Which likelihood ratio?

(Comment on “Why the effect of prior odds should accompany the likelihood ratio when reporting DNA evidence”, by Ronald Meester and Marjan Sjerps)

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Meester and Sjerps (2004) (henceforth M&S) draw attention to an ambiguity in the definition of “the likelihood ratio” that can arise with forensic multiple transfer evidence. Here I argue that this is a mathematical artifact and a logical red herring; in particular, for their data and assumptions only the likelihood ratio associated with their “first set of hypotheses” is suitable for presentation in court.

1 Background

Suppose that, prior to introducing the DNA evidence, we can accept the following background information $K$:

$K_1$: The crime was committed by two people.

$K_2$: Two stains were found, one on a pillow and the other on a sheet, each of which was left by one or other of the perpetrators.

$K_3$: John Smith (S) is accused of being one of the perpetrators.

$K_4$: John Smith’s DNA profile $\chi_S$ has been measured and found to be of type $A$.

$K_5$: DNA profiles $\chi_1$, $\chi_2$ have been obtained from the pillow stain and sheet stain respectively.

These are almost the same assumptions as made by M&S, except that I do not suppose a priori that the two stains must originate from different people. I do assume ($K_1$) that it is known in advance just how many perpetrators there
were, but only because not to do so would add too many complications. Also for simplicity I am regarding as firm what would in practice be an uncertain link between presence of $S$ at the scene of the crime and the ultimate issue of $S$'s guilt.

We start the analysis at the point where the impact of all previous information $I$, including $K$, has already been accounted for. We now wish to introduce the new evidence: the pair of measured values for the DNA stains $(\chi_1, \chi_2)$ found at the crime-scene.

## 2 Hypotheses

In the light of $K_2$ the following hypotheses are mutually exclusive and exhaustive.

- $H_1$: $S$ is the donor of the pillow stain only.
- $H_2$: $S$ is the donor of the sheet stain only.
- $H_3$: $S$ is the donor of both stains.
- $H_4$: There is a single donor of both stains, who is not $S$; the other perpetrator is $S$.
- $H_5$: There is a single donor of both stains, who is not $S$; the other perpetrator is not $S$.
- $H_6$: There are two distinct stain donors, neither of whom is $S$.

However, the hypothesis of direct legal interest,

$$G: \text{"John Smith is guilty"},$$

is none of these. Rather (in the light of $K_1$ and $K_2$) $G$ is the union of $H_1$ to $H_4$; similarly the hypothesis $\overline{G}$ of innocence is the union of $H_5$ and $H_6$. To address the impact of the evidence on such composite hypotheses we need to introduce prior probabilities $q_i = \Pr(H_i | I)$ over the simple hypotheses $H_i$ ($i = 1, \ldots, 6$).

## 3 Alternative data

To avoid unnecessary obfuscations, we first analyse the problem using different data than M&$S$. Thus suppose the DNA findings were $(\chi_1, \chi_2) = (A, A)$ (: $\xi'$, say); both stains are of type $A$. Table 2 gives the likelihood function over the hypotheses $H_1, \ldots, H_6$ based on these data, together with the associated prior and posterior probabilities (here $p_A = 1/R_A$ denotes the “match probability” for profile $A$, etc.). The posterior odds for $G = H_1 \cup H_2 \cup H_3 \cup H_4$ against $\overline{G} = H_5 \cup H_6$ are thus

$$\frac{\Pr(G | \xi', I)}{\Pr(G | \xi', I)} = \frac{q_3 + (q_1 + q_2 + q_4)p_A}{p_A(q_3 + q_6 p_A)} \quad (1)$$

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Table 1: Likelihoods, priors and posteriors for simple hypotheses: data \((A, A)\)

<table>
<thead>
<tr>
<th>Hypothesis (H_i)</th>
<th>Likelihood (p_A)</th>
<th>Prior (q_1)</th>
<th>Posterior (q_1 p_A)</th>
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We could, if desired, go on to divide these posterior odds by the corresponding prior odds \(\delta/(1 - \delta)\), where \(\delta = q_1 + q_2 + q_3 + q_4\) denotes the prior probability of guilt \(G\), to obtain the appropriate value for the likelihood ratio (again, as between \(G\) and \(\overline{G}\)) produced by the evidence \(\xi\). We draw particular attention to the fact that, for these data, the specific problem highlighted by M&S, namely which hypotheses to address, simply does not arise: there is no alternative but to compare \(G\) and \(\overline{G}\).

