W_j(\mathbf{L}^*)) \end{aligned}$$

The asymptotic approximation to the edge-level response in Lemma 6.2 implies

$$\begin{aligned} n^{1/2}P(D_{ji} = 1|W_j(\mathbf{L}^*)) &= n^{1/2}\mathbb{E}[\Phi(j, i_1, \ldots, i_r|\mathbf{z}^*_j)|W_j(\mathbf{L}^*)\mathbb{1}\{i \in \{i_1, \ldots, i_r\}\}] \\ &= n^{1/2}\sum_{r\geq 0}\sum_{i_1,\ldots,i_r} \Phi(j, i_1, \ldots, i_r|\mathbf{z}^*_j)\mathbb{1}\{i \in \{i_1, \ldots, i_r\}\} \end{aligned}$$

$$= \sum_{r\geq 0} \frac{(r+1)!}{r!} \frac{\exp\{U^*(x_j, x_i; (r, s'_{2j})', s_i, t_{ji}) + U^*(x_i, x_j; s_i, (r, s'_{2j})', t_{ji})\}(I^*_j)^r}{(1 + I^*_j)^{r+2}} + o_p(1) < \infty$$

Since the last expression is uniformly bounded in s_i and $I^*_j \geq 0$, it follows that

$$n^{-1/2}\sum_{i\in W_j(\mathbf{L}^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\}P(D_{ji} = 1|W_j(\mathbf{L}^*)) = o_p(1)$$

noting that by Lemma B.2, $|\{j : i \in W_j(\mathbf{L}^*)\}|/n \to 0$ almost surely. Hence the contribution of nodes j such that $i \in W_j(\mathbf{L}^*)$ to the inclusive value is negligible to first order.

Next consider the nodes j such that $i \notin W_j(\mathbf{L}^*)$. Note that in that case, a link proposal to i does not result in a new link, and therefore D_{ji} does not affect the network structure. Hence, for given values of s_i, s_j, t_{ij} and payoff shocks, the link proposal indicator D_{ji} is uniquely determined. Hence, using Lemma 6.2 again,

$$\begin{aligned}\hat{H}_n^*(x_i; s_i) &= n^{-1/2} \sum_{i \notin W_j(\mathbf{L}^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\} P(D_{ji} = 1 | W_j(\mathbf{L}^*)) + o_p(1) \\ &= \frac{1}{n} \sum_{j=1}^{n} \frac{s_{1j,+i}^* \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij}) + U^*(x_j, x_i; s_j, s_i, t_{ij})\}}{1 + I_j^*} + o_p(1)\end{aligned}$$

where the last expression depends on the empirical distribution of endogenous network characteristics given exogenous traits. Using Lemma 6.3 and noting that under Assumption 4.3 (iv), the effect of edge-specific interaction effects through $U^*(\cdot, t_{ij}) - U^*(\cdot, t_0)$ on the inclusive value is negligible in the limit, we can now write

$$\hat{H}_n^*(x_i; s_i) = \frac{1}{n} \sum_{j=1}^{n} \frac{s_{1j,+i}^* \exp\{U^*(x_i, x_j; s_i, s_j, t_0) + U^*(x_j, x_i; s_j, s_i, t_0)\}}{1 + \hat{H}_n^*(x_j; s_j)} + o_p(1) \tag{B.7}$$

Substituting in the definition of $\hat{\Psi}_n$ in (6.2), we obtain pointwise convergence in x, s. Uniformity follows from the Glivenko-Cantelli property of the systematic payoff functions, noting that I_j^* and $\hat{H}_n^*$ are guaranteed to be nonnegative □

B.6.5. *Proof of Corollary 6.1:* Given part (i) of Proposition 6.1, it is sufficient to show that $\mathbb{E}[s_{1i}|x_i = x]$ is uniformly bounded for $x \in \mathcal{X}$. To this end, notice that for payoffs of the form $U^*(x_1, x_2; s_1, s_2) = U^*(x_1, x_2)$, the inclusive value function only depends on x, i.e. $H^*(x; s) = H^*(x)$. Furthermore, the edge-level response is unique so that the conditional degree distribution given $x_i = x$ has p.d.f. $P(s_{1i} = s | x_i = x) = \frac{H^*(x)^s}{(1+H^*(x))^{s+1}}$. Hence, the conditional expectation of s_{1i} is given by

