

# The Analytic Theory of a Monetary Shock\*

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## Abstract

We propose a new method to analyze the propagation of a once and for all shock in a broad class of sticky price models. The method is based on the eigenvalue-eigenfunction representation of the cross-sectional process for price adjustments and provides a thorough characterization of the *entire* impulse response function of *any moment* or function of interest, in response to a once-and-for-all aggregate shock (*any* displacement of the initial distribution). We use the method (i) to discuss a general analytic characterization of the “selection effect” in sticky-price models, (ii) to show that the response of the cross-sectional dispersion of prices to a small shock is zero at all horizons, (iii) to derive a parsimonious representation of the output response to monetary shocks, and the key parameters determining its shape, (iv) to study the propagation of monetary shocks after a change in volatility.

*JEL Classification Numbers: E3, E5*

*Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection, Volatility.*

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# 1 Introduction

Economists are often faced with the analysis of dynamic high dimensional objects, such as cross sectional distributions of incomes, assets, markups, and other economic variables of interest. This is the case for instance when studying impulse response functions, resulting from the dynamics of selected moments computed on the distribution of interest. We present a powerful method for such analyses, which typically require solving the partial differential equation that characterizes the evolution of the distribution of interest. The method is the eigenvalue-eigenfunction decomposition that allows to solve the partial differential equation through a neat separation of the time-dimension from the state-dimension, providing a tractable solution to a non-trivial problem. Two examples of use of eigenfunction-eigenvalues in economics are [Gabaix, Lasry, Lions, and Moll \(2016\)](#), who use it to study the transition of the cross-section distribution of incomes, and [Hansen and Scheinkman \(2009\)](#) who use it to define and study long run risk in asset pricing.<sup>1</sup>

We apply this method to analytically compute the entire impulse response function to a once and for all monetary shock in a broad class of sticky price models including versions of [Taylor \(1980\)](#), [Calvo \(1983\)](#), [Golosov and Lucas \(2007\)](#), a version of the “CalvoPlus” model by [Nakamura and Steinsson \(2010\)](#), a generalization of the Calvo plus model to arbitrary random menu costs as in [Dotsey, King, and Wolman \(1999\)](#), multi-product models as in [Midrigan \(2011\)](#), [Bhattarai and Schoenle \(2014\)](#) and [Alvarez and Lippi \(2014\)](#), and the model with “price-plans” as in [Eichenbaum, Jaimovich, and Rebelo \(2011\)](#) and [Alvarez and Lippi \(2019\)](#). In these models firms are hit by idiosyncratic shocks and face a price setting problem featuring (possibly random) menu costs, as well as “price plans” (i.e. the possibility of choosing 2 prices instead of a single one upon resetting). As in most of the general equilibrium literature on the topic we abstract from strategic complementarities to retain tractability. These problems are typically computationally intensive and numerical solutions

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<sup>1</sup> See also [Caballero \(1993\)](#) for an early use eigenvalue-eigenfunctions to analyze the dynamics of a cross-section distribution and [Krieger \(2002\)](#) for an early exploration of the dynamics of a high-dimensional state in macroeconomic models using *eigenstates*.

may hinder a clear understanding of the mechanism at work. The approach we propose greatly facilitates the solution of such models, which in many cases has a simple-to-derive analytic form, while at the same time unveiling the key forces and deep parameters behind the results.

Our method delivers an analytic representation of the *whole profile* of the impulse response function, as opposed to previous analytic results on the impact effect of shocks, such as Caballero and Engel (2007), or analytic results on the cumulated impulse response to shocks such as Alvarez, Le Bihan, and Lippi (2016) and the extension developed by Baley and Blanco (2019a) for problems with asymmetries. The results also provide straightforward characterizations for several features of interest for a large class of sticky price models, such as duration analysis and the dynamics of *any* moment of interest after an aggregate shock. Moreover the analysis applies to *shocks of any size* as well as to *shocks to higher moments*, such as uncertainty shocks, differently from previous analytic investigations focusing mostly on approximations for small monetary shocks, as in e.g. Gertler and Leahy (2008); Alvarez, Le Bihan, and Lippi (2016); Baley and Blanco (2019b).

The paper is organized as follows: Section 2 presents the setup of the analysis, the mathematical definition of impulse response and the benchmark economy. Section 3 establishes some preliminary results that are useful to clarify the scope of the analytic results that follow. We show that while our main results are derived under a benchmark model with zero inflation and a symmetric return function, this benchmark provides an accurate description of economies where inflation is low and the firm's return function features a moderate asymmetry, i.e. we show a local insensitivity of the impulse response function to both drift and asymmetry. Section 4 presents our main result, namely the analytic representation of the impulse response function which neatly separates the effect of the shock (shape of the initial impulse), the effect of the shape of the function or moment of interest, and the effect of the time horizon. The rest of the paper illustrates the power of the method by discussing three substantive economic applications that we briefly describe next.

First we provide an analytic characterization of the “selection effect”, which is one of the main reasons why different sticky price models yield different real effects (Section 5). The selection effect, first discussed by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. We present an analytic result showing how the selection effect creates a wedge between the duration of price spells and the duration of the aggregate output response. The two durations coincide when there is no selection. We show that such a wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of aggregate output. In the same section we present a surprising result showing that, following a small monetary shock, the response of even centered moments is zero at all horizons. This applies for instance to the variance of price gaps, to the cross-sectional price dispersion, to the kurtosis of the price changes, as well as to the survival function. Since the cross-sectional price dispersion is the main measure of inefficiency in models with price stickiness, this result implies that in these models such a cost should be measured in an “average” sense, and not as a consequence of a particular shock. The analytic approach clearly identifies the forces behind this surprising result.

Second, we analyze the shape of the impulse response function, and discuss the accuracy of an approximate characterization obtained by using “selected” eigenvalues (Section 6). The latter question arises in, for example, Gabaix et al. (2016), where a single eigenvalue, namely the “dominant eigenvalue”, is used to provide a convenient description of the speed of the dynamic adjustment. It is thus interesting to ask whether a single eigenvalue can be found to approximate the impulse response function. We present several results. We show that an interesting special case exists: the canonical menu cost model can be effectively summarized

by a single eigenvalue-eigenfunction pair. We explore the accuracy of approximations that use a single “leading” eigenvalue in the Calvo-plus model. We show that in general it is not possible to summarize the impulse response function, nor its asymptotic behavior, with a single eigenvalue-eigenfunction pair. As a concrete example of a case where no single eigenvalue can be used to characterize the IRF we discuss a class of sticky price models that gives rise to a hump-shaped impulse response function.<sup>2</sup> We provide analytic conditions for the hump-shaped impulse response to arise.

Third we study how the propagation of monetary shocks is affected by the aggregate shocks to the volatility faced by firms (Section 7). Bloom (2009) analyzed the macroeconomic impact of volatility shocks, and other scholars have shown that recessions are times in which the volatility of shocks is higher. We use our method to analyze whether monetary shocks propagate more quickly in times when volatility is high. The question relates to the recent quantitative investigation by Vavra (2014) and the empirical studies on volatility and the propagation of monetary shocks by Tenreyro and Thwaites (2016); Eickmeier, Metiu, and Prieto (2019). We show that the answer crucially depends on the time elapsed since the volatility shock occurred: we show that if the monetary shock and the volatility shock occur nearly simultaneously, a case that we label the “short run”, then the cumulative effect of a monetary shock is, to first order, unaffected. Indeed this is the combination of two forces, faster propagation coming from higher volatility, but lower initial effect of the monetary shock because sS bands have become wider due to the option value effect. However, if the monetary shock occurs long enough after the increase in volatility, so that the cross-sectional distribution of firms has settled at the new invariant distribution, a case that we call the “long-run”, then the result changes. In the long run the shock propagation is faster and the cumulative effect of a monetary shock is smaller by a factor proportional to the increase of volatility. This is because after a long period of high volatility, the cross sectional distribution settled to have a wider support consistent with the wider sS bands. Our method allows us

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<sup>2</sup>This hump-shaped behavior emerges in economies where the price setting technology features “price plans” and some randomness in the cost of price changes (weak selection).

to quantify the time that it takes for the long-run effect to settle in. We show that this time is very short, i.e. a very small fraction of the typical length between price changes, a finding that suggests that higher volatility does lead to “less powerful” monetary policy in most realistic cases. The result of the short high life is, at least for the pure menu cost model, independent of any parameter values.

[Section 8](#) concludes by discussing developments and avenues for future work.

## 2 Set up

This section introduces the main objects of the analysis. First we set up a standard mathematical definition of the impulse response. Second, we present a benchmark sticky-price model that is used to illustrate several applications of interest.

The standard set up is made by the following objects: the law of motion of the Markov process  $\{x(t)\}$  for each individual firm, the function of interest  $f(x)$ , the cross-sectional initial distribution of  $x$ , denoted by  $P(x; 0)$ . At this general level the set-up and definition of an impulse response is closely related to the one in [Borovicka, Hansen, and Scheinkman \(2014\)](#). The law of motion for the process  $f(x)$ , with  $x \in X \equiv [\underline{x}, \bar{x}]$ , is also Markov and is described using

$$\mathcal{H}(f)(x, t) = \mathbb{E}[f(x(t)) | x(0) = x] \tag{1}$$

where the operator  $\mathcal{H}$  computes the  $t$  period ahead expected value of the function  $f : X \rightarrow \mathbb{R}$  conditional on the state  $x = x(0)$ . Next we describe the initial distribution of  $x$ , which we denote by  $P(\cdot; 0) : X \rightarrow \mathbb{R}$ . This represents the measure of firms that start with value smaller or equal than  $x$  at time  $t = 0$ , each of them following the stochastic process described in  $\mathcal{H}(f)$ , with independent realizations. We allow the distribution  $P(x; 0)$  to have mass points. In particular  $P$  has a piecewise continuous derivative (density) which we extend to the entire domain, so that  $p(\cdot, 0) : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$ , where  $P$  can have countably many jump discontinuities (mass points), denoting the difference between the right and left limits by

$p_m(\cdot; 0) : \{x_k\}_{k=1}^{\infty} \rightarrow \mathbb{R}$ , so that  $x_k$  is the location of the mass points.<sup>3</sup> While the possibility of handling a distribution with mass point is of theoretical interest, most of the economic applications that we discuss have no mass points and hence the density  $p$  will suffice. Summarizing, we assume that the “shocks” are idiosyncratic, and that the initial condition is given by a cross sectional distribution  $P(\cdot, 0)$ .

We are interested in the standard impulse response function  $H$  defined for each  $t > 0$  as:

$$H(t; f, P - \bar{P}) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x, t) [dP(x; 0) - d\bar{P}(x)] \quad (2)$$

where  $\bar{P}$  is the invariant distribution of  $x$ . The impulse response  $H$  is the expected value of the cross-sectional distribution of  $f$  *in deviation from its steady state value*, where each  $x(t)$  has followed the Markov process associated with  $\mathcal{H}(\cdot)$  and whose cross sectional distribution at time zero is given by  $P(\cdot; 0)$ .<sup>4</sup> In other words, for ergodic processes we are forcing the impulse response to go to zero as  $t$  diverges. Since we evaluate  $H$  only for the difference between two measures, i.e. only for signed measures, we introduce the convenient notation  $\hat{P} \equiv P - \bar{P}$  and likewise for the densities  $\hat{p} = p - \bar{p}$ . Thus

$$\hat{P}(x) \equiv P(x, 0) - \bar{P}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \quad (3)$$

so that  $\hat{P}$  defines the “initial condition” for the impulse response, an object that we further discuss below.

We note that the impulse response can also be written in terms of the evolution of the cross sectional distribution  $P(\cdot, t)$ , namely:  $H(t; f, P - \bar{P}) = \int_{\underline{x}}^{\bar{x}} f(x) [dP(x; t) - d\bar{P}(x)]$ . The two definitions are equivalent. To characterize [equation \(2\)](#) requires solving a Kol-

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<sup>3</sup>Given our assumption on  $P$  we can write the expectations of any function  $\nu(x)$  as:

$$\int_{\underline{x}}^{\bar{x}} \nu(x) dP(x; 0) = \int_{\underline{x}}^{\bar{x}} \nu(x) p(x; 0) dx + \sum_{k=1}^{\infty} \nu(x_k) p_m(x_k; 0).$$

<sup>4</sup>The ergodicity of  $\{x\}$  implies that we can write  $H(t; f, P - \bar{P}) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x, t) dP(x; 0) - \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x)$ .