4 Original data

The above form of analysis should now help us to steer a clear path through the minefields sown by M&S for the case that the observed data are \((\chi_1, \chi_2) = (A, B)\) (\(\xi\), say). Table 2 gives the full prior-to-posterior analysis for these data.

<table>
<thead>
<tr>
<th>Hypothesis (H_i)</th>
<th>Likelihood (p_B)</th>
<th>Prior (q_1)</th>
<th>Posterior (q_1 p_B)</th>
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<td>(H_6)</td>
<td>(p_A p_B)</td>
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<td>(q_6 p_A p_B)</td>
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Table 2: Likelihoods, priors and posteriors for simple hypotheses: data \((A, B)\)

We now find that the posterior odds for comparing \(G\) with \(\overline{G}\) are

\[
\frac{\text{pr}(G \mid \xi, I)}{\text{pr}(\overline{G} \mid \xi, I)} = \frac{q_1}{q_6} \times \frac{1}{p_A}.
\] (2)
The corresponding likelihood ratio (for comparing $G$ with $\overline{G}$) is obtained by dividing the posterior odds by the corresponding prior odds:

$$LR = \frac{q_1/\delta}{q_0/(1 - \delta)} \times \frac{1}{p_A}.$$  \hspace{1cm} (3)

M&S consider other definitions of “likelihood ratio”, relating to hypotheses that are conditionally equivalent to $G$ and $\overline{G}$. But the very existence of such hypotheses is no more than an accidental mathematical artifact of the pattern of 0’s in Table 2. Exactly as in §3, our fundamental task remains that of comparing $G$ and $\overline{G}$: there was initially, and still remains after seeing the data, no reason to be interested in variant forms of these hypotheses.

In general (3) depends on the prior probabilities, but a few reasonable assumptions will eliminate this. We first note that

$$q_1/\delta = \Pr(S \text{ left the pillow stain only } | \text{ S is guilty})$$  \hspace{1cm} (4)

$$q_0/(1 - \delta) = \Pr(\text{there are two stain donors } | \text{ S is not guilty}).$$  \hspace{1cm} (5)

Now suppose:

(a). The number of distinct stain donors is independent of whether or not John Smith is guilty.

(b). Given that there are two stain donors, each is equally likely to have left either stain.

Let $\lambda$ denote the prior probability that there were two distinct donors. It then follows from (4) and (5) that $q_1/\delta = \frac{1}{2} \lambda$, $q_0/(1 - \delta) = \lambda$ and the likelihood ratio becomes

$$LR = \frac{1}{2} \times \frac{1}{p_A}.$$  \hspace{1cm} (6)

M&S appear to assume from the outset that $\lambda = 1$, i.e. that each culprit has left a stain, along, implicitly, with (b) above, (a) being vacuously true in this case. Then $q_3 = q_4 = q_5 = 0$, and the hypothesis $G = H_1 \cup H_2 \cup H_3 \cup H_4$ that John Smith is guilty becomes unconditionally equivalent to their hypothesis $H_1 \cup H_2 \cup H_3$ that John Smith was one of the crime stain donors; in particular the prior probability for their hypothesis is indeed the prior probability $\delta$ of guilt. But it should be noted that, under the more reasonable assumption $\lambda < 1$, none of the hypotheses considered by M&S is unconditionally equivalent to the only hypothesis of genuine interest, namely $G$. Before seeing the data $(A, B)$ there would typically be no good reason to eliminate the possibility that both stains had the same donor, i.e. to take $\lambda = 1$ (and I do not understand on what grounds, other than an illicit peek at the data, M&S have thought fit to do this). But in any event, so long as we can argue for the reasonableness of (a) and (b) prior to peeking at the new data, we can justify the expression (6) for the likelihood ratio in favour of $G$, John Smith’s guilt, based on observing $(\chi_1, \chi_2) = (A, B)$. 

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5 What to report?

Ideally the expert witness would simply report her posterior probability of John Smith’s guilt. As M&S verify, this must, reassuringly, be unchanged if we replace guilt and innocence by any other hypotheses that (given the new data) are conditionally equivalent to them.

