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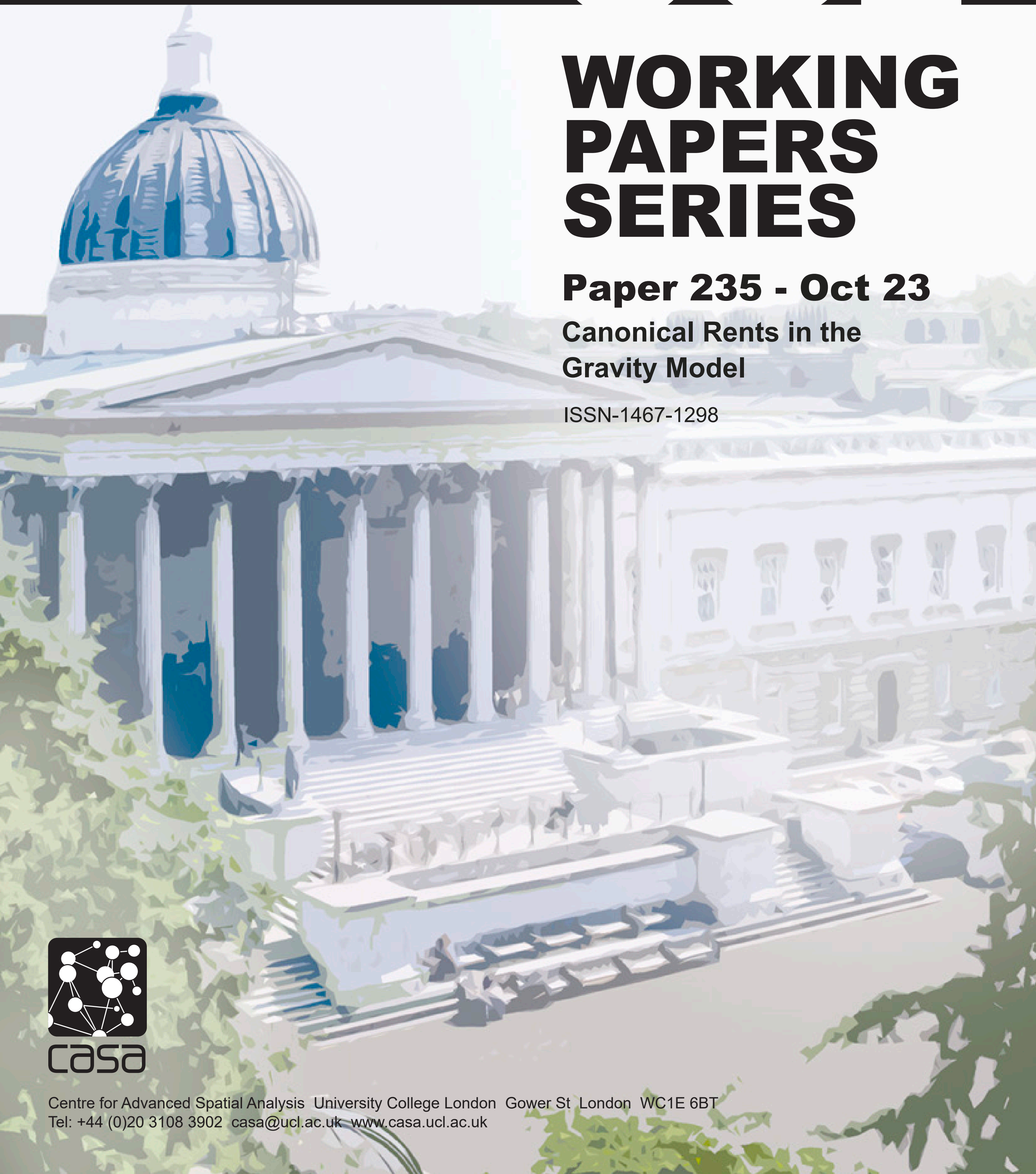

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Canonical Rents in the Gravity Model

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Canonical Rents in the Gravity Model

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Abstract

Some of the rents identified in the Maximum Entropy(ME) gravity model are always negative. We show this is not true for the Minimum Mutual Information(MMI) formulation of the model or for the Uniform prior (UP). Both methods give identical positive rents lending weight to the idea that these rents are invariant to the choice of baseline. Furthermore, we show that the differences in origin rents and in destination rents between ME and the UP and MMI models are both constant which preserves the ME trip matrix and mean trip cost.