Kolmogorov backward (KB) equation, while the latter requires solving a Kolmogorov forward (KF) equation. In spite of the equivalence, there are two reasons why [equation \(2\)](#) is sometimes preferred. The first reason depends on how well behaved are  $\hat{P}$  vs  $f$ . If we start with a distribution with mass points, as will be the case after a large shock, then the initial condition will not be a density. The dynamics of the distribution will then involve Dirac functions. Instead, using the KB we do not need to deal with this. Notice the asymmetry:  $f$  are functions, where  $\hat{P}$  is a generalized function, i.e. can include Dirac functions. Second, even if we do not deal with generalized functions, so that both  $f$  and  $\hat{P}$  are functions, the boundary conditions can be trickier in the KF than in the KB. The boundary conditions for the KF are stated in terms of right and left derivatives, and indeed the corresponding eigenfunctions are not differentiable at the reinjection point. While using the KB the boundary conditions are standard -just as the ones in a Bellman equation- and the eigenfunctions are differentiable at the reinjection point.

We define another impulse response function, which uses a stopping time  $\tau$ , and a modified expectation operator  $\mathcal{G}$  defined as:

$$\mathcal{G}(f)(x, t) = \mathbb{E} [1_{\{t \leq \tau\}} f(x(t)) | x(0) = x] \quad (4)$$

The operator  $\mathcal{G}$  computes the  $t$  period ahead expected value of the function  $f : X \rightarrow \mathbb{R}$  starting from the value of the state  $x = x(0)$ , conditional on  $x$  surviving. The indicator function  $1_{\{t \leq \tau\}}$  becomes zero when the first adjustment following the shock occurs at the stopping time  $\tau$ .

In the context of the price setting models with  $sS$  rules we refer to the operator  $\mathcal{H}$  as the one for the problem with “rejection”, i.e. one in which the operator follows a firm forever, i.e. does not stop keeping track of the firm after the first adjustment occurs (at time  $\tau$ ). In contrast, we refer to the operator  $\mathcal{G}$  as one for the problem without “rejection”, i.e. tracking the firm until the first adjustment. For example, in the sticky price models discussed below the stopping time  $\tau$  will be given by the occurrence of a price adjustment.



We define the impulse response function  $G$  for each  $t > 0$  as:

$$G(t; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} \mathcal{G}(f)(x, t) [dP(x; 0) - d\bar{P}(x)] \quad (5)$$

The interpretation of  $G(t)$  is the expected value of the cross-sectional distribution of  $f$ , conditional on surviving, where each  $x(t)$  follows the Markov process, and as before the cross-sectional distribution at time zero is given by  $P(\cdot; 0)$  so that the initial condition is  $\hat{P} \equiv P(\cdot; 0) - \bar{P}$ .

While  $H$  is the impulse response as commonly defined, it turns out that  $G$  is simpler to characterize. Moreover, the next section will discuss two useful results: **Proposition 1** establishes conditions under which the impulse response  $G(t; f, \hat{P})$  coincides with  $H(t; f, \hat{P})$  for all  $t$ . **Proposition 3** shows that under even weaker conditions the cumulative impulse response of *any moment of interest* computed on  $G(t; f, \hat{P})$  coincides with the cumulative response  $H(t; f, \hat{P})$ .

An equivalent, and perhaps simpler, representation of the impulse response function  $G(t)$  is obtained by using the transition function  $Q_t(y|x) = \Pr\{x(t) < y, t < \tau | x(0) = x\}$ , with density function  $q_t(y|x) = \partial_y Q_t(y|x)$ . The impulse response function is

$$G(t; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} f(y) q_t(y|x) dy d\hat{P}(x) \quad (6)$$

Our analytical characterization of  $G(t)$ , which will be given in **Proposition 4**, can be easily understood by using a finite-dimensional version of **equation (6)**, as we will discuss after **Corollary 1**.

In general our interest is to compute  $H(t; f, \hat{P})$  by using the simpler operator  $G(t; f, \hat{P})$  which takes as an argument the *signed* measure  $\hat{P}$ . Nevertheless, we will also use  $G(t; f, \hat{P})$  to construct other objects of interest such as the survival function of price changes or the conditional distribution of the state  $q_t$ . In these cases we abuse notation and let  $\hat{P}$  denote a probability measure.

**The initial condition.** Our setup encodes the impulse in the initial condition,  $\hat{P}(x) \equiv P(x, 0) - \bar{P}(x)$ , which denotes the distribution of the state variable  $x \in (\underline{x}, \bar{x})$  at time zero in deviation from the invariant distribution. In particular  $P(\cdot; 0)$  describes the cross-sectional distribution of the state immediately after the shock. As time elapses the initial distribution will converge to the invariant distribution  $\bar{P}(x)$ , tracing out the impulse response for the function of interest  $f(x)$ . We mentioned above that our method allows the initial distribution to have mass points. This can be useful, for instance, if the initial shock is large enough to displace a non-negligible mass of agents onto the return point  $x^*$ , to compute the survival function of price changes, or to compute the conditional distribution of the state  $q_t$ . Also notice that our formulation allows us to handle a variety of shocks. Several papers have focussed on a small uniform displacement  $\delta > 0$  of the whole distribution relative to the invariant  $\bar{P}(x)$ , what [Borovicka, Hansen, and Scheinkman \(2014\)](#) label the “marginal response function”, a case that has been studied in the literature, as discussed in the introduction, which we formalize next. We stress, however, that our method can handle any type of initial condition, such as one triggered by a large shock, or the one triggered by a higher-order shock.

**Marginal impulse response function.** Starting from the invariant distribution  $\bar{p}(x)$  and considering a small uniform displacement  $\delta > 0$  of the whole distribution so that  $\hat{p}(x) = \bar{p}(x + \delta) - \bar{p}(x) = \bar{p}'(x)\delta + o(\delta)$  we have the following definition

**DEFINITION 1.** Let the marginal impulse response function to a monetary shock be:

$$Y(t; f) \equiv \frac{\partial}{\partial \delta} H(t; f, \bar{p}(x + \delta) - \bar{p}(x)) \Big|_{\delta=0} \text{ for all } t \geq 0 \quad (7)$$

The marginal impulse response function is the first order expansion of  $H(t; f, \bar{p}(x + \delta) - \bar{p}(x))$  with respect to  $\delta$ . For future reference notice that since  $\bar{p}(x + \delta) - \bar{p}(x) = \delta \bar{p}'(x) + o(\delta)$  we

will often use  $H(t; f, \bar{p}'(x)\delta)$  instead of  $H(t; f, \hat{p}(x))$ . Since  $H(t; f, \mathbf{0}) = 0$ , where  $\mathbf{0}$  denotes the zero function, thus the expansion gives:

$$H(t; f, \bar{p}(x + \delta) - \bar{p}(x)) = Y(t; f)\delta + o(\delta)$$

where the marginal impulse response  $Y(t)$  is measured per unit of the monetary shock.

## 2.1 The firm's price setting problem

This section lays out the price setting problem solved by a firm in the “Calvo plus” model. In this model, which can be seen as one with random menu costs, the firm is allowed to change prices either by paying a fixed menu cost or upon receiving a random free adjustment opportunity (a menu cost equal to zero). The setup nests several models of interest, from the canonical menu cost problem to the pure Calvo model.

**The firm sS problem in the Calvo plus model.** We describe the price setting problem for a firm in steady state. The firm cost follows a Brownian motion with variance  $\sigma^2$  and drift  $\mu$ , where the latter is due to inflation. The firm can change its price at any time paying a fixed cost  $\psi > 0$ . At exogenous random times, which occur with a constant probability per unit of time  $\zeta$ , the firm faces a zero menu cost. The price gap  $x$  is defined as the price currently charged by the firm relative to the price that will maximize current profits, which is proportional to the firm cost (measured as the log of the ratio between these prices). The optimal policy is to change the price when the gap  $x$  reaches either of two barriers,  $\underline{x} < \bar{x}$ , or when the menu cost is zero. In either case, at the time of a price change, the firm sets a new price which determines a price gap  $x^*$ , which is the optimal return point after the adjustment. Thus price changes are given by  $x^* - x(\tau)$ , where  $\tau$  is the time at which either one of the barriers is hit, or a free adjustment opportunity arrived.

The flow cost of the firm is given by  $R(x)$ . The firm maximizes expected discounted profits, using a discount rate  $r > 0$ . Thus, the optimal policy can be describe by three

numbers:  $\underline{x} < x^* < \bar{x}$ , whose values can be found by minimizing the value function in the inaction region:

$$(r + \zeta)v(x) = R(x) + \mu v'(x) + \frac{\sigma^2}{2}v''(x) + \zeta v(x^*) \text{ for } x \in [\underline{x}, \bar{x}] \quad (8)$$

and imposing the relevant boundary conditions: value matching, smooth pasting, and optimality of the return point:

$$v(\bar{x}) = v(\underline{x}) = v(x^*) + \psi, \text{ and } v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0 \quad . \quad (9)$$

The density  $\bar{p}$  of the invariant distribution for price gaps generated by the policy  $\{\underline{x}, x^*, \bar{x}\}$  and the law of motion of  $x$  is the solution to the Kolmogorov forward equation:

$$\zeta \bar{p}(x) = -\mu \bar{p}'(x) + \frac{\sigma^2}{2} \bar{p}''(x) \text{ for } x \in [\underline{x}, \bar{x}], x \neq x^* \quad (10)$$

with boundary conditions at the exit points, unit mass, and continuity requirements:

$$\bar{p}(\underline{x}) = \bar{p}(\bar{x}) = 0, \int_{\underline{x}}^{\bar{x}} \bar{p}(x) dx = 1, \text{ and } \bar{p} \text{ continuous at } x = x^* \quad .$$

The boundary conditions at the exit points are immediate in  $sS$  models with fixed costs since no mass can accumulate at the boundary of the inaction region as long as  $\sigma > 0$ .

### 3 Scope of the analysis

This section presents three results that are useful to frame the scope of the main results of this paper. We start by defining the notion of a *symmetric sS problem*. Informally, this amounts to assuming that the firm problem has zero drift and that the firm's return function is symmetric. The first result shows that, for symmetric  $sS$  problems,  $H(t) = G(t)$  for all horizons  $t \geq 0$ . This is useful because while  $H$  is the object of interest, the function  $G$ , the

simple IRF computed stopping after the first adjustment, turns out to be much easier to characterize. The second result shows that the impulse response for symmetric  $sS$  problems provides a good approximation to the impulse response for non-symmetric problems. In particular, when the lack of symmetry comes from either the presence of a drift or from an asymmetric objective function, we show that the deviations in the marginal impulse response are of second order in the strength of asymmetries. The third result shows that the *cumulative* impulse response  $H$  coincides with the *cumulative* impulse response  $G$ , even if the problem is not symmetric and/or shocks are not marginal.

**DEFINITION 2.** *Symmetric  $sS$  Calvo plus problem.* Assume that the unregulated state has zero drift, i.e.  $\mu = 0$ , and that the optimal return point  $x^*$  is equidistant from the barriers, i.e.  $(\bar{x} + \underline{x})/2 = x^*$ . Let  $\{x(t)\}$  be the regulated state,  $g(x; t, x^*)$  be the density of distribution of  $x(t) = x$ , conditional on  $x(0) = x^*$ . In this case the density satisfies  $g(z + x^*; t, x^*) = g(-z + x^*; t, x^*)$  for all  $z \geq 0$ .

One can show that the *optimal* decision rule for the problem in [equation \(8\)](#) have barriers that are equidistant from  $x^*$  if the flow return  $R(x)$  is symmetric in  $x$  and if  $\mu = 0$ . The symmetry on  $g(\cdot; t, x^*)$  follows from the combination of the symmetry of the distribution of a BM without drift, and the symmetry of the boundaries relative to  $x^*$ .

While we have chosen the Calvo plus problem as our benchmark case to preserve a simple, yet economically interesting, framework, we know that most of our results apply to a more general class of  $sS$  problems featuring analogous symmetric properties. This can be seen by noticing that for the results of the next two subsections, symmetry can be defined purely in terms of the conditions on the density  $g$ . Examples of more general  $sS$  problems where all the results presented in this section will apply are the price-plan model developed in [Alvarez and Lippi \(2019\)](#) and briefly described in [Section 6](#), or the multiproduct price-setting problem developed in [Alvarez and Lippi \(2014\)](#) and described in [Appendix B](#), which is a generalization of the Calvo-plus model that considers a firm controlling a vector of  $n$  prices instead of a single one. In [Appendix C](#) we consider another extension of the benchmark model that

allows for a non-degenerate distribution of random fixed costs, which generalizes the seminal analysis of [Dotsey, King, and Wolman \(1999\)](#).