$$\begin{aligned}\mathbb{E}[s_{1i}|x_i = x] &= \sum_{s=0}^{\infty} s \frac{H^*(x)^s}{(1 + H^*(x))^{s+1}} = \frac{1}{1 + H^*(x)} \sum_{s=0}^{\infty} s \left(\frac{H^*(x)}{1 + H^*(x)} \right)^s \\ &=: \frac{1}{1 + H^*(x)} \sum_{s=0}^{\infty} s \delta^s = \frac{1}{1 + H^*(x)} \frac{\delta}{(1 - \delta)^2} = H^*(x)\end{aligned}$$

where $\delta := \frac{H^*(x)}{1+H^*(x)}$. Finally, it remains to be shown that $H^*(x)$ is uniformly bounded: from the fixed-point condition (3.2),

$$\begin{aligned}\Psi[H, M](x) &= \int \frac{s_{1j} \exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1 + H(x_j)} M(s_j | x_j, x) w(x_j) ds_j dx_j \\ &= \int \frac{H^*(x_j) \exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1 + H(x_j)} M(s_j | x_j, x) w(x_j) ds_j dx_j \\ &\le \exp\{2\bar{U}\}\end{aligned}$$

where $\bar{U} < \infty$ is the bound in Assumption 4.1. Hence the range of Ψ_0 is uniformly bounded, so that the fixed point H^* also has to satisfy this bound □

B.6.6. *Proof of Lemma 6.5.* To accommodate the general set-valued case, we state and prove a strengthened version of the Lemma - let the set $\hat{\mathcal{M}}_n^*$ be a solution to the fixed point problem

$$\hat{\mathcal{M}}_n^* = \text{conv}\left(\bigcup_{M\in\hat{\mathcal{M}}_n^*} \text{core } \hat{\Omega}_n[H,M]\right) \tag{B.8}$$

Also, as in equation (A.4) let $\mathcal{M}^*$ be a set solving

$$\mathcal{M}^* = \text{conv}\left(\bigcup_{M\in\mathcal{M}^*} \text{core } \Omega_0[H,M]\right)$$

Note also that the union $\mathcal{M}^*_{max}$ of all sets satisfying (A.4) is in turn a solution to that fixed point condition.

We now state and prove the following strengthened version of Lemma 6.5:

Lemma B.3. *Suppose that Assumptions 4.1-4.5 hold. Then for any stable network, the inclusive value function $\hat{H}_n^*(x;s)$ satisfy the fixed-point conditions in (6.3), and the reference distributions are in a set $\hat{\mathcal{M}}_n^*$ solving (B.8). Moreover, there exist H^*, M^* satisfying the population fixed-point conditions in (3.2) and (A.4) such that $\|\hat{H}_n^* - H^*\|_\infty = o_p(1)$ and $\sup_{M\in\mathcal{M}_n^*} d(M,\mathcal{M}_0^*) = o_P(1)$.*

Proof: For the first claim of the Lemma, notice that the fixed point condition (6.3) is a direct consequence of Lemmas 6.3 and 6.4. Furthermore, (B.8) holds by construction of the capacity $\hat{\Omega}_n$, where the exact form of the fixed-point mapping has to be derived separately for the problem at hand. For the proof of the second claim, we first state the following Lemma:

Lemma B.4. *Suppose the conditions for Proposition 6.1 hold. Then the mapping*

$$\hat{\Psi}_n[H,M](x;s) \stackrel{p}{\to} \Psi_0[H,M](x;s)$$

uniformly in $H\in\mathcal{G}$, $M\in\mathcal{U}$, and $(x',s)'\in\mathcal{X}\times\mathcal{S}$ as $n\to\infty$.

This result is a straightforward extension of Lemma B.6 in Menzel (2015), a separate proof will therefore be omitted.

For the remainder of the proof of Lemma B.3, note that we can rewrite the fixed-point condition (A.4) in a more compact vector form, $M^* \le \Omega_0[H^*,M^*]$, where the respective components $M^*(W;\mathbf{r}|\mathbf{x}) := \int_W M^*(\mathbf{w};\mathbf{r}|\mathbf{x})d\mathbf{w}$ and $\Omega^*[H^*,M^*](\mathbf{x};\mathbf{r},W)$ are indexed by $x\in\mathcal{X}$, $\mathbf{r}\in\mathcal{R}$, and $W\subset\mathcal{S}\times\mathcal{T}^{d_\cap}$, and we continue to use the notation introduced in the proof of Theorem 4.1.