1 Introduction

Negative rents arise in the Maximum Entropy negative exponential gravity model because the balancing factors may be greater or less than one meaning that under the total flow constraint other factors must be less or greater than one than one. In the Maximum Entropy model negative rents arise when the balancing factors are greater than one. They serve to correct flows that are otherwise too low to conform to the constraints. The converse is true for balancing factors less than one. In the monopolar von Thünen model the boundary of the model is taken to be where rents go to zero. This is the point at which agricultural supply to the central city ceases because the cost of travel is too high. Beyond this cost boundary is von Thünen's wilderness. This analysis is not appropriate in the multipolar case since the cost of travel to one pole may be too high but others may be reachable and, for most study areas, there is no wilderness. Nevertheless, the rents in the gravity model are of the von Thünen type albeit they are multipolar (Morphet, 2015). The interpretation of the balancing factors as functions of rent has long been suggested (Dieter, 1962)and known(Cesario, 1977) and the solution for the problem of negative rents has been taken to be adding to all rents an amount slightly greater than than the most negative rent(Williams and Senior, 1978; Shabrina and Morphet, 2022). This is a somewhat informal solution which forces uniform arbitrary, uplifts on origin and destination rents as but preserves the trip matrix and hence the average trip cost. In this working paper we investigate the use of a model derived datum and show how this allows the estimation of rents which are canonical in the sense that they are endogenous and hence comparable across and between study areas. We compare three doubly constrained models, the standard Maximum Entropy (ME) model,the Minimum Mutual Information (MMI) model and the Uniform Prior (UP) model. We use two data sets the first with a 5x5 trip matrix is essentially a toy model(Kirby, 1970).This model is exact in that its elements are chosen to fit exactly a given value of the deterrence function exponent. This permits an examination of the structure of the models uncluttered by any random sampling effects. The second is the Arcadia Model (Batty, 2009) which, with a 1767x1767 trip matrix based on Census data shows random effects and reflects some of the variation faced in practical analyses. It is therefore, more realistic and covers an area approximating the Outer Metropolitan Area of London and South East England. We also demonstrate a method for checking the validity of the sign conventions used in calculating the rents. The results (see the Appendix) show that positive rents can be relied upon when using the Minimum Mutual Information and Uniform Prior models and further, that the differential between these rents and those derived from the Maximum Entropy model is a constant. This ensures that the trip matrices are unaffected by the method of rent estimation and thus emphasise the consistency between the model types whilst rendering unnecessary the questionable use of an arbitrary uplift.

The importance of the rents relates to both their absolute values and to changes in those values under the influence of transport and density changes. The differential values remain useful in analysing the effects of transport investments including their relative distributional effects. The non-negative values are easier to explain and are useful in relating the derived rents to observed rents as part of validation.The absolute values are also of use in analysing the nature of the property market as it relates to access, incomes and the location of benefits arising from transport investments.

The gravity model seeks to represent the reaction to travel costs of travellers seeking interactions over space. Since travel costs are an impediment to travel they represent a restraint on interaction. Where this interaction is trade in goods, services, ideas and information, travel costs are a restraint on trade and accordingly the model is a model of imperfect competition. This results in prices within the market being studied, being higher or lower than those of perfect competition. The reasons for this are discussed in section 6.