### 3.1 Equivalence of IRF's $H$ and $G$ for symmetric $sS$ problems

Next we present a proposition for symmetric problems establishing conditions for the standard IRF  $H(t)$  to coincide with  $G(t)$ . The conditions for this to happen are that either the function of interest  $f$ , or that the initial condition  $\hat{P}(\cdot)$  are antisymmetric, where a function  $\nu(x)$  is antisymmetric about  $x^*$  if it satisfies  $\nu(x - x^*) = -\nu(x^* - x)$  for all  $x \in [\underline{x}, \bar{x}]$ .

**PROPOSITION 1.** Consider, for simplicity, the Calvo plus model. Assume the firm's problem is  $sS$  symmetric as in [Definition 2](#). Then if either

- (i) the function of interest  $f : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$  is antisymmetric and  $\hat{P}(\cdot)$  is arbitrary
- (ii) the signed measure (initial condition)  $p(\cdot, 0) - \bar{p}(\cdot) : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$  as well as its mass points  $p_m(\cdot, 0) - \bar{p}_m(\cdot) : \{x_k\}_{k=1}^{\infty} \rightarrow \mathbb{R}$  are antisymmetric and  $f(\cdot)$  is arbitrary

we have that  $G(t; f, \hat{P}) = H(t; f, \hat{P})$  for all  $t$ .

See [Appendix A](#) for the proof. The proposition's requirement that either the function of interest  $f$ , or the initial condition  $\hat{P}$ , is anti-symmetric is not that restrictive for our applications. The main function of interest for the paper, used to compute the IRF for output, for example  $f(x) = -x$  in the Calvo-plus model, is anti-symmetric in the class of models we analyze. Also, our benchmark case in this class of models is that the density  $\hat{P}$  is anti-symmetric when we consider a small monetary shock..

### 3.2 Local insensitivity to asymmetries of output's IRF

In this section we show that the output's impulse response to a monetary is locally insensitive to two forms of deviations from the assumptions defining a symmetric  $sS$  problem. The first concerns deviations from the zero drift, or zero inflation, assumption. The second concerns

the deviation from the symmetry of the firm's objective function. The insensitivity applies to the impulse response of any antisymmetric function  $f$ , not just the one for output. This gives the precise sense in which our result for symmetric  $sS$  problems extend to a range of small inflation rates.

We let  $\mu$  be the steady state inflation. We also let  $a$  be a coefficient that measures the degree to which the firms's period return function is asymmetric. In particular, we let  $R(x, a)$  be the return function which satisfies

$$R(x, a) = R(-x, -a) \text{ for all } x, a \quad (11)$$

Note that  $a = 0$  implies that the return function is symmetric. For  $a \neq 0$ , the function can be interpreted as the sum of a symmetric and antisymmetric function. An example of this is

$$R(x, a) = Bx^2 + ax^3$$

which can represent a third order expansion of the original profit function around the static maximizing choice  $x = 0$ . In this case  $a$  is 6 times the third derivative of the original profit function at  $x = 0$ . When either  $a \neq 0$  or  $\mu \neq 0$  the thresholds of the optimal  $sS$  rule will not be equidistant from the optimal return point  $x^*$ .

The next proposition computes two derivatives of the output's IRF to a small monetary shock at any horizon  $t$  and shows that they are both zero. One is the derivative with respect to steady state inflation, and the other one is the derivative with respect to the degree of asymmetry of the objective function. Indeed we prove a more general result, that the IRF of any antisymmetric function  $f$ , instead of just  $f(x) = -x$ , to a monetary shock is locally insensitive to inflation or asymmetries in the objective function.

For this proposition we need to consider the general case with re-injection, since we don't have the required conditions of [Proposition 1](#) that ensure that  $H = G$ . Also note that the steady state distribution  $\bar{p}$  will be, in general, also a function of the drift and the

degree of asymmetry  $(\mu, a)$ , and hence we include them as argument on the initial impulse  $\hat{p}(x; \delta, \mu, a)$ , as well as the function  $H$  itself, since this function depends on the process for  $x$ , and in particular the value of its drift, optimal return point  $x^*$ , and thresholds  $\underline{x}, \bar{x}$ . We also explicitly include the shock  $\delta$  as an argument of  $\hat{p}$ , which will clarify the arguments below. The initial condition for a monetary shock are thus  $\hat{p}(x; \delta, \mu, a) \equiv \bar{p}(x + \delta; \mu, a) - \bar{p}(x; \mu, a)$ . Thus the marginal impulse response is:

$$Y_{\mu,a}(t; f) \equiv \frac{\partial}{\partial \delta} H(t; f, \hat{p}(\cdot, \delta, \mu, a), \mu, a) \Big|_{\delta=0} \text{ for all } t \geq 0 \quad (12)$$

The next proposition states that impulse response  $Y_{\mu,a}(t)$  at any horizon  $t$  is approximately the same as our benchmark case with zero inflation and with symmetric return function. Formally we have:

**PROPOSITION 2.** Consider, for simplicity, the Calvo plus model. Let  $\mu$  be the drift of the state (the inflation rate), and  $a$  the index of asymmetry in the return function  $R(x, a)$ . Then:

$$\frac{\partial}{\partial \mu} Y_{\mu,a}(t; f) \Big|_{\mu=0, a=0} = 0 \text{ for all } t \geq 0 \quad (13)$$

$$\frac{\partial}{\partial a} Y_{\mu,a}(t; f) \Big|_{\mu=0, a=0} = 0 \text{ for all } t \geq 0 \quad (14)$$

for any antisymmetric function  $f$ .

See [Appendix A](#) for the proof. Notice that the proof holds for any function of interest  $f$  that is antisymmetric. So this holds true for the output impulse response, given by  $f(x) = -x$ , as well as for other antisymmetric functions.

While [Proposition 2](#) shows the insensitivity of deviations to inflation and asymmetries in the objective function for infinitesimal changes, combining the results in [Appendix D](#) and [Appendix E](#) we can analyze the effect on the impulse response for deviations of any size. We relegate these results to appendices because characterizing the entire impulse response function requires extending the techniques used for the baseline case. From the results on



these appendices we conclude that for moderate levels of inflation and asymmetries in the objective function, the results are materially the same.

### 3.3 Equivalence of Cumulative IRF's for $H$ and $G$ .

We use the impulse response function  $H(t; f, \hat{P})$  to define the discounted cumulative response, given by the (discounted) area under the impulse response function:

$$\mathbf{H}(r; f, \hat{P}) = \int_0^{\infty} e^{-rt} H(t; f, \hat{P}) dt. \quad (15)$$

Although the cumulative response is not informative about *the shape* of the impulse response function, which is the main focus of this paper, its characterization gives an informative (scalar!) summary of the economy's response to a permanent shock.

We have introduced the notion of cumulative impulse response for output following a monetary shock in [Alvarez, Le Bihan, and Lippi \(2016\)](#). We have shown that  $\mathbf{H}$  can be analytically computed by solving an ordinary differential equation that gives the cumulated impulse response for an agent with initial condition  $x \in (\underline{x}, \bar{x})$ .<sup>5</sup> Integrating across the different values of  $x$  after the shock, using the displaced distribution  $\hat{P}$ , provides an analytic solution for  $\mathbf{H}$ . The approach is extremely useful since it does not require to solve (or simulate) the whole impulse response function. Our initial use of the cumulative response was restricted to problems with zero drift (inflation). In [Alvarez and Lippi \(2019\)](#) we have extended the result to the cumulated output response for problems featuring non-zero inflation. In this section we extend our original proposition using a result from [Baley and Blanco \(2019b\)](#), to compute the cumulative response  $\mathbf{H}$  for *any function of interest*  $f$ . The payoff is to go beyond the analysis of the output effect typically obtained by using  $f(x) = -x$ . For instance we can now analyze the cumulative effect of the dispersion of markups,  $f(x) = x^2$ , in economies with inflation or other sources of drift.<sup>6</sup>

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<sup>5</sup>The logic behind this differential equation is analogous to the one used to compute expected values using a Bellman (or Kolmogorov) equation.

<sup>6</sup>An earlier draft of this paper restricted the result to  $f(x) = -x$ , i.e. to the impulse response of output.

Next let us define the cumulative effect as the area under the impulse response  $G(t)$ , namely the one for the problem without reinjections defined in [equation \(5\)](#):

$$\mathbf{G}(r; f, \hat{P}) = \int_0^\infty e^{-rt} G(t; f, \hat{P}) dt. \quad (16)$$

We now discuss a useful result concerning the undiscounted (i.e.  $r = 0$ ) cumulated impulse response function defined in [equation \(15\)](#) and [equation \(16\)](#). The result states that for any moment of interest  $f(x)$ , and for any parametrization of the Calvo-plus model described in the previous section, the cumulated impulse response for the problem without reinjection coincides with the cumulated impulse response for the problem with reinjection. We state the result as follows (see [Appendix A](#) for the proof):

**PROPOSITION 3.** Consider the Calvo plus problem described above with a rate  $\zeta$  of free adjustment opportunities, decision rule given by  $\{\underline{x}, x^*, \bar{x}\}$ , and the law of motion of the state  $x$  given by parameters  $\{\mu, \sigma\}$ . Let the economy be subject to an initial shock  $\hat{P} \equiv P(x, 0) - \bar{P}$  (displacement of the initial distribution) and the function of interest be  $f$ . Let  $m(x; f)$  be the expected cumulated value of  $f(x)$  in deviation from its mean  $\bar{f}$ , computed between time zero, where  $x(0) = x$ , and the first time of adjustment  $\tau$ :

$$m(x; f) = \mathbb{E} \left( \int_0^\tau (f(x(t)) - \bar{f}) dt \mid x(0) = x \right) \quad \text{where} \quad \bar{f} \equiv \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x)$$

We have

$$\mathbf{H}(0; f, \hat{P}) = \mathbf{G}(0; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} m(x; f) dP(0, x) .$$

The result is surprising to us. The decision rule  $\{\underline{x}, x^*, \bar{x}\}$  need not be the optimal one, for instance it could fail to satisfy value matching and/or smooth pasting. The problem may feature asymmetric  $sS$  bands, and the state may have drift  $\mu \neq 0$ . Yet we can compute

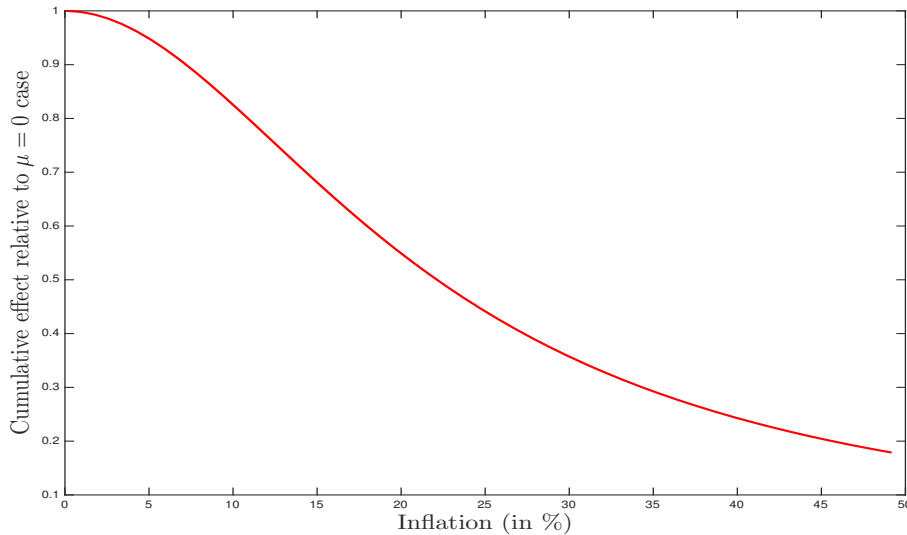
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We are indebted with Isaac Baley for pointing out Proposition 2 in [Baley and Blanco \(2019b\)](#).

the cumulative IRF as in the symmetric case with no drift.<sup>7</sup> We make several comments to this result. First, notice that the proposition holds for an arbitrary function  $f$ . Second, we stress that the proposition holds for any initial condition  $\hat{P}$ . This allows us to study either small shocks, such as a marginal displacement of the invariant distribution  $\bar{P}$ , as well as large shocks that give rise to any type of initial signed mass  $\hat{P} = P(x, 0) - \bar{P}$ .