However, this posterior probability depends not only on the new evidence but also on the prior probabilities, and these are quite properly regarded as in the province of the court rather than that of the witness: current recommended good practice is that a forensic expert may testify to the likelihood ratio issuing from the data, but not to her posterior probability of guilt. But, as noted by M&S, a potential difficulty with this recommendation is that switching between different conditionally equivalent hypotheses can deliver different likelihood ratio values. Then the concept of “the” likelihood ratio appears empty.

This does not however mean that anything goes: there can be very good reasons for preferring one set of hypotheses, and their associated likelihood ratio value, to another. In particular I have argued above that, in the two-trace problem, the only natural hypothesis to consider is $G$, that John Smith is guilty as charged. The court should, explicitly or implicitly, be presented with the likelihood ratio value (e.g. (6)) relevant to this specific hypothesis. (Of course, it must also be made clear that this is what is being done).

Rather than any likelihood ratio, M&S recommend presenting to the court a table giving the transformation from prior to posterior engendered by the evidence, leaving the court to decide on an appropriate value to insert for the prior. Such a table can certainly be of value in helping the court to understand the impact of a specific likelihood ratio value on the relevant hypotheses. However, this issue is quite distinct from the main problem M&S raise, of the shifting value of the likelihood ratio as we move between conditionally equivalent forms of the hypotheses. Because of the counterbalancing effect between prior and likelihood, this changes the prior probability, without having any effect on the posterior probability: a property that would not seem to be easily displayed by a tabular representation. The tabular approach thus does nothing to eliminate the need to specify carefully exactly which hypotheses are under consideration, which I find unclear in M&S’s description. I would regard it as potentially helpful only when used to ease the passage specifically from the prior probability of guilt $G$ to its posterior probability.

6 To double or not to double?

I have argued in favour of using $LR$, the likelihood ratio in favour of guilt, as an appropriate likelihood ratio figure to offer in court. For the case analysed of data $(A, B)$ under assumptions (a) and (b), $LR = \frac{1}{2} \times (1/p_A)$.

One further issue needs to be considered. Even though it may not be appropriate for an expert witness to offer evidence as to the value of the prior probability $\delta$ of John Smith’s guilt $G$, the fact $K_1$ that it is known that there
were two perpetrators must be highly relevant — typically it could be expected, roughly, to double whatever value might have been appropriate had there been only one. Since the treatment of single-perpetrator cases may be more familiar, this prior doubling requirement may not be easy for the court to appreciate or take into account. To assist the court, one might instead incorporate this doubling directly into a quoted “surrogate likelihood ratio”, leaving the court to assess a prior for $G$ as if the case were of the standard single-perpetrator kind. For the case analysed this would lead to quoting $1/p_A$. Although this happens to agree with the likelihood ratio for M&S’s “second set of hypotheses”, in which we directly compare $H_1$ and $H_0$, that is purely accidental. Indeed, for different data, e.g. $(A,A)$, this second set of hypotheses would not even be introduced, but the argument for doubling the original likelihood ratio would be unaffected.

Whether or not this doubling would be a good idea (and I admit that it could easily add confusion rather than clarity) will be a matter, not of logic, but of psychology — especially, whether or not the court would be likely to neglect to take proper account of the impact of the prior evidence $K_1$. If so, then doubling could provide a crude form of correction for this. But it would be preferable, when practicable, to discuss in detail all the evidence behind the prior assessment of guilt, and give some guidance as to how this might be quantified. With an appropriate value for $\delta$, no doubling is called for.

7 Dependence of likelihood ratios on the prior

Although the particular likelihood ratio (6) is independent of prior probabilities, this is not always so. For example, consider again the data $\xi$ of §3. We might again accept (a) and (b), and even add:

(c). When there is only one stain donor, this is equally likely to be either of the two perpetrators.