Now let

$$\mathcal{Z}^* := \{(H^*,M^*): H^*\in\Psi_0[H^*,M^*], M^*\in\mathcal{M}^*_{max}\}$$

be the set of fixed points of (3.2) and (A.4). Since by Assumption 4.5 (ii), the respective ranges of Ψ_0 and Ω_0 are contained in $\mathcal{G}$ and $\mathcal{U}$, respectively, any fixed points must be in $\mathcal{G}\times\mathcal{U}$, so that it is sufficient to consider the fixed-point mapping restricted to that compact space.

Now fix $\delta>0$ and define

$$\eta := \inf\left\{\inf_{M\in\mathcal{M}^*_{max}}\sup_{\mathbf{x},\mathbf{r},W}|M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[H,M](\mathbf{x};\mathbf{r},W)|_+ + \sup_{x,s}|\Psi_0[H,M](x;s) - H(x;s)| : d((H,M),\mathcal{Z}^*)\ge\delta\right\}. \tag{B.9}$$

By definition of $\mathcal{Z}^*$, we must have that either

$$\inf_{M\in\mathcal{M}^*_{max}}\sup_{\mathbf{x},\mathbf{r},W}|M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[H,M](\mathbf{x};\mathbf{r},W)|_+ > 0$$

or

$$\sup_{x,s} |\Psi_0[H,M](x;s) - H(x;s)| > 0$$

for any $(H,M) \notin \mathcal{Z}^*$. Furthermore the δ-enlargement $(\mathcal{Z}^*)^\delta := \{(H,M) \in \mathcal{G} \times \mathcal{U} : d((H,M),\mathcal{Z}^*) < \delta\}$ is open, so that its complement is closed. Since any closed subset of a compact space is compact, the set $\{(H,M) \in \mathcal{G} \times \mathcal{U} : d(H,M) \geq \delta\}$ is compact. Since furthermore the quantities $\inf_{M \in \mathcal{M}^*_{max}} \sup_{\mathbf{x},\mathbf{r},W} |M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[H,M](\mathbf{x};\mathbf{r},W)|_+$ and $\sup_{x,s} |\Psi_0[H,M](x;s) - H(x;s)|$ are continuous in H, M, the infimum in the definition of η in (B.9) is attained, which implies that $\eta > 0$.

Finally, by Lemma B.4 and Assumption 4.5 (iii), the fixed-point mappings $\hat{\Omega}_n$ and $\hat{\Psi}_n$ converge uniformly to the respective limits, Ω_0 and Ψ_0. In particular, for any $\varepsilon > 0$, we can find $n_\varepsilon < \infty$ such that for all $n \geq n_\varepsilon$, $\sup_{M,H} \|\hat{\Omega}_n[H,M] - \Omega_0[H,M]\| < \eta/2$ and $\sup \|\hat{\Psi}_n[H,M] - \Psi_0[H,M]\| < \eta/2$ with probability greater than $1-\varepsilon$. It follows that as n increases, any point $(\hat{H}^*, \hat{M}^*)$ satisfying the fixed point conditions (6.3) and (6.5) is contained in $(\mathcal{Z}^*)^\delta$ w.p.a.1, establishing the second claim $\square$

B.7. **Proof of Theorem 4.2.** Consider a pair of nodes i,j drawn from $\mathcal{N}$ independently and uniformly at random. By Lemma B.2, the number of link opportunities available to either node is bounded from above by $n^{1/2}\exp\{\bar{U} + B_T\}$. By Lemma 6.1, i and j's taste shifters are asymptotically independent of availability, so that by Lemma 6.2 the number $R_{ij} := |\mathcal{N}_0|$ of nodes $l \in \mathcal{N}$ such that $\bar{U} + \eta_{kl} \geq MC_k$ and $\bar{U} + \eta_{lk} \geq MC_l$ for any $k \in \{i,j\}$, is asymptotically tight.

Let l be a node drawn at random from the uniform distribution over $\mathcal{N}$. From the definition of the reference distribution, it follows that node l's attributes, including the potential values for s_l, are distributed according to a p.d.f. $\hat{M}^*_l(s_l|x_{ijl})w(x_l)$, where $\hat{M}^*_l(s_l|x_{ijl}) \in \hat{\mathcal{M}}^*_n$. By Lemma B.3, $d(\hat{M}_l, \mathcal{M}^*) = o_P(1)$ for some set $\mathcal{M}^*_0$ satisfying the condition (A.4) and $d(\hat{H}^*_n, H^*) = o_P(1)$ for an inclusive value function H^* satisfying condition (3.2).