**Proposition 3** yields a straightforward analysis of an otherwise computationally intensive question. The last equality states that we can obtain the cumulative IRF by simply keeping track of each firm until the time it makes the first adjustment following the shock. This is convenient both for analytical computation as well as for simulations. If one is interested in analytic solutions, the function  $m(x; f)$  is readily computed by solving the differential equation  $0 = f(x) + m'(x; f)\mu + m''(x; f)\sigma^2/2 - \zeta m(x; f)$ , where primes denote derivatives with respect to  $x$ , and with boundaries  $m(\underline{x}; f) = m(\bar{x}; f) = 0$ , as in [Alvarez, Le Bihan, and Lippi \(2016\)](#).

Figure 1: Cumulative output response at different inflation rates



Note: cumulated output response triggered by a marginal monetary shock, at different inflation rates, computed using **Proposition 3** with a return function  $f(x) = -x$ . The cumulated output, measured on the vertical axis, is normalized relative to the one at zero inflation rate ( $\mu = 0$ ).

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<sup>7</sup>An identical result holds for the model with price plans discussed in [Section 6](#).

Figure 1 illustrates this point by exploring how the the cumulated output response changes with the inflation rate. For simplicity we focus on a marginal shock by postulating that the distribution right after the shock is equal to a small displacement of the invariant, namely that  $p(x, 0) = \bar{p}(x + \delta)$ , so that  $\hat{p}(x) = p(x, 0) - \bar{p}(x) \approx \bar{p}'(x)\delta$ , where  $\delta$  is a (small) aggregate monetary shock. The figure shows that the cumulated output effect is not responsive to inflation when inflation is low (it is easy to prove that the function has a zero derivative at  $\pi = 0$ ). As inflation increases however the real (cumulated) effect of policy vanishes fast. At an inflation rate equal to 50% per year the effect is about 1/5th of the effect at low inflation.

## 4 Analytic Impulse Response Functions

This section presents an analytic solution for the operator  $\mathcal{G}(f)(x, t)$  defined in equation (4). The main assumption of this section is that the problem is symmetric as defined in Definition 2. By Proposition 1 we then know that the impulse response is equivalent to the one of a problem without reinjections and is given by  $G(t, f, \hat{P})$ . As argued in Section 3 the symmetric setup is a convenient starting point for the analysis and provides an accurate approximation to problems with small drift or small asymmetries, assumptions that are widely used in sticky-price models.

We assume the process for the firm's price gap is given by a drift-less brownian motion, with instantaneous variance per unit of time  $\sigma^2$ . The stopping time, i.e. the time at which prices are changed, is given by the first time at which  $x(t)$  hits either  $\underline{x}$  or  $\bar{x}$ , or that a Poisson counter, with instantaneous rate  $\zeta$ , changes its value. The definition of  $\mathcal{G}(f)(x, t)$  as an expected value implies that this function must satisfy the following partial differential

equation:

$$\partial_t \mathcal{G}(f)(x, t) = \frac{\sigma^2}{2} \partial_{xx} \mathcal{G}(f)(x, t) - \zeta \mathcal{G}(f)(x, t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t > 0 \quad (17)$$

with boundary conditions

$$\mathcal{G}(f)(\underline{x}, t) = \mathcal{G}(f)(\bar{x}, t) = 0 \text{ for all } t > 0 \text{ and} \quad (18)$$

$$\mathcal{G}(f)(x, 0) = f(x) \text{ for all } x \in [\underline{x}, \bar{x}] \quad (19)$$

where boundary conditions for  $t > 0$  are an implication from  $\bar{x}$  and  $\underline{x}$  being exit points, and hence close to them the survival rate tends to zero. The boundary condition at  $t = 0$  follows directly from the definition of  $\mathcal{G}(f)$ .

Two key steps, based on the properties of eigenvalues and eigenfunctions proved below, allow us to solve for  $\mathcal{G}(f)$ . First, that the function  $f$  can be represented by a linear combination of eigenfunctions, with typical member  $\varphi_j$ , as follows  $f(x) = \sum_{j=1}^{\infty} b_j[f] \varphi_j(x)$ , where  $b_j[f]$  are coefficients. Second, that the solution for  $\mathcal{G}(\varphi_j)$  for each eigenfunction is multiplicatively separable in  $(x, t)$ :

$$\mathcal{G}(\varphi_j)(x, t) = e^{\lambda_j t} \varphi_j(x)$$

for some constant  $\lambda_j$ . Thus the partial differential equation in (17) becomes the following ordinary differential equation:

$$\lambda_j \varphi_j(x) = \varphi_j''(x) \frac{\sigma^2}{2} - \zeta \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ with boundary conditions } \varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0 .$$

Note that the boundary condition for  $\mathcal{G}(\varphi_j)$  at  $t = 0$ , in equation (19), holds by construction. We will denote the  $\varphi_j$  as eigenfunctions and the corresponding  $\lambda_j$  as eigenvalues. The eigenvalues and eigenfunctions are:

$$\lambda_j = - \left[ \zeta + \frac{\sigma^2}{2} \left( \frac{j \pi}{\bar{x} - \underline{x}} \right)^2 \right] \text{ and } \varphi_j(x) = \frac{1}{\sqrt{(\bar{x} - \underline{x})/2}} \sin \left( \frac{[x - \underline{x}]}{[\bar{x} - \underline{x}]} j \pi \right) \text{ for } j = 1, 2, 3, \dots \quad (20)$$

This is the solution to a well known problem which has been studied extensively.<sup>8</sup> A key property of the set of eigenfunctions  $\{\varphi_j\}$  is that they form an orthonormal base for the functions  $f : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$  and for which  $\int_{\underline{x}}^{\bar{x}} [f(x)]^2 dx < \infty$  so that we can write:

$$\hat{f}(x) \equiv \sum_{j=1}^{\infty} b_j[f] \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

where  $\int_{\underline{x}}^{\bar{x}} (\hat{f}(x) - f(x))^2 dx = 0$  and the projection coefficients  $b_j[f]$  are:

$$b_j[f] = \int_{\underline{x}}^{\bar{x}} f(x) \varphi_j(x) dx \text{ for all } j \geq 1, \quad (21)$$

which uses that  $\{\varphi_j\}$  is an orthonormal base. Next we present a Lemma for the representation of the projections (expectations) of the function of interest  $f$ :

**LEMMA 1.** Assume that  $f : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$  is piece-wise differentiable, with countably many discontinuities, and  $\int_{\underline{x}}^{\bar{x}} [f(x)]^2 dx < \infty$  then:

$$\mathcal{G}(f)(x, t) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f] \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

Now we turn to the impulse response. We define the projection coefficients for the initial displaced distribution  $\hat{P}(\cdot)$ , defined in [equation \(3\)](#), as follows:

$$b_j[\hat{P}] = \int_{\underline{x}}^{\bar{x}} \varphi_j(x) d\hat{P}(x) \equiv \int_{\underline{x}}^{\bar{x}} \varphi_j(x) \hat{p}(x) dx + \sum_{k=1}^{\infty} \varphi_j(x_k) \hat{p}_m(x_k) \quad (22)$$

so that if  $\hat{P}$  has no mass points, it coincides with the definition for a function  $f$ . Using [Lemma 1](#) and the definition of the projection coefficients  $b_j[\hat{P}]$  we can write the impulse response function in [equation \(5\)](#) as follows:

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<sup>8</sup>That this is the solution of the o.d.e. follows since  $\sin(0) = \sin(j\pi) = 0$  for all  $j \geq 1$  and also because  $\sin''(x) = -\sin(x)$  for all  $x$ . Matching coefficients in the o.d.e. gives the eigenvalues.

PROPOSITION 4. Assume that  $f : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$  is piece-wise differentiable, with countably many discontinuities, and  $\int_{\underline{x}}^{\bar{x}} [f(x)]^2 dx < \infty$ . Furthermore assume that  $\hat{P}$  has a piecewise continuous density and at most countably many mass points, then:

$$G(t; f, \hat{P}) = \sum_{j=1}^{\infty} e^{\lambda_j t} \beta_j \quad \text{where} \quad \beta_j \equiv b_j[f] b_j[\hat{P}] . \quad (23)$$

where the projection coefficients  $b_j[f]$  and  $b_j[\hat{P}]$  were defined in equation (21) and equation (22), respectively.

Proposition 4 provides a tractable analytic representation of the impulse response function that separates the effect of the function of interest  $f$ , encoded in the projection coefficient  $b_j[f]$ , the effect of the initial impulse  $\hat{P}$ , encoded in the projection coefficient  $b_j[\hat{P}]$ , and the effect of time, encoded in the eigenvalues  $\lambda_j$ . For instance, we will use  $f(x) = -x$  to analyze the impulse response of output, since the contribution of output of each firm is inversely proportional to their price gap. Various shocks  $\hat{P}$  will be considered, with the marginal impulse response given by  $\hat{P} = \delta \bar{P}'$ . It is also straightforward to use the proposition to compute the cumulative impulse response defined in equation (16):

$$\mathbf{G} \equiv \int_0^{\infty} G(t) dt = \sum_{j=1}^{\infty} \frac{\beta_j}{-\lambda_j} .$$

The result in Proposition 4 can be readily understood by considering the discrete-time and discrete-state of the representation in equation (6). In this case  $q_t(y|x)$  becomes the  $t$ -th power of the 1-period transition matrix. Diagonalizing this matrix delivers the eigenvalue and eigenvectors that can be used to compute projections that are equivalent to the ones in equation (23). It is straightforward to apply Proposition 4 to derive the density for the transition function  $q_t(y|x)$  defined in equation (6):

COROLLARY 1. Let  $(x, y) \in [\underline{x}, \bar{x}]^2$ , let  $f(y, \epsilon) = \mathbf{1}_{\{x \in (y-\epsilon, y+\epsilon)\}} / (2\epsilon)$ , and  $\hat{P}(\cdot)$  a CDF representing a degenerate random variable concentrated at  $x$  so that, by equation (22),

$\hat{p}_m(x) = 1$ . Taking the limit as  $\epsilon \rightarrow 0$ , using [equation \(23\)](#) and computing the  $b_j[f]$  and  $b_j[\hat{P}]$  projections we have

$$q_t(y|x) = \sum_{j=1}^{\infty} \varphi_j(y)\varphi_j(x)e^{\lambda_j t} \quad . \quad (24)$$

We finish this subsection with a discussion of the generality of the representation in [Proposition 4](#), and a brief explanation of why we concentrated on such a case. While we have used the idea of symmetry for our analysis, the applicable more general concept is whether the operator  $\mathcal{G}$  is self-adjoint or not. Indeed one can consider an abstract positive operator  $\mathcal{G}$  defined in a general vector space and still obtain a result essentially identical to [Proposition 4](#). In particular, without entering in all details, if the law of motion for this state  $x$  can be described by a semigroup family (e.g. [Anderson, Hansen, and Sargent \(2003\)](#)), so that we can use the infinitesimal version of its law of motion, and if the resulting operator is compact, then the spectral theorem immediately gives the existence of a version of [equation \(23\)](#). The drawback of such an approach is that one may not have a simple characterization of the eigenfunctions and eigenvalues. We can consider a much simpler case than this abstract one, and yet more general than our benchmark case, in which  $x$  follows a general one dimensional diffusion in a closed interval. The Sturm-Liouville theory gives not only the representation in [equation \(23\)](#), but also gives many of the properties for the eigenvalues and eigenfunctions.<sup>9</sup> One requirement in the Sturm-Liouville theory, which is also present in our benchmark case, is the type of boundary condition.<sup>10</sup> This requirement is satisfied for the process of  $x$  stopped at the first price change, but it is not satisfied for the problem with re-injection. For the general class of the Sturm-Liouville problems the eigenfunctions and eigenvalues may not be all known in closed form, but there are ways to approximate them and also to efficiently compute them.

Finally there are many cases, besides our benchmark one, where eigenvalues and eigen-

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<sup>9</sup>Examples of these properties are indexing the eigenfunctions by the number of zeroes, and having eigenvalues discretely separated, etc.

<sup>10</sup> To be specific, to be of the Neumann or Dirichlet type.



functions are known and are tractable. One example is in [Appendix B](#) where we examine the case of the operator representing a firm solving a multiproduct price setting problem, and give the eigenvalues and eigenfunctions on this n-dimensional problem. Another example is given in [Appendix C](#) where we consider an extension of the benchmark model that allows for a non-degenerate distribution of random fixed costs, as in e.g. [Dotsey, King, and Wolman \(1999\)](#). In both cases we characterize the eigenfunctions and eigenvalues.