2 The Models

We look at three closely related models. The Maximum Entropy model which has a deterrence function of $e^{-\beta c_{ij}}$, the Minimum Mutual Information model with a deterrence function of $p_i p_j e^{-\beta c_{ij}}$ and the Uniform Prior model with a deterrence function of $n^{-2}e^{-\beta c_{ij}}$ where n is the number of zones. All three models give the same trip matrix and the same value of β when calibrated. It should be noted that the rent values derived from the balancing factors are rents per trip just as the costs are costs per trip.

2.1 Maximum Entropy

The Maximum Entropy model is derived by maximising the entropy in the constrained Lagrangian as follows:

$$\mathcal{L} = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} ln p_{ij} - \lambda_0 \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} - 1 + \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} p_{ij} - p_{i*} - \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} p_{ij} - p_{*j} + \beta \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} c_{ij} - \bar{c} \tag{1}$$

Differentiating $\mathcal{L}$ with respect to p_{ij} and setting the result to zero delivers the Maximum Entropy model thus

$$p_{ij} = e^{-\lambda_0} e^{-\lambda_i} e^{-\lambda_j} e^{-\beta c_{ij}} \tag{2}$$

In this model λ_0 is a normalising factor usually designated as the partition function and $e^{-\lambda_i}$ and $e^{-\lambda_j}$ are origin and destination balancing factors. The rents in this case are given by $-\frac{1}{\beta} ln\lambda_i$ and $-\frac{1}{\beta} ln\lambda_j$ respectively (Morphet, 2015).

2.2 Minimum Mutual Information

Similarly the Mutual Information is minimised in the constrained Lagrangian as follows:-

$$\mathcal{L} = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} ln \frac{p_{ij}}{p_i p_j} - \lambda_0 \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} - 1 - \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} p_{ij} - p_{i*} - \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} p_{ij} - p_{*j} - \beta \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} c_{ij} - \bar{c} \tag{3}$$

Differentiating $\mathcal{L}$ with respect to p_{ij} and setting the result to zero delivers the Minimum Mutual Information model thus

$$p_{ij} = p_i p_j e^{-\lambda_0} e^{-\lambda_i} e^{-\lambda_j} e^{-\beta c_{ij}} = e^{-\lambda_0} e^{-\lambda_i + lnp_i} e^{-\lambda_j + lnp_j} e^{-\beta c_{ij}} \quad (4)$$

In this model the $p_i p_j$ factor acts as a prior corresponding to a rent of $-\frac{1}{\beta} ln(p_i p_j)$ which is the joint rent corresponding to zero trip cost i.e the rent at the von Thünen pole (see Figure 1). This joint rent or Zero Trip Cost Rent(ZTCR) factors into $ZTCR_i$ an origin rent $-\frac{1}{\beta} ln(p_i)$ and $ZTRC_j$ a destination rent $-\frac{1}{\beta} ln(p_j)$. In the derivation of the relationship between rent in the von Thünen model and rent in the gravity model the term $ln(p_i) - \lambda_i$ in the Minimum Mutual Information model plays a similar but not identical role to the balancing factor $-\lambda_i$ in the Maximum Entropy model. The former term results in a rent of $\frac{-1}{\beta}(ln(p_i) - \lambda_i)$ which is positive for $-ln(p_i) > \lambda_i$ which is the case when p_i is less than the balancing factor. In the Minimum Mutual Information model the λ values act as corrections to the $\log(p_i)$ or ZTCR values.The two elements in the Minimum Mutual Information balancing factors may be seen to reflect spatial pattern ($ln(p_i)$) and spatial interaction ($-\lambda_i$) in a manner rather simpler than those reviewed by Oshan (2021).