Moreover, combining Lemmas 6.1 and 6.2, we have that for node l the probability $n^{1/2}P(\bar{U} + \sigma\eta_{li} \geq MC_l) \to \frac{s_{1l}\exp\{\bar{U}\}}{1+H^*(x_l,s_l)}$, and the conditional probability $P(\tilde{U} + \sigma\eta_{li} \geq MC_l | \bar{U} + \sigma\eta_{li} \geq MC_l) \to \exp\{\tilde{U} - \bar{U}\}$ for any $\tilde{U} \leq \bar{U}$. Since $|\mathcal{N}_0|$ is asymptotically tight and the conclusion of Lemma 6.1 holds after conditioning on finitely many link opportunity sets $\{\mathcal{W}^*_l : l \in \mathcal{N}_0\}$, availability is conditionally asymptotically independent across all nodes in $\mathcal{N}_0$.

To complete the stochastic representation of $\mathcal{F}^*_0$, let η^*_{km} and η^*_{k0j} be i.i.d. draws from the extreme-value type I distribution for $k = i,j$, $m \in \mathcal{W}^*_k$ and $j = 1,\dots,J_k$. It then follows from the main result in Dagsvik (1994) that the availability probabilities of the form $\frac{s_{1k}\exp\{U^*(x_k,x_l;s_k,s_l,t_{kl})\}}{1+H^*(x_k,s_k)}$ can be represented as the probability that $U^*(x_k,x_l;s_k,s_l,t_{kl}) + \eta^*_{kl}$ is among the s_{1k} highest order statistics of the sample $\{U^*(x_k,x_m;s_k,s_m,t_{km}) + \eta^*_{km}, \eta^*_{k0j} : m \in \mathcal{W}^*_k, j \leq J_k\}$ conditional on $|\mathcal{W}^*_k| \geq s_{1k} + 1$, which completes the proof $\square$

B.8. **Proof of Theorem 4.3.** For this proof, note that the *tangent cone* to a set $K \subset \mathcal{Z}$ (say) is defined as the set $T_K(z) := \limsup_{h\downarrow 0} \frac{1}{h}(K - z)$ where $K - z := \{(y - z) : y \in K\}$. In particular, the tangent cone at a point z in the interior of K relative to $\mathcal{Z}$ is all of $\mathcal{Z}$. The proof relies on a fixed point theorem for inward mappings, where the mapping Υ_0 is said to be *inward* on a convex set $K \subset \mathcal{Z}$ if $\Upsilon_0[z] \cap (z + T_K(z)) \neq \emptyset$ for any $z \in K$ and $T_K(z)$ denotes the tangent cone to K in $\mathcal{Z}$.

Since the contingent derivative of the mapping $\Upsilon_0[\mathbf{z}] - \mathbf{z}$ is surjective by assumption, we can use Lemma C.1 in Menzel (2016) to conclude that Υ_0 is an inward mapping when restricted to a neighborhood of any of its fixed points. Furthermore, Υ_0 and $\hat{\Upsilon}_n$ are also convex-valued mappings since the sets $\hat{\Psi}_n$ and Ψ_0 and core Ω_0 are convex by standard properties of the core. Finally, $\hat{\Upsilon}_n$ converges uniformly to Υ_0 by Lemma

B.4, so that w.p.a.1 $\hat{\Upsilon}_n$ is also locally inward. In complete analogy to the proof for Theorem 3.1 part (b) in Menzel (2016), local existence of a fixed point then follows by Theorem 3.2.5 in Aubin and Frankowska (1990), noting that this fixed point result applies to general Banach spaces $\square$

Proof of Proposition 5.1. For notational simplicity, we let $U_{i0} := MC_i$ which we can approximate by $U_{i0} = \log J + \sigma\eta_{i0}^*$ as J grows large, where η_{i0}^* is a random draw from the same distribution as $\eta_{i1}, \dots, \eta_{iJ}$. We therefore let $\tilde{U}_{i0} := \log J$. Consider the case in which degree is equal to s, so that MC_i is the $(s+1)$ highest order statistic. Following Assumption 4.3 (ii), let $J = n^{1/2}$ and denote the number of elements in $W_i(\mathbf{L}^*)$ with J_W. Note also that by Lemma B.2, $J_W = O_P(n^{1/2})$, and by Lemma 6.1, taste shifters η_{ij} are asymptotically independent of $W_i(\mathbf{L}^*)$.