#### 4.1 Application to the canonical menu cost model.

We illustrate [Proposition 4](#) using the canonical menu-cost model, obtained by setting  $\mu = 0$  and  $\zeta = 0$  in the problem of [Section 2.1](#), which yields the symmetric inaction region  $\underline{x} = -\bar{x}$  with optimal return  $x^* = 0$ . To compute the impulse response of output we use  $f(x) = -x$  since the contribution of a firm to the deviation of output (relative to steady state) is inversely proportional to its price gap. Integrating  $f(x)$  against  $\varphi_j(x)$  we find the projection coefficients  $b_j[f]$  in [equation \(23\)](#):

$$b_j[f] = \frac{4\bar{x}^{3/2}}{j\pi} \text{ for } j = 2, 4, 6, \dots, \text{ and } b_j[f] = 0 \text{ otherwise.} \quad (25)$$

In this example we consider a *marginal* monetary shock so that  $\hat{p}(x) = \delta\bar{p}'(x)$ , as discussed above. The invariant distribution for this model is readily derived from [equation \(10\)](#) and the associated boundary conditions, which gives the triangular density  $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$  for  $x \in (-\bar{x}, \bar{x})$ . It is apparent that  $\bar{p}'(x)$  is a step function, equal to  $1/\bar{x}^2$  for  $x \in [-\bar{x}, 0)$  and equal to  $-1/\bar{x}^2$  for  $x \in (0, \bar{x}]$ . We thus construct the projection coefficients  $b_j[\bar{p}']$  by integrating  $\bar{p}'(x)$  against  $\varphi_j(x)$ . This gives

$$b_j[\bar{p}'] = \frac{8}{j\pi\bar{x}^{3/2}} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \dots, \text{ and } b_j[\bar{p}'] = 0 \text{ otherwise.} \quad (26)$$

Thus the impulse response  $b_j[\bar{p}']b_j[f]$  coefficients for [equation \(23\)](#) are:

$$b_j[\bar{p}']b_j[f] = \frac{32}{(j\pi)^2} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \dots, \text{ and } b_j[\bar{p}']b_j[f] = 0 \text{ otherwise.}$$

and the (marginal) output impulse response as defined in [equation \(7\)](#) is:

$$Y(t) = \sum_{i=0}^{\infty} \frac{32}{((2+4i)\pi)^2} e^{-N \frac{((2+4i)\pi)^2}{8} t} \quad (27)$$

where the expression has a single parameter given by  $N = \sigma^2/\bar{x}^2$ , namely the average number of price changes per period.

## 5 The selection effect and price dispersion

This section applies [Proposition 4](#) to illustrate why different sticky-price models display different degrees of “selection”. The notion of selection, coined by Golosov and Lucas, refers to the fact that firm that adjust prices following a monetary shock are selected from a particular set. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Taylor or the Calvo model. It is known that different amounts of selection critically affect the propagation of monetary shocks.

We present an analytic result showing that selection creates a wedge between the duration of price changes and that of output. The two durations coincide when there is no selection. In this case the frequency of price changes is a sufficient statistic for the output effect of monetary policy. We show that such wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of output. We illustrate this result using the Calvo-plus model, a model that nests several

special cases featuring different degrees of selection, from Golosov-Lucas to the pure Calvo model. The result also holds in several other models, featuring multiproduct firms or price plans.

This section also presents an important result, in [Proposition 5](#), showing that the response of even centered moments to a marginal monetary shocks is zero at all horizons. This applies for instance to the variance of price gaps, or cross-sectional price dispersion, the kurtosis of the price changes, as well as to the survival function. The result is relevant because e.g. the cross-sectional price dispersion is the main measure of inefficiency in models with price stickiness. This result implies that cross-sectional price dispersion is not sensitive to small monetary shocks, so that its fluctuations won't be detectable in estimated impulse responses, but that instead it reacts to changes with the steady state level of inflation, or more generally with the monetary policy regime. An advantage of the analytic approach is to clearly illustrate the forces behind this surprising result.

While the remaining of this section works out our benchmark case, the Calvo Plus model, all the results –appropriately restated– hold for the multiproduct model in [Appendix B](#) and the general random fixed cost model in [Appendix C](#).

**Application to the Calvo-plus model.** Next we use the decision problem defined in [Section 2.1](#), and assume zero inflation ( $\mu = 0$ ) and a quadratic profit function  $R(x) = x^2$ . It is straightforward that  $\bar{x} = -\underline{x} > 0$  and that the optimal return is  $x^* = 0$ . Given the policy parameters  $\{-\bar{x}, \bar{x}\}$  and the law of motion of the state  $dx = \sigma dW$  it is immediate that the eigenvalues-eigenfunctions of the problem are those computed in [equation \(20\)](#). Since the eigenvalues depend on the speed at which prices are changed, we find it convenient to rewrite them in terms of the average number of price changes per unit of time. To this end we compute the expected number of adjustments per unit of time, the reciprocal of the expected time until an adjustment,  $N = \frac{\zeta}{1 - \text{sech}(\sqrt{2}\phi)}$  where  $\phi \equiv \frac{\zeta \bar{x}^2}{\sigma^2}$ . Note that as  $\bar{x} \rightarrow \infty$  then  $N \rightarrow \zeta$ , which is the Calvo model where all adjustment occur after an exogenous poisson shock. As  $\zeta \rightarrow 0$  then  $N \rightarrow \sigma^2/\bar{x}^2$  so that the model is Golosov and Lucas. This single parameter

$\phi \in (0, \infty)$  controls the degree to which the model varies between Golosov-Lucas and Calvo. Note that with this parameterization we can distinguish between  $N$  and the importance of the randomness in the menu cost  $\zeta$  vs the width of the barriers,  $\bar{x}^2/\sigma^2$ . Indeed  $\zeta/N$ , the share of adjustment due to random free-adjustments, depends only on  $\phi$ . We let  $\frac{\zeta}{N} = \ell(\phi)$  where this function is defined as  $\ell(\phi) = 1 - \operatorname{sech}(\sqrt{2\phi})$ . The function  $\ell(\cdot)$  is increasing in  $\phi$ , and ranges from 0 to 1 as  $\phi$  goes from 0 to  $\infty$ . Using the formula for  $N$  and [equation \(20\)](#) we have:

$$\lambda_j = -\zeta - \frac{\sigma^2 (j\pi)^2}{\bar{x}^2} \frac{1}{8} = -\zeta \left[ 1 + \frac{(j\pi)^2}{8\phi} \right] = -N \ell(\phi) \left[ 1 + \frac{(j\pi)^2}{8\phi} \right] \quad . \quad (28)$$

**Interpretation of the Dominant Eigenvalue.** The dominant eigenvalue has the interpretation of the asymptotic hazard rate of price changes. In particular, let  $h(t)$  be the hazard rate of price spells as a function of the duration of the price spell  $t$ . Let  $\tau$  be the stopping time for prices, i.e.  $\tau$  is the first time at which  $\sigma W(t)$ , which started at  $W(0) = 0$ , either hits  $\bar{x}$  or  $\underline{x} = -\bar{x}$ , or that the Poisson process changes. Let  $S(t)$  be the survival function, i.e.:  $S(t) = \Pr\{\tau \geq t\}$ . Notice that the function of interest to compute the survival function is the indicator  $f(x) = 1$  for all  $t < \tau$ . The hazard rate is defined as  $h(t) = -S'(t)/S(t)$ . Application of [Proposition 4](#) gives the following:

**COROLLARY 2.** The Survival function  $S(t)$  depends only on the odd-indexed eigenvalues-eigenfunctions, i.e.  $(\lambda_i, \varphi_i)$  for  $i = 1, 3, 5, \dots$ . Let  $h(t)$  be the hazard rate of price changes. Then, the dominant eigenvalue  $\lambda_1$  is equal to the asymptotic hazard rate, i.e.

$$S(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1} t} \beta_{2j-1} \quad \text{and} \quad -\lambda_1 = \lim_{t \rightarrow \infty} h(t)$$

where  $\beta_j \equiv b_j[1] \varphi_j(0)$  where we use [equation \(22\)](#) where  $\hat{P}$  represents a degenerate random variable concentrated at  $x = 0$  so that  $\hat{p}_m(0) = 1$ .

**Irrelevance of dominant eigenvalue for output IRF.** Next we show that the dominant eigenvalue  $\lambda_1$ , as well as all other odd-indexed eigenvalue-eigenfunction pairs, play no role in the output impulse response. Consider the output coefficients in the impulse response, given by [equation \(25\)](#). It is apparent that the coefficients  $b_j[f]$  for all the odd-indexed eigenvalues-eigenfunctions ( $j = 1, 3, \dots$ ) are zero, i.e. the loading of these terms are zero. This implies that the coefficient corresponding to the dominant eigenvalue  $\lambda_1$  is zero. The first non-zero term, which we call the “leading” eigenvalue, involves  $\lambda_2$ . This is because  $\varphi_j(\cdot)$  is symmetric around  $x = 0$  for  $j$  odd, and antisymmetric for  $j$  even. Thus:

$$\int_{\underline{x}}^{\bar{x}} \varphi_j(x) f(x) dx = 0 \implies b_j[f] = 0 \text{ for } j = 1, 3, \dots$$

This happens since all the odd-indexed eigenfunctions  $\varphi_j$  ( $j = 1, 3, \dots$ ) are symmetric functions, and thus the projection onto them of an asymmetric function, such as  $f(x) = -x$ , yields a zero  $b_j$  coefficient. We summarize this result in the next corollary:

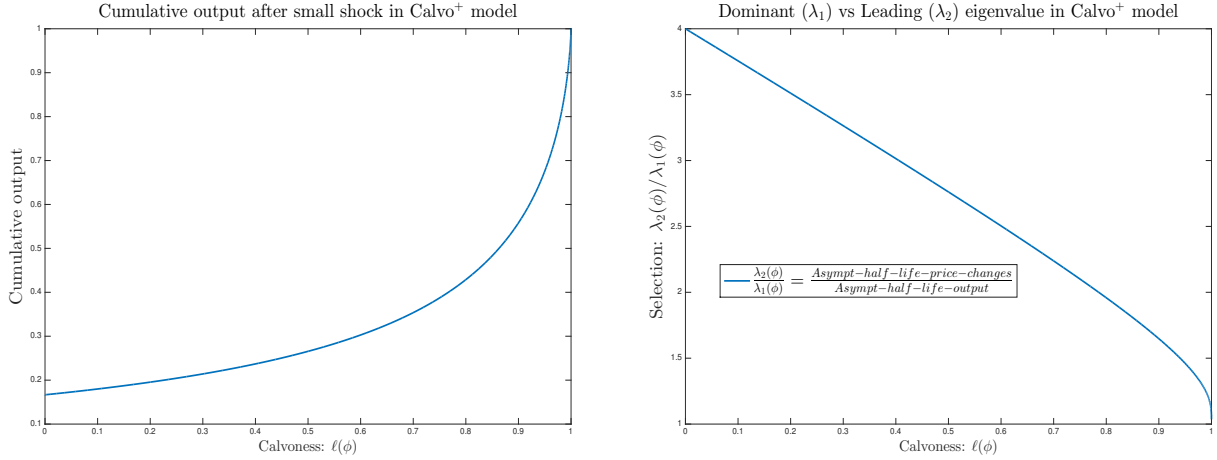
**COROLLARY 3.** The output impulse response function for the Calvo-plus model depends only on the even-indexed eigenvalue-eigenfunctions  $(\lambda_j, \varphi_j)$ , and has zero loadings on the odd-indexed ones, such as the dominant eigenvalue. Thus the first leading term corresponds to the second eigenvalue  $\lambda_2$ :

$$G(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j}t} \beta_{2j} \quad \text{and} \quad \lambda_2 = \lim_{t \rightarrow \infty} \frac{\log Y(t)}{t}$$

where  $\beta_j \equiv b_j[f] b_j[\hat{p}]$ .