2.3 Uniform Prior

The UP model resembles the MMI model but with the replacement of the $p_i p_j$ factor by n^{-2} where n is the number of zones. The model is derived as in equations (3) and (4).

$$p_{ij} = n^{-2} e^{-\lambda_0} e^{-\lambda_i} e^{-\lambda_j} e^{-\beta c_{ij}} = e^{-\lambda_0} e^{-\lambda_i - \ln(n)} e^{-\lambda_j - \ln(n)} e^{-\beta c_{ij}} \quad (5)$$

3 The causes of negative rents

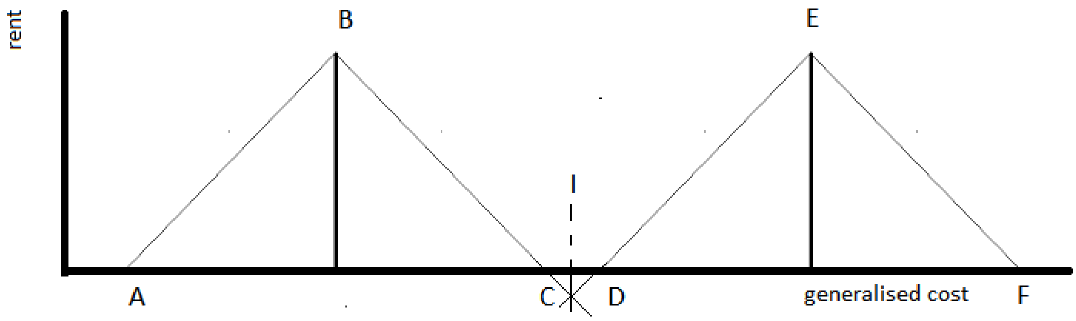

Figure 1: von Thünen Negative Rent

It is important to consider the causes of negative rents in reality and distinguish them from those caused by the method of deriving the model and its method of calculation.

3.1 Negative rents in reality

In reality a site may have a negative rent or value because of its associated costs of remediation arising from pollution or the costs of providing access and services. Such negative rents arise irrespective of the distribution of trip ends and population. Negative rents may also arise in cases where the rent paid is regulated to be less than a market rent. This is a defining feature of social housing and reflects not just the basic need for shelter but also the dependence of the housing/commercial/land market on a service labour force for which it does not fully pay. Although such reasons are of practical and policy importance they will only be reflected in the model by the extent to which they influence origins and destinations. However, if negative rents are caused by components of reality reflected in the model i.e trip costs and trip ends, then they should be regarded as identifying areas in need of greater access and/or development. Where the model is one of journey to work then negative rents may flag up areas disconnected from the labour market being modelled. In this case a negative rent may mean that rents are accruing outside the modelled area. We may also regard a negative rent for a particular origin zone as the subsidy needed for workers to pay a rent sufficicient to join the labour market being modelled. Similarly, negative destination rents may reflect the subsidy needed for employers to locate in a given zone.

We distinguish between negative rents which should not arise in the model and those rents which are greater or lesser than the rents we might expect under perfect competition. Such rents will arise in the model as a result of trip and location decisions being taken in an imperfect market characterised by an oligopolic supply of transport services, imperfect information and lags in the housing and land markets. This is considered further in section 6.

3.2 Negative rents in the model

The difficulty in measuring rents in the model lies in the definition of a zero rent baseline. The lowest modelled local rent will be defined by the intersection of the von Thünen cones with each other (see Figure 1) as analysed by Launhardt (1885). In the model neither the level von Thünen plain nor the von Thünen wilderness are defined. The rents themselves are generated by the interaction of costs and the map pattern of trip origins and destinations. Indeed we can view the model as being that for a change of pattern of the origin map into the destination map. In the ME model we expect negative rents when the balancing factors are greater than one and this gives us two potential explanations which both depend on whether the balancing factor is greater or less than one.

1. the balancing factor is a multiplier of p_i or p_j implying that the level of trips is lower or higher than that justified by the trip cost
2. the balancing factor is a multiplier of $e^{-\beta c_{ij}}$ implying that trip costs are too low or too high for the given number of trips.

It may also be that both mechanisms operate simultaneously.