In the following we let $A_i^+(r;s)$ denote the event that payoffs support $s_i = s$ and network degree $s_{1i} \geq r$. From the law of iterated expectations, the partial mean of the rth order statistic $U_{i;r}$ given $A_i^+(r;s)$ is

$$\begin{aligned}
\mathbb{E} \quad & \left[\left(U_{i;r} - \frac{1}{2}\log n\right) \mathbb{1}\{A_i^+(r;s)\}\right] = \sum_{j=1}^{J_W} \mathbb{E}[(\tilde{U}_{ij} + \sigma\eta_{ij} - \log J)\mathbb{1}\{B_i^+(r;s), U_{i;r} = \tilde{U}_{ij} + \sigma\eta_{ij}\}] \\
= \quad & \frac{1}{J^{r-1}} \sum_{j=1}^{J_W} \sum_{j_1 \neq \dots \neq j_{r-1} \neq j} \int_{-\infty}^{\infty} s \left(\prod_{q=1}^{r} J(1 - G(s - \sigma^{-1}\tilde{U}_{ij_q})\right)\left(\prod_{q=r+1}^{J_W} G(s - \sigma^{-1}\tilde{U}_{ij_q})\right) \\
& \times g(\sigma^{-1}(s - \tilde{U}_{ij} + \log J)ds + o(1) \\
= \quad & \frac{1}{(r-1)!}\int_{-\infty}^{\infty} s \left(\frac{1}{J}\sum_{j=1}^{J_W} \exp\{\tilde{U}_{ij}\}\right)^r \exp\left\{-rs - e^{-s}\left(1 + \frac{1}{J}\sum_{k=1}^{J_W}\exp\{\tilde{U}_{ik}\}\right)\right\} + o(1)
\end{aligned}$$

where the last step follows from the approximation in equation (B.4).

Now note that for any $\lambda \geq 0$ we can write

$$\frac{d}{d\lambda} v^\lambda \bigg|_{\lambda=r} = \log(v) v^\lambda \bigg|_{\lambda=r} = \log(v) v^r$$

Hence, if we define $a := \frac{1}{J}\sum_{j=1}^{J_W} \exp\{\tilde{U}_{ij}\}$, and after a change of variables $v = e^{-w}$, we have

$$\begin{aligned}
\mathbb{E}\left[\left(U_{i;r} - \frac{1}{2}\log n\right)\mathbb{1}\{A_i(r;s)\}\right] &= \frac{1}{(r-1)!}\int_{-\infty}^{\infty} a^r w \exp\{-rw - (1+a)e^{-w}\} dw + o(1) \\
&= -\frac{1}{(r-1)!}\left(\frac{a}{1+a}\right)^r \int_0^{\infty} [\log(v) - \log(1+a)]\, v^{r-1} e^{-v} dv + o(1) \\
&= \left(\frac{a}{1+a}\right)^r \left(\frac{1}{(r-1)!}(\log(1+a)\Gamma(r) - \Gamma'(r)) + o(1)\right) \\
&= \left(\frac{a}{1+a}\right)^r \left(\log(1+a) + \gamma - \sum_{q=1}^{r-1}\frac{1}{q} + o(1)\right)
\end{aligned}$$

where $\Gamma(r+1) := \int_0^\infty v^r e^{-v} dv$ denotes the Gamma function. Since $P(A_i^+(r;s)) = \left(\frac{a}{1+a}\right)^r$, it follows that

$$\lim_n \mathbb{E}\left[\left(U_{i;r} - \frac{1}{2}\log n\right)\bigg| A_i^+(r;s)\right] = \log(1+a) + \gamma - \sum_{q=1}^{r-1}\frac{1}{q}$$

Finally note that by Lemmas 6.3 and 6.5, $\frac{1}{J}\sum_{j=1}^{J_W}\exp\{\tilde{U}_{ij}\} \xrightarrow{p} H^*(x_i; s_i)$. Since the draws $U_{i:1}(s), \dots, U_{i:J_W}(s)$ are independent, this also establishes the first claim of the Lemma.

Similarly, the partial mean of MC_i given that MC_i is the $(s+1)$th order statistic is given by

$$\mathbb{E}\left[\left(MC_i - \frac{1}{2}\log n\right)\mathbb{1}\{A_i(t;s)\}\right] = \frac{1}{s!}\int_{-\infty}^{\infty} a^s w \exp\{-(s+1)w - (1+a)e^{-w}\}dw + o(1)$$
$$= \frac{a^s}{(1+a)^{s+1}}\left(\log(1+a) + \gamma - \sum_{q=1}^{s}\frac{1}{q} + o(1)\right)$$

where $P(A_i(r;s)) = \frac{a^s}{(1+a)^{s+1}}$, which establishes the second claim $\square$

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