The corollary states that only half of the eigenvalues (those with an even index) show up in the output impulse response function. The largest eigenvalue is  $\lambda_2$ , which we call the “leading” eigenvalue of the output response function. It is interesting to notice that the dominant eigenvalue  $\lambda_1$  does not appear in the impulse response for output. Notice the difference with the survival function where the only eigenvalues that appear are those with

Figure 2: Selection effect in Calvo<sup>+</sup> model



an odd-index. The right panel of [Figure 2](#) plots the ratio between the leading eigenvalue for output  $\lambda_2$  and the dominant eigenvalue  $\lambda_1$ . It is straightforward to see that the ratio is  $\frac{\lambda_2}{\lambda_1} = \frac{8\phi+4\pi^2}{8\phi+\pi^2}$  depends only on  $\phi$ , so that it can be immediately mapped into the “Calviness” of the problem  $\ell(\phi) \in (0, 1)$ . It appears that the ratio, which can also be interpreted as the ratio between the asymptotic duration of price changes over the asymptotic duration of the output impulse response, is monotonically decreasing in  $\ell$ , and converges to 1 as  $\ell \rightarrow 1$ . The economics of this result is that the shape of the impulse response of output depends on the differential impact of the aggregate shock on price *increases* and price *decreases*. Instead the dominant eigenvalue controls the asymptotic behavior of price *changes*, both increases and decreases. As  $\ell \rightarrow 1$  selection disappears from the model and the two durations coincide. The left panel of the figure uses the particular case of a small monetary shock (developed in detail in the next subsection) to illustrate that as  $\ell$  increases the cumulated output effect becomes larger due to a muted selection effect.

**IRF of cross-sectional dispersion of prices to a monetary shock.** We conclude this section with a discussion of a relevant topic in monetary models that concerns the welfare effect of shocks. In many monetary models the presence of price stickiness implies that welfare

of the representative consumer depends on the dispersion of prices, or markups (i.e. prices relative to a flexible price benchmark where dispersion is nil). It is therefore of interest to analyze how the dispersion of markups behaves following a small monetary shock of size  $\delta$ .<sup>11</sup> Let  $M_k(t; \delta) \equiv \mathbb{E} [x(t) - \mathbb{E}(x(t))]^k$ , for  $k = 2, 4, \dots$ , denote the  $k$ -th even centered moment of  $x$ , measured  $t$  periods after the monetary shock  $\delta$  hits the economy at the steady state. In particular,  $M_2(t; \delta)$  will denote the cross sectional variance of markups  $t$  periods after the monetary shock. We have the following result:

**PROPOSITION 5.** Assume the initial condition  $\hat{p}$ , the signed mass right after the aggregate shock, is antisymmetric. Then the initial impulse  $\hat{p}$  does not have a first-order effect on any even centered moment  $M_k(t; \delta)$  with  $k = 2, 4, \dots$

The proposition implies that a small (marginal) monetary shock does not have a first order impact on the dispersion of markups at all  $t > 0$  after the monetary shock. It also shows that a zero first-order effect is predicted for all even centered moments of the distribution of markups (such as Kurtosis). Instead, uneven moments, such as the the mean markup (proportional to total output) or the the skewness of the distribution display a non-zero first order effect following a marginal monetary shock. As mentioned at the beginning of the section this result matters because even moments, such as the dispersion of price gaps, map directly into the efficiency of the economy. In terms of measurement the result implies one should look at the effect of varying the level of inflation on price dispersion as in [Alvarez et al. \(2019\)](#) or [Nakamura et al. \(2018\)](#). Likewise, the hazard rate of price changes should not react to a small monetary shock. This gives theoretical support to the estimation of hazard rates of price changes using time series evidence without controlling for monetary shocks.<sup>12</sup> Finally, we present the result on Kurtosis because elsewhere –[Alvarez, Le Bihan, and Lippi \(2016\)](#)– we have argued that the *steady state* kurtosis and frequency of price changes characterized the

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<sup>11</sup>We are thankful to Nobu Kiyotaki for posing this question to us.

<sup>12</sup>In [Alvarez et al. \(2019\)](#) it is shown theoretically that price dispersion and the average frequency of price changes should not be affected by inflation around zero inflation, but that it should be responsive at higher levels. The same paper provides evidence on these two variables.

*cumulative* IRF to a small monetary shock. This result says that kurtosis does not change with monetary shocks, i.e. that remains approximately constant at its steady state value after a monetary shock.

## 6 On the shape of the impulse response function

This section discusses whether it is possible to approximate the impulse response function in a parsimonious way, a question that is naturally related to the shape of the impulse response. A natural candidate would be to analyze the impulse response associated to the leading eigenvalue as defined in [Section 5](#), namely the largest eigenvalue associated with non-zero projection coefficient  $b_j$  in [equation \(23\)](#), for a case in which the IRF is close to exponential. We analyze this question by focusing on a small monetary shock that causes a marginal displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo<sup>+</sup> model as well as for a model with price plans.

**Initial Condition  $\hat{p}$  for the impulse response to a monetary shock.** The invariant density function  $\bar{p}$  solves the Kolmogorov forward  $\zeta\bar{p}(x) = \sigma^2/2\bar{p}''(x)$  in the support, except at  $x = 0$ , integrates to one and it is zero at  $\pm\bar{x}$ . This gives:

$$\bar{p}(x) = \frac{\theta [e^{\theta(2\bar{x}-|x|)}e - e^{\theta|x|}]}{2(1 - e^{\theta\bar{x}})^2} \text{ for } x \in [-\bar{x}, \bar{x}] \quad \text{where } \theta \equiv \sqrt{2\zeta/\sigma^2} \quad . \quad (29)$$

(note that to simplify notation we define a new parameter  $\theta$ ). The invariant distribution  $\bar{p}(\cdot)$  is symmetric with  $\bar{p}(x) = \bar{p}(-x)$  and  $\bar{p}'(x) = -\bar{p}'(-x)$  for all  $x \in [-\bar{x}, 0)$ . The density of the distribution is continuous but non-differentiable at the injection point  $x = 0$ . For concreteness we focus below on the response to a marginal monetary shock  $\delta$ , starting at the steady state. For such a small shock we can disregard the fraction of firms that change prices



on impact, i.e. this effect is of order  $\delta^2$ . Thus the initial condition is

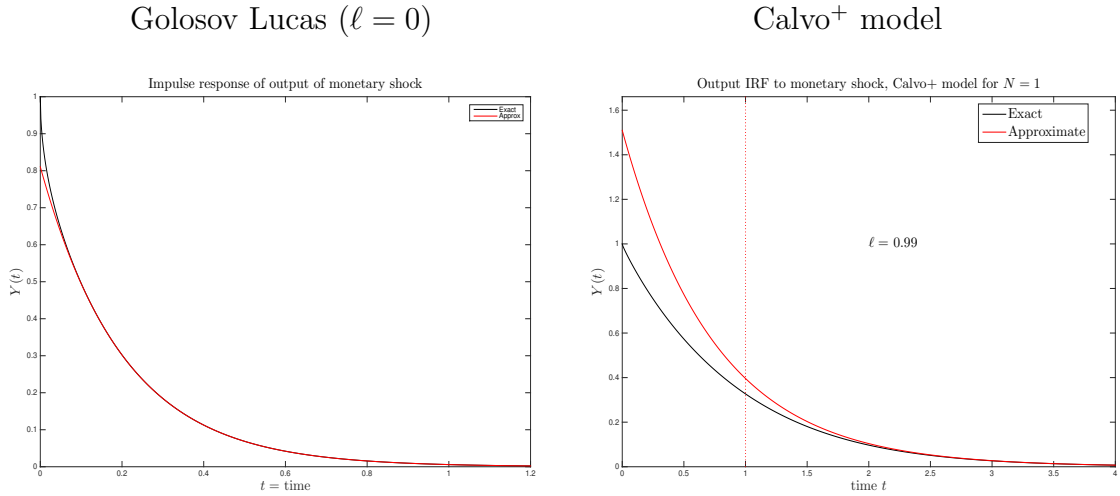
$$\hat{p}(x) = \bar{p}'(x)\delta + o(\delta) \text{ for all } x \in [-\bar{x}, 0) \text{ and } x \in (0, \bar{x}]$$

As done above, we find it convenient to express the output IRF  $Y(t)$  per unit of the shock  $\delta$  by using

$$Y(t) \equiv \frac{\partial}{\partial \delta} G(t, \delta) \Big|_{\delta=0} = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} b_{2j}[f] b_{2j}[\bar{p}']$$

Application of **Proposition 4** gives the following:

Figure 3: Exact vs approximate IRF



**PROPOSITION 6.** The coefficients for the impulse response to a small monetary shock in the Calvo<sup>+</sup> model are given by:

$$\beta_j(\phi) \equiv b_j[\bar{p}'] b_j[f] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 \left[ \frac{1 + \cosh(\sqrt{2}\phi)}{1 - \cosh(\sqrt{2}\phi)} \right] \left[ \frac{1}{1 + \frac{j^2 \pi^2}{8\phi}} \right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2 \left[ \frac{1}{1 + \frac{j^2 \pi^2}{8\phi}} \right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \quad (30)$$

Recall that  $\phi \equiv \sigma^2 \zeta / \bar{x}^2 \in (0, \infty)$  is a single parameter that locates the Calvo<sup>+</sup> model between the Golosov-Lucas ( $\phi = 0$ ) and the Calvo ( $\phi \rightarrow \infty$ ) model. Each of these cases can be easily computed by simple calculus while keeping  $N$  constant. The solution for the first case,  $\phi \rightarrow 0$ , was given in [equation \(25\)](#) and [equation \(26\)](#). The second case,  $\phi \rightarrow \infty$ , is peculiar because in this limit the spectrum is no longer discrete. To see this notice that each of the eigenvalues  $\lambda_j \rightarrow -\zeta$  and:

$$\lim_{\phi \rightarrow \infty} \beta_j(\phi) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Note that in this case, using just the “leading” eigenvalue, i.e. the term with the first non-zero weight, gives an impulse response  $G(t)$  that is twice as large than the true one, for each  $t$ . This is in stark contrast with the case when  $\phi \rightarrow 0$  where the approximation is extremely accurate. The difference is less than 1.5%, to see this note that when  $\phi \rightarrow 0$  then  $\mathbf{G} \rightarrow 1/(6N) \approx 0.1677/N$ . On the other hand, using only the term corresponding to the second eigenvalue we obtain  $b_2/(-\lambda_2) = 16/(\pi^4)/N \approx 0.1643/N$ .

The next proposition gives a characterization of the ratio between the true area under the output impulse response and the approximate one, computed using only the leading eigenvalue:

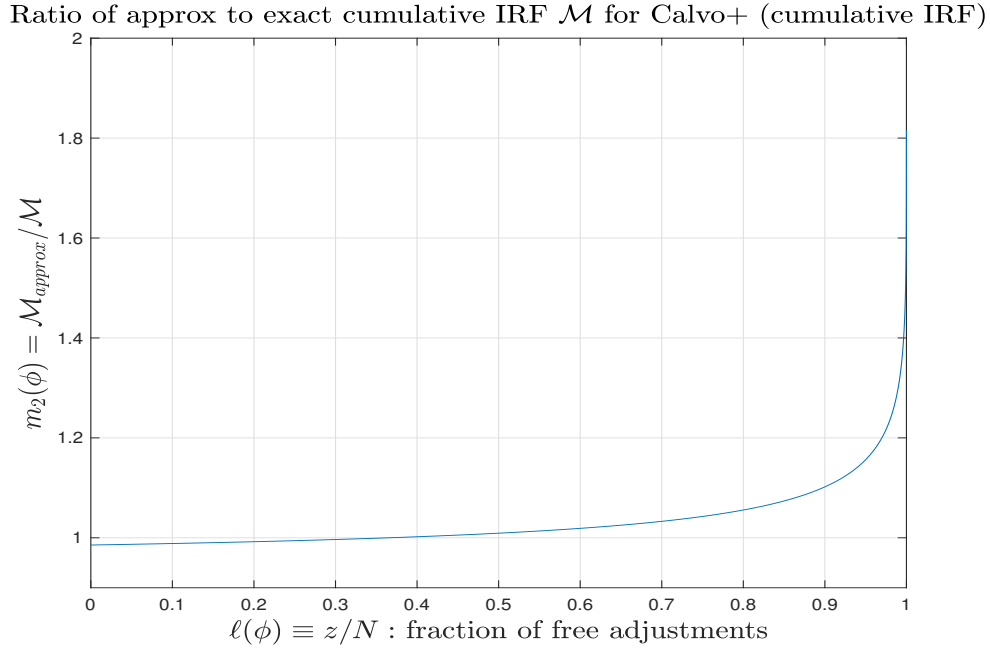
**PROPOSITION 7.** Consider the marginal impulse response for output, so that  $f(x) = -x$  and  $\hat{p} = \delta \bar{p}'$ . Define the ratio of the approximate cumulative impulse response based on the leading eigenvalue relative to the area under the impulse response as:

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} \beta_j(\phi)/\lambda_j(\phi)} = 2 \frac{[1 + \cosh(\sqrt{2\phi})]}{[\cosh(\sqrt{2\phi}) - 1 - \phi] \left[1 + \frac{\pi^2}{2\phi}\right]^2}$$

We note that  $m_2(0) = \frac{16}{\pi^4}6 \approx 0.98$ ,  $m_2'(\phi) > 0$  and  $m_2(\phi) \rightarrow 2$  as  $\phi \rightarrow \infty$ .