The choice of explanations depends on whether we concentrate on trips or trip costs. We work mainly with the first as it focusses on the trip matrix margins as do the estimated rents. It is also implicit in the Minimum Mutual Information and Uniform Prior model structures since the balancing factors together with p_i, p_j or n^{-2} factor into origin and destination effects whereas c_{ij} does not and of course, c_{ij} is considered as fixed in the model. If we find surprisingly low or negative rents in the model it is likely to reflect areas with a dearth of public or private transport. This is unlikely in a 2001 model of South East England as presented below. It may however, be found in areas of the UK where public transport services do not offer full coverage. It does not of course, mean that there are individuals receiving the benefits of a negative rent but rather they are unable to derive a living within the labour market and must rely on other means of income support. In practice negative rents will not arise if we use the Minimum Mutual Information formulation of the model but will certainly be seen when we use the Maximum Entropy version. The rents calculated are based on accessibility levels and it may be the case that costs of land remediation are such as to result in a negative vauation. Further, In agricultural areas access to labour markets may not be the driver of rent values and agricultural land values based on fertility and market access may dominate.

4 Model comparison

In this section we compare the Maximum Entropy, Minimum Mutual Information and Uniform Prior models in their characterisation of rents. We use a 5x5 model (Kirby, 1970) as this allows visual inspection of the results. The cost matrix is symmetrical and the trip matrix has a β value of 0.1 with trip matrix and origins

Table 1: Cost data

	Z1	Z2	Z3	Z4	Z5
Z1	10.0	14.1	14.1	14.1	14.1
Z2	14.1	10.0	20.0	28.3	20.0
Z3	14.1	20.0	10.0	20.0	28.3
Z4	14.1	28.3	20.0	10.0	20.0
Z5	14.1	20.0	28.3	20.0	10.0

Table 2: Zero trip cost interaction rents

	Oi	Dj	Z1	Z2	Z3	Z4	Z5
Z1	29.9573	6.9315	36.8888	41.9971	52.9832	59.9146	59.9146
Z2	29.9573	12.0397	36.8888	41.9971	52.9832	59.9146	59.9146
Z3	12.0397	23.0259	18.9712	24.0795	35.0656	41.9971	41.9971
Z4	6.9315	29.9573	13.8629	18.9712	29.9573	36.8888	36.8888
Z5	23.0259	29.9573	29.9573	35.0656	46.0517	52.9832	52.9832

and destinations chosen to agree with this. This gives an exact model with a level of exactitude that is useful in determining relationships that exist in the model. The small size of the model allows element by element inspection of its workings. In practice the observed trips would be from a sample that is randomly selected over a given period so exact results as shown below are unlikely to be reproduced in reality unless they represent invariant properties of the model. Models based on observed data will tend to propagate any random variations in the observations. Later in section 5 we carry out a full scale comparison that parallels the 5x5 test.

As rents decline with increasing distance the zero trip cost or perfect competition rents(ZTCR) arise when the trip cost is zero giving $p_{ij} = p_i p_j$ with origin and destination rents of $-\frac{1}{\beta}ln(p_i)$ and $-\frac{1}{\beta}ln(p_i)$ respectively. They are necessarily positive as p_i and p_j are both less than unity. Under perfect competition we might expect ZTCR to be a maximum rent but under imperfect competition this is no longer the case (see section 6)

4.1 Base Data

The test model has a 5x5 matrix of trip costs (Table 1), a set of origins and destinations (Table 3). The value of β is 0.1 giving an average trip cost of 16.38. The flow matrix, $p_i p_j$, corresponding to a zero trip cost is shown in Table 2 and acts as a premutiplier to $e^{-\beta c_{ij}}$ in the Minimum Mutual Information model.

4.2 Maximum Entropy Model

For this model the deterrence function is $e^{-\beta c_{ij}}$ For each model we show a flow matrix (Tables 4 and 7),mean costs (Tables 5 and 8) and balancing factors and rents (Tables 6 and 9).