Then the values of $q_1, \ldots, q_b$ are respectively $\frac{1}{2} \lambda \delta$, $\frac{1}{2} \lambda \delta$, $\frac{1}{2} (1 - \lambda) \delta$, $\frac{1}{2} (1 - \lambda) \delta$, $\frac{1}{2} (1 - \lambda) \delta$, $\frac{1}{2} (1 - \lambda) \delta$, and the posterior odds (1) become

$$
\left( \frac{\delta}{1 - \delta} \right) \times \frac{1}{2} \left\{ \frac{1}{p_A} + \frac{1}{1 - \lambda + \lambda p_A} \right\}.
$$

(7)

The likelihood ratio in favour of guilt is obtained by deleting the first term in (7) — but what remains will still be sensitive to prior assessments: specifically, to the prior probability $\lambda$ that both perpetrators left stains.

This feature raises important issues of calculation and reporting that it may not be easy for an expert witness to address, especially in still more realistic cases where we cannot even assume we know the number of perpetrators. Such issues are, however, entirely unrelated to our principal question of which hypotheses to address, and need not be considered further here.
8 The database problem

M&S point to some parallels between the two-stain problem and the database search problem of their Example 2. In the latter, we assume as background that the criminal left the DNA crime trace, and entertain a collection of simple hypotheses $H_i$: “Individual $i$ is the criminal”. Then (assuming no laboratory error etc.) the search findings allow us to exclude. a posteriori, all those individuals in the database whose DNA does not match the crime trace. When John Smith ($S$) is the only match found, the compound hypothesis $D$ that the culprit is in the database is then conditionally equivalent to the simple hypothesis $H_S$ that Smith is the culprit. As emphasised by Stockmarr (1999) the likelihood ratios relevant to these two can be very different; but (David 2001) the differences in the priors exactly compensate for this. As a matter of logic, there can be no ambiguity over the calculation of the quantity of fundamental importance: the posterior probability that John Smith is guilty.

Although these points may seem very close to those discussed by M&S, there are some important differences between the two-stain problem and the database problem.

In the two-stain problem, the hypotheses $G$ and $\overline{G}$ of direct legal interest are composite. Given data $(A,B)$ it so happens that they can be replaced by the conditionally equivalent simple hypotheses $H_1$ and $H_0$ — but, of itself, this latter comparison was not, and is not, of any intrinsic interest.

In contrast, in the database search case we are genuinely interested in the various simple hypotheses $H_i$. It turns out that, following a search resulting in a single match to John Smith, the simple hypothesis $H_S$ of principal interest becomes conditionally equivalent to the compound hypothesis $D$ that the perpetrator is in the database; but this time it is the compound hypothesis that is not of any direct interest in itself. (To see this, suppose we were to allow a non-zero probability that any DNA profile in the database has been incorrectly recorded. Then $H_S$ and $D$ would no longer be conditionally equivalent — and now we would no longer have any reason to care about $D$.) Thus if we are required to quote a likelihood ratio, this should again relate to the hypotheses of substantive interest to the court: the guilt or innocence of John Smith. The appropriate value\footnote{Contra Stockmarr (1999), this is not a “data-dependent hypothesis”, but merely reflects one possible focus of our interest: there is no reason to refrain from considering the possible guilt $H_i$ of other specified individuals $i$, whether within or without the database. This would be a non-trivial consideration even for $i$ within the database if there were more than one match, or if we could not exclude laboratory error.} is $LR = (1/p_A) \times (1 - \delta)/(1 - \pi)$ — which then needs to be combined with the prior probability $\delta$ that $S$ is guilty.

Logically this procedure is essentially the same as would be appropriate for analysing any other kind of evidence, where the possibility of conditional equivalence did not even arise. However, as in §7, we again face the problem of the dependence of $LR$ on the prior probabilities $\delta$ and $\pi$. This effect can be ignored\footnote{Note that the use of $\delta$ and $\pi$ by M&S is opposite to that of David (2001). However, there was an unfortunate error in equation (3) of David (2001), where these two symbols were transposed: and this has resulted in a corresponding error in equation (8) of M&S.}
for $\pi$ very much smaller than 1, since we can then (conservatively) approximate $LR = (1/p_A)$. Otherwise it might well be helpful to present the impact of the evidence in a form such as M&S’s Table 2, having first made some simplifying assumptions to express $\pi$ as a function of $\delta$. Again this presentational issue is independent of the principal question of which hypotheses we should be addressing. I would argue that such a table is only of interest if it relates specifically to the effect of the evidence on the probability of John Smith’s guilt.

References