**Proposition 7** shows that the leading eigenvalue provides an accurate approximation of the total cumulative IRF for most variants of the calvo-plus model. At values of  $\ell \approx 0.7$  the approximate function is close to 95% of the true effect. **Figure 4** shows that accuracy degenerates as the model converges towards a pure Calvo model  $\ell \rightarrow 1$ .

Figure 4: Calvo-plus model



We can also use the expression for the coefficients of the impulse response to show that the slope of  $Y$  at  $t = 0$  is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

**PROPOSITION 8.** The derivative of the IRF with respect to  $t$  at  $t = 0$  is given by:

$$\left. \frac{\partial}{\partial t} Y(t) \right|_{t=0} = -\infty \quad \text{for} \quad 0 \leq \phi < \infty.$$

Note that when  $\phi \rightarrow \infty$ , so we get the pure Calvo model, then the impulse response becomes  $Y(t) = \exp(-Nt)$ , and thus  $Y'(0)$  is finite.<sup>13</sup>

**Price Plans and the hump-shaped Output IRF.** The model with price plans assumes that upon paying the menu cost the firm can choose two, instead of one price. At any point in time the firm is free to charge either price within the current plan, but changing to a new plan (another pair of prices) is costly. The idea was first proposed by [Eichenbaum, Jaimovich, and Rebelo \(2011\)](#) to model the phenomenon of temporary price changes (prices that move from a reference value for a short period of time and then return to it). In [Alvarez and Lippi \(2019\)](#) we provide an analytic solution to this problem and characterize the determinants of  $\bar{x}$ , the threshold where a new plan is chosen, as well as the optimal prices within the plan, named  $\tilde{x}$  and  $-\tilde{x}$ . When  $x \in [-\bar{x}, 0]$  the firm charges  $-\tilde{x}$  and when  $x \in (0, \bar{x}]$  it charges  $\tilde{x}$ . The invariant density of price gaps is still given by [equation \(29\)](#). For a given threshold  $\bar{x}$  the value of  $\tilde{x}$  is given by:

$$\tilde{x} = \bar{x} \left[ \frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi} (e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2)} \right] \equiv \bar{x} \rho(\phi) > 0 \quad \text{where} \quad \phi = \bar{x}^2 \zeta / \sigma^2$$

and the function  $\rho(\phi)$  gives the optimal price within the plan as a function of the adjustment threshold, namely  $\tilde{x} = \rho(\phi)\bar{x}$ , as a function of  $\phi$ . Simple analysis shows that the images of the function  $\rho(\phi)$  lie in the interval  $(0, 1/3)$ , that it is decreasing, and that it converges to  $1/3$  as  $\phi \rightarrow 0$  (see [Alvarez and Lippi \(2019\)](#)).

In the model with plans the contribution to the aggregate of a firm with output price gap  $x$  is, instead of  $f(x) = -x$ , the following function:

$$\tilde{f}(x) = \begin{cases} -x - \tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ -x + \tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

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<sup>13</sup> See the web Appendix ?? analyzes the the output IRF for the pure Calvo model.

By the linearity of Fourier series we can add to the coefficients of the function  $f(x) = -x$ , shown in [equation \(25\)](#) the ones of the step function:

$$f_0(x) = \begin{cases} -\tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ +\tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

Importantly, we note the function  $\tilde{f}(x) = f(x) + f_0(x)$  is still an asymmetric function. The function  $f_0$  has Fourier sine coefficients equal to:

$$b_j[f_0] = -\frac{8\bar{x}^{3/2}\rho(\phi)}{j\pi} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \dots, \text{ and } b_j[f_0] = 0 \text{ otherwise}$$

From here we conclude that:

$$\beta_j^0(\phi) = b_j[f_0]b_j[\bar{p}] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -4\rho(\phi) \left[ \frac{1+\cosh(\sqrt{2\phi})}{\cosh(\sqrt{2\phi})-1} \right] \left[ \frac{1}{1+\frac{j^2\pi^2}{8\phi}} \right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ 0 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \quad (31)$$

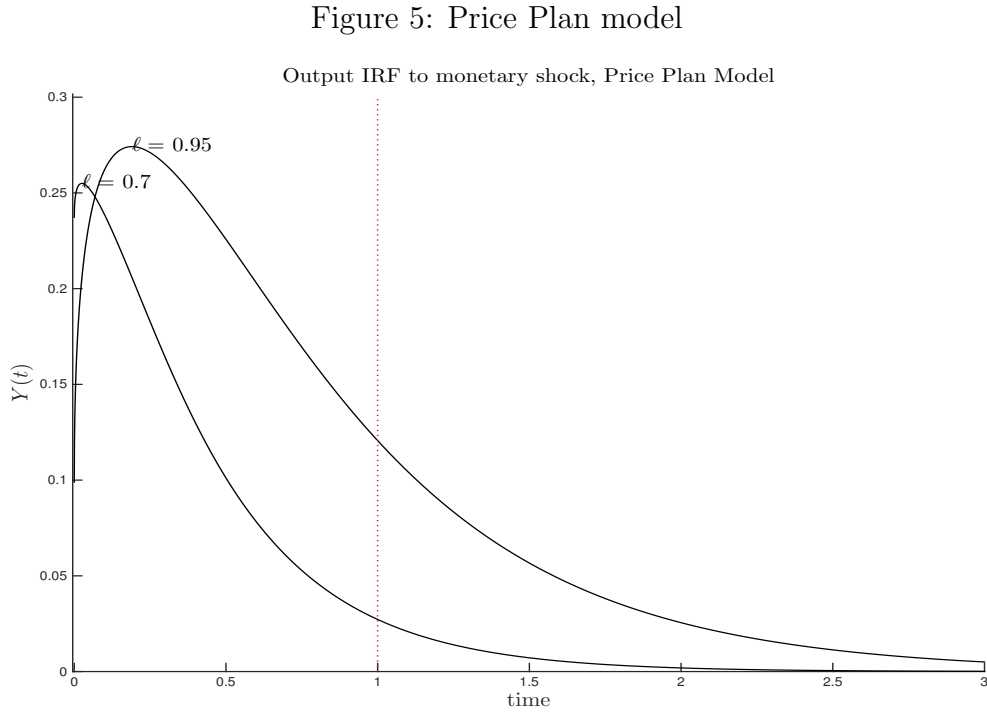
Thus the impulse response is given by:

$$Y_{\text{Plan}}(t) = Y_{\text{Calvo}^+}(t) + \sum_{j=1}^{\infty} \beta_j^0(\phi) e^{\lambda_j(\phi)t}$$

While the impulse response is monotone decreasing in the Calvo<sup>+</sup> model, in the price plan model the impulse response can be hump shaped. Indeed, as the  $\phi$  increases, the impulse response goes from decreasing to humped shaped. The reason for this difference is that in the price plan model there is a non-negligible impact effect, due to the non-negligible set of firms that within the plan adjust from one price to the other. The difference across models, as we increase  $\phi$ , is that invariant distribution has more firms with price gaps close to zero, and hence more firms that can change from one price to the other within the plan. This is

because the higher is the impact effect on prices, the smaller is the effect on output. The next proposition indeed shows that when  $\phi = 1$ , so that  $\zeta = \sigma^2/\bar{x}^2$ , i.e. the number of plan changes in a pure-barrier model equals the one in a pure-random-plan model, are equal. For  $\phi = 1$  the impulse response has a “hump”, but it is infinitesimal.

**PROPOSITION 9.** For  $0 \leq \phi \leq 1$ , the impulse response  $Y(t)$  is decreasing. For  $\phi > 1$ , the impulse response is hump shaped. The value for  $t_0 N$  at which the maximum is reached,  $Y'(t_0 N) = 0$ , increases relative to expected time to adjustment  $1/N$ .



Given that the impulse response is hump-shaped any single eigenvalue cannot approximate the impulse response. Even for low values of  $\phi$ , where the impulse response is monotone, using  $\lambda_2$  gives a very bad approximation.

## 7 Volatility shocks and the propagation of monetary impulses

This section discusses the effect that changes to the volatility of shocks exert on the propagation of monetary shocks. The issue matters to e.g. the effectiveness of monetary policy in recessions vs boom, when the state of the economy is assumed to feature, respectively, high vs low volatility of shocks as in [Vavra \(2014\)](#). Our method provides a sharp analytic answer to this question.

For concreteness we illustrate the problem by using the pure menu cost model (without Calvo adjustment i.e.  $\zeta = 0$  so that  $\phi = \ell = 0$ ), whose output response to a small monetary shock was given in [equation \(27\)](#). We conduct a comparative static exercise to analyze how the propagation is affected by an innovation of the “volatility shocks”, namely a permanent change in the common value of the idiosyncratic volatility  $\sigma$ .<sup>14</sup>

We start with a steady state for the model with idiosyncratic volatility  $\sigma$ . We characterize the effect of a small monetary shock,  $\delta > 0$ , which occurs  $s \geq 0$  periods after a change in idiosyncratic volatility from  $\sigma$  to  $\tilde{\sigma}$ , so that  $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma}) \sigma$ . In particular we let  $Y(t; s, d\sigma/\sigma)\delta$  denote the output’s IRF  $t \geq 0$  periods after an unexpected monetary shock of size  $\delta$  starting with a cross sectional distribution that has evolved  $s$  periods since the change in  $\sigma$ .<sup>15</sup>

While we characterize  $Y$  for all  $t > 0$  and  $s \geq 0$ , two interesting cases are worthwhile to mention separately: the short-run and the long-run effect of volatility. The short-run effect, defined as  $Y(t; 0, d\sigma/\sigma)$  or  $s = 0$ , consists of considering a simultaneous permanent change of both  $\sigma$  (to  $\tilde{\sigma}$ ) and  $\delta > 0$ . After the shock the forward looking firm’s decision rules adjusts immediately to the new volatility  $\tilde{\sigma}$ , while the initial distribution of price gaps corresponds to the stationary distribution obtained under the old decision rule. The long-run effect, denoted

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<sup>14</sup>For simplicity and clarity of the results we consider here once and for all shocks to volatility. It is simple to modify the setup to consider a two-state Markov switching volatility process and to solve the associated firm’s decision rules.

<sup>15</sup>We will keep using the notation of  $Y$  as the output’s IRF per unit of monetary shock, and then omit the  $\delta$  in the expressions below.

by  $Y(t; \infty, d\sigma/\sigma)$  or  $s \rightarrow \infty$ , is equivalent to computing the effect of a monetary shock  $\delta$  for a new steady state with volatility  $\tilde{\sigma}$ . We refer to this as the *long-run* effect since it is the effect of an unanticipated monetary shock once the distribution of price gaps has achieved its new invariant distribution. In this case the firm's decision rule corresponds to the new volatility  $\tilde{\sigma}$  and the economy is described by the new invariant distribution of price gaps.

The general case characterizes an IRF whose coefficients are indexed by the parameter  $0 < s < \infty$ . The key feature of this case is that the monetary shock  $\delta$  occurs  $s$  periods after the volatility shock, thus displacing a cross-section distribution of price gaps that is in a transition towards the new invariant distribution. Our analytic method allows us to exactly compute the evolution of this distribution and hence the effect of a monetary shock.

The next proposition uses the notation introduced above, where  $Y(t; 0, 0)$  denotes the impulse before any change in volatility occurs, which we use as a benchmark. Also, the difference  $Y(t; s, \frac{d\sigma}{\sigma}) - Y(t; \infty, \frac{d\sigma}{\sigma})$  is the correction to the long run effect of a volatility shock  $d\sigma/\sigma$  due to a finite duration  $s$ .