Table 3: Origins and Destinations by zone

	Z1	Z2	Z3	Z4	Z5
Origins	500	500	3000	5000	1000
Destinations	5000	3000	1000	500	500

Table 4: MaxEnt Flows by zone

	Z1	Z2	Z3	Z4	Z5
Z1	215	211	37	13	25
Z2	143	319	20	3	14
Z3	1305	1069	505	66	56
Z4	2882	1029	410	395	283
Z5	455	372	28	23	122

Table 5: MaxEnt Mean Costs

	Trips	Orent	Drent
	16.38	4.516835	3.977007

4.3 Minimum Mutual Information Model

For this model the deterrence function is $p_i p_j e^{-\beta c_{ij}}$. A comparison of the flow tables 4 and 7 shows the well known result that they are identical with the consequence that their mean trip costs shown in table 5 and 8 are also identical.

```
## [1] "Sum ln(Pij) for 5x5 model=-108.169968559108"
```

```
## [1] "Sum of cost data for 5x5 model=436"
```

```
## [1] "Estimated 5x5 beta value=0.0953784043109038"
```

The differences all arise in the balancing factors and rents with the average rents of the Minimum Mutual Information model being substantially greater than the Maximum Entropy model rents as shown in tables 5 and 8.

In Table 9 the columns Orent and Drent are those elements of rent derived from the λ values alone.They are the corrections to the ZTCR rents which result in the Minimum Mutual Information rents. It will be seen that adding the corrections to the zero tripcost rents results in all rents being positive (Table 9). In figure 2 we compare the MaxEnt and MMI rents for origins and for destinations. It will be seen that the relationship is linear with the MMI rents being larger but with a similar range.The slope of the line is 1 in both cases indicating thatt they differ only by a constant. The zero trip cost trip distribution is one of perfect competition as the cost of travel is no longer an impediment to trade. We may thus regard the λ_i and λ_j as measures of the extent of the difference between perfect competition and the imperfect competition represented in the Minimum Mutual Information model.

We can see from figures 2 and 3 that the minimumum information rents are positive and we detail below (section 6) why this is so.

Table 6: MaxEnt Factors

	BFi	BFj	Orent	Drent	OriginZTCR	DestinationZTCR
Z1	0.160	1.860	18.30	-6.23	30.00	6.93
Z2	0.161	2.750	18.30	-10.10	30.00	12.00
Z3	1.470	0.479	-3.82	7.37	12.00	23.00
Z4	3.240	0.170	-11.70	17.70	6.93	30.00
Z5	0.511	0.331	6.72	11.10	23.00	30.00

Table 7: Min Inf Flows

	Z1	Z2	Z3	Z4	Z5
Z1	215	211	37	13	25
Z2	143	319	20	3	14
Z3	1305	1069	505	66	56
Z4	2882	1029	410	395	283
Z5	455	372	28	23	122

Table 8: MMI Mean Costs

	Trips	Orent	Drent
	16.38	12.23657	11.70617

Table 9: Min Inf Factors by zone

	BFi	BFj	Orent	Drent	Origin ZTCR	Destination ZTCR	MI Origin rent	MI Destination rent
Z1	0.599	0.777	5.120	2.5200	30.00	6.93	35.10	9.45
Z2	0.603	1.910	5.060	-6.4900	30.00	12.00	35.00	5.55
Z3	0.915	0.997	0.893	0.0268	12.00	23.00	12.90	23.10
Z4	1.210	0.707	-1.920	3.4700	6.93	30.00	5.01	33.40
Z5	0.956	1.380	0.447	-3.2000	23.00	30.00	23.50	26.80