**PROPOSITION 10.** Let  $Y(t; s, \frac{d\sigma}{\sigma})$  denote the time- $t$  value of the output marginal impulse response that occurs  $s$ -periods after a volatility increase from  $\sigma$  to  $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma})\sigma$ . The *long run* effect ( $s \rightarrow \infty$ ) of the volatility shock  $\frac{d\sigma}{\sigma}$  on the impulse response of output to a monetary shock is:

$$Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right) \quad \text{for all } t \geq 0. \quad (32)$$

The *short run* effect ( $s \rightarrow 0$ ) of the volatility shock  $\frac{d\sigma}{\sigma}$  on the impulse response of output to a monetary shock is:

$$Y\left(t; 0, \frac{d\sigma}{\sigma}\right) = \left(1 + \frac{d\sigma}{\sigma}\right) Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right) \quad \text{for all } t \geq 0. \quad (33)$$

The deviation from the long run response as a function of  $s$  is given by:

$$Y\left(t; s, \frac{d\sigma}{\sigma}\right) - Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k}t} b_{2k}[f] b_{2k}[\hat{p}'(\cdot, s)] \quad \text{for all } t, s \geq 0. \quad (34)$$



where  $\hat{p}'(\cdot, s)$  is the initial condition (i.e. a displaced cross section) at the time of the monetary shock,  $s$  periods after the change in volatility, whose projection coefficients are given by:

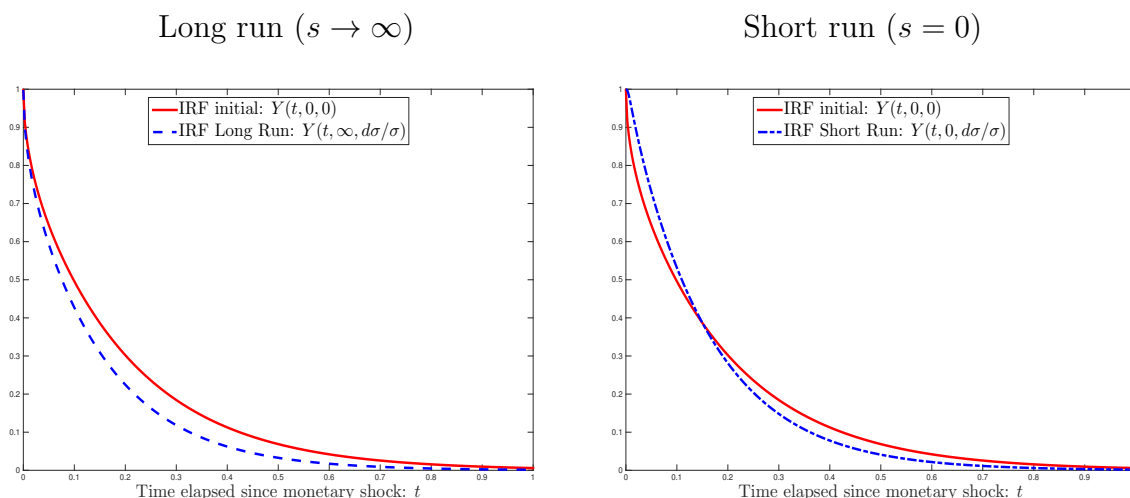
$$b_{2k}[\hat{p}'(\cdot, s)] = \frac{d\sigma}{\sigma} \frac{1}{\bar{x}^{\frac{3}{2}}} \sum_{j=1,3,5,\dots}^{\infty} e^{\lambda_j s} \left( 2 \frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\pi)^2} \right) \left( \frac{4kj}{(4k^2 - j^2)} \right) , \quad k = 1, 2, 3, \dots \quad (35)$$

and where  $b_{2k}[f] = 2\bar{x}^{3/2}/(k\pi)$  as in equation (25).

A few comments are in order.

(i) Figure 6 illustrates the difference between the short run and long run effect of an increase in volatility on the output's response to a monetary shock. The left panel compares the IRF with no change in volatility,  $Y(t; 0, 0)$  to the one where the volatility increase has occurred  $s \rightarrow \infty$  periods ago, i.e.  $Y(t; \infty, d\sigma/\sigma)$  the long run effect. The right panel compares the IRF with no change in volatility,  $Y(t; 0, 0)$  to the one where the volatility increase has occurred at the same time as the monetary shock  $s = 0$  periods ago, i.e.  $Y(t; 0, d\sigma/\sigma)$  the short run effect.

Figure 6: Short-run and long-run IRF vs. IRF before volatility increases



Note:  $N = 1$  (one price adjustment per unit of time, on average) and  $d\sigma/\sigma = 0.1$ .

(ii) For this proposition we use the form of the decision rules for the threshold  $\bar{x}$ , which as the discount rate goes to zero is  $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$  where  $\psi$  is the fixed cost –as fraction of the

frictionless profit and  $B$  is the curvature of the profit function around the frictionless profit. This implies that the elasticity of  $\bar{x}$  to  $\sigma$  is  $1/2$ . This elasticity is the so called “option value” effect on the optimal decision rules.

(iii) The rescaling of time in  $Y(t(1 + \frac{d\sigma}{\sigma}); 0, 0)$  in the expressions for the long and short run effect of volatility reflects the change in the eigenvalues, which depend on the value of  $N$ , the implied average number of price changes per unit of time, as  $\lambda_j = -N(\pi j)^2/8$  (see [equation \(28\)](#) for  $\zeta = 0$ ). Recall that  $N = (\sigma/\bar{x})^2$ , and hence all the eigenvalues change proportionally with  $\sigma$ .

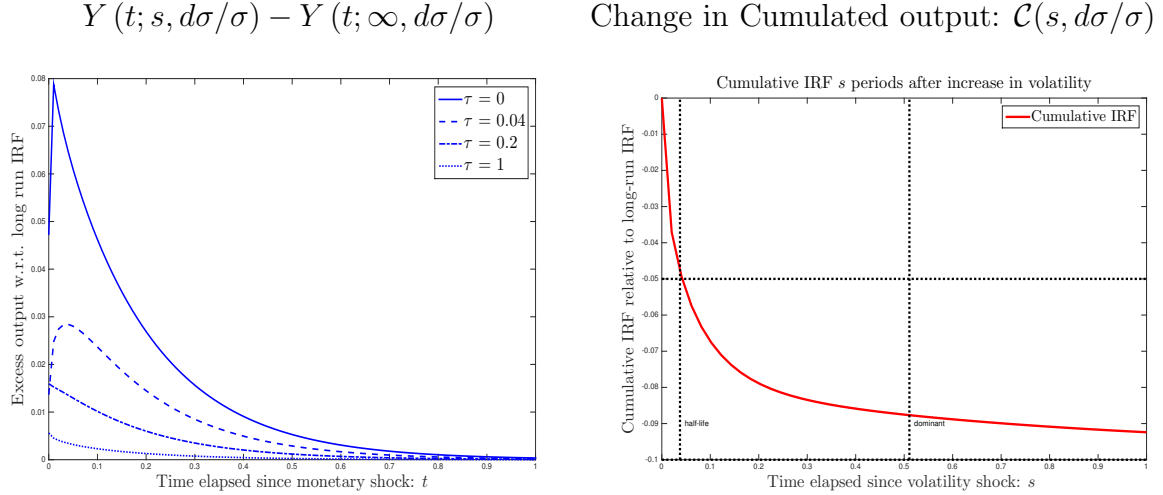
(iv) For the case of the impact effect and in which  $\tilde{\sigma} > \sigma$ , the invariant distribution just before the monetary shock is narrower than the range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level  $(1 + \frac{d\sigma}{\sigma})$  in the impact effect in [equation \(33\)](#): since firms have price gaps that are discretely away from the inaction bands, then prices react more slowly, generating the extra effect on output. The logic for the case where  $\tilde{\sigma} < \sigma$  is similar.

(v) In [equation \(34\)](#) we use only the even terms for the projections, i.e. the index for the projection  $b_{2k}[\cdot]$  runs on  $2k$  because  $f$  is antisymmetric. For these coefficients, as was the case without volatility shocks, the eigenvalues that control the effect of the horizon  $t$  in the IRF are the even ones, i.e.  $\lambda_2, \lambda_4, \dots$ , starting with the leading one  $\lambda_2$ .

(vi) The expressions in [equation \(34\)](#) and [equation \(35\)](#) show that what governs the difference between the long run and the short run volatility effects are the odd eigenvalues, i.e.  $\lambda_1, \lambda_3, \dots$ , since these are the only elements where  $s$  affect the expressions. In particular,  $\lambda_1$  is the dominant eigenvalue.

(vii) We note that the expression for the correction term in [equation \(34\)](#) involves no parameter for the model with the exception of  $N$ , which enters only in the eigenvalues  $\lambda_j = -N(j\pi)^2/8$ . This gives a meaning to the units of  $t$  and  $s$ , which are measured relative to the (new) steady state duration of price changes  $1/N$ . This remark is needed to interpret the time units in the horizontal axes of both panels of [Figure 7](#).

Figure 7: The propagation of monetary shocks as  $s$  grows



Note:  $N = 1$  (one price adjustment per unit of time, on average) and  $d\sigma/\sigma = 0.1$ .

(vii) To illustrate the general case of  $0 < s < \infty$  in [Figure 7](#) we display two plots. First, the left panel of [Figure 7](#) plots [equation \(34\)](#), evaluated at 4 values of  $s$ . It is apparent that as  $s$  becomes bigger monetary policy becomes less effective and gradually converges to the long run value. This can be seen by comparing the correction for any given  $t$  across the four values of  $s$ . Second, the right panel, plots the cumulated IRF of a monetary shock  $s$  periods after the volatility shock, relative to the cumulative IRF of a monetary shock when there is no volatility shock. In particular we plot:

$$\mathcal{C}(s, d\sigma/\sigma) \equiv \frac{\int_0^\infty Y(t, s, d\sigma/\sigma) dt}{\int_0^\infty Y(t, 0, 0) dt} - 1$$

We use the cumulated IRF to obtain a simple one-dimensional summary of this effect across all times  $t$ . Notice the following properties of  $\mathcal{C}$ : for all  $s$  we have  $\mathcal{C}(s, d\sigma/\sigma) = (d\sigma/\sigma)\mathcal{C}(s, 1)$ , since it is based on a derivative, and for extreme values of  $s$  we have  $\mathcal{C}(\infty, d\sigma/\sigma) = -d\sigma/\sigma$ , and  $\mathcal{C}(0, d\sigma/\sigma) = 0$ . From [Figure 7](#) it is clear that the transition to the higher volatility occurs very fast, a cumulative effect of  $\mathcal{C}$  half as large as half of the one in  $s \rightarrow \infty$  will occur when  $s_{1/2} \approx 0.05/N$ , a half-life indicated by a vertical bar in the right panel. More

precisely,  $s_{1/2}$  is defined as  $\mathcal{C}(s_{1/2}, d\sigma/\sigma) = -(1/2)(d\sigma/\sigma)$ . This effect is much faster than the half-life corresponding to the dominant eigenvalue  $\lambda_1 = -N\pi^2/8$ , which is given by  $t_{1/2} \equiv -8 \log(0.5)/(N\pi^2) \approx 0.56/N$ , and it is indicated by another vertical bar in the right panel. The ratio of the two times is very large:  $t_{1/2}/s_{1/2} \approx 12$ , and it is independent of any parameter of the model.<sup>16</sup> From this comparison we conclude that for this particular model using exclusively the dominant eigenvalue  $\lambda_1$  to approximate the time it takes for the distribution to converge after the change in volatility will be misleading. Summarizing, in the Golosov-Lucas model the short run effect of the volatility change is only relevant when the monetary shock occurs almost immediately after the volatility change.

## 8 Conclusion and Future Work

We have analytically characterized the impulse response function for a large class of sS models applied to price setting, using eigenfunctions-eigenvalues techniques. We have illustrated the usefulness of this method to study the effect of several shocks of macroeconomic interest. We have relied on assumptions of symmetry and lack of drift to derive many of the results. We established that for price setting problems in economies with low inflation deviating from such assumptions only has second-order consequences for the results. Our characterization also uses general equilibrium environment and price setting problem for the firm which implies that there are no first order strategic complementarities in the firm's optimal decision rules.

Open questions for future research will involve exploring setups featuring some combinations of asymmetries, large drift, and non-negligible strategic complementarities. For instance asymmetries are important in inventory models, such as those applying to the management of liquid assets, a large drift is important in models of capital relocation with frictions (where capital depreciation rates are large), and strategic complementarities can be important depending on the general equilibrium setup faced by firms. Technically, incorporating asym-

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<sup>16</sup>The vertical distance on the correction between  $s = 0$  and  $s \rightarrow \infty$  plotted on the right panel is  $d\sigma/\sigma$ , which is 0.1 for this example. For other values, the vertical axis scales proportionally.

metries and drift requires working out the dynamics of problems with reinjection, for which we already derived some result in this paper (see [Appendix D](#) and [Appendix E](#)) but more is needed. Incorporating strategic complementarities requires extending the result to endogenous moving boundary problems. We think that the use of similar techniques to analyze such case is feasible, but we believe it should be the subject of a work on its own.

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