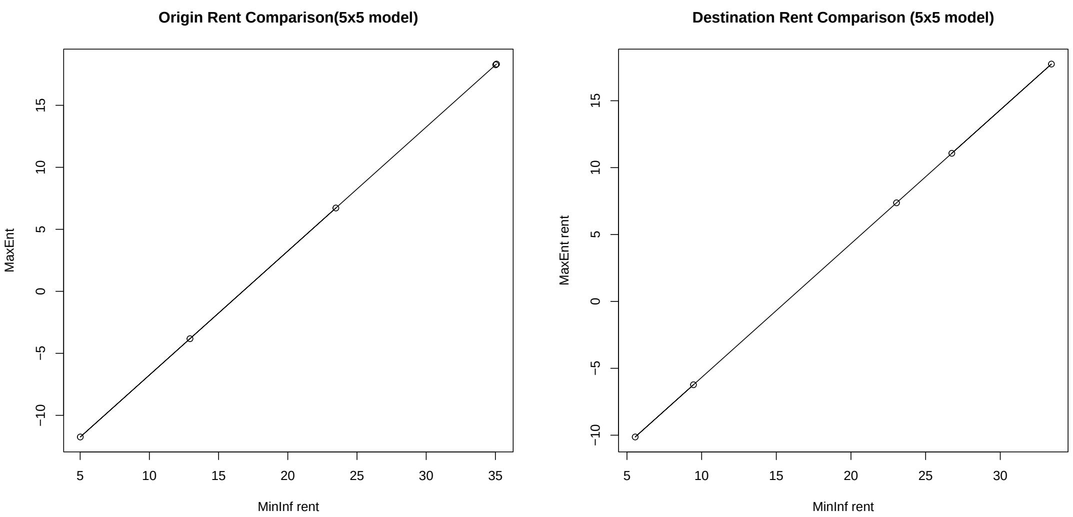

Figure 2: Origin and Destination Comparison

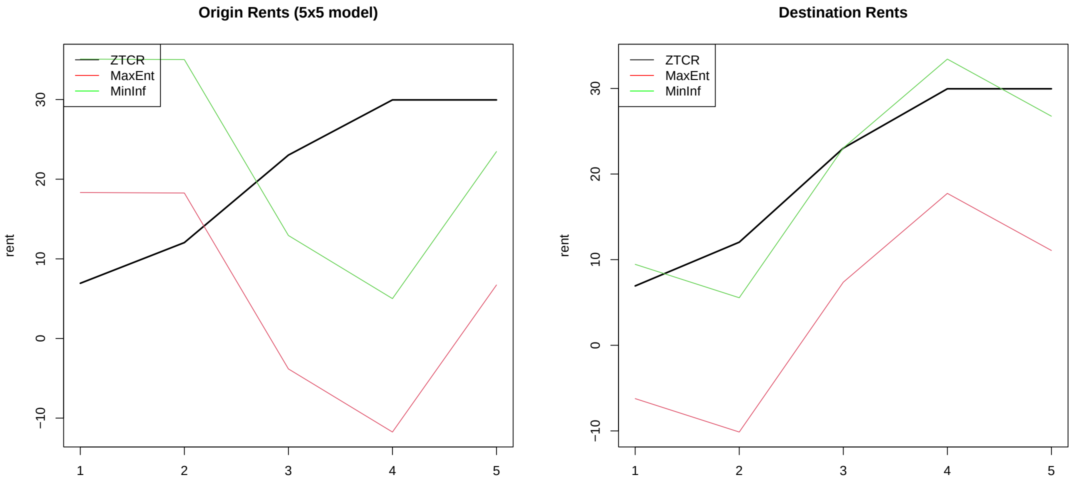

Figure 3: Origin and Destination Profile Comparison

Table 10: Energy Balance

	Entropy	Mean Trip Cost	Rent	Free Energy
MaxEnt	24.20065	16.38	-8.493842	-16.31449
MMI	24.20065	16.38	23.942745	16.12209

A negative rent requires that $p_i e^{-\lambda_i} > 1$ i.e. that $ln(p_i) - \lambda_i > 0$ but $ln(p_i)$ is always negative and $-\lambda_i$ is negative. A problem may arise when we have a positive λ_i expression. In that case $ln(p_i) + \lambda_i > 0$ if $\lambda_i > |ln(p_i)|$. This is a situation that becomes more likely as the number of zones increases and the average value of p_i falls and the absolute value of $ln(p_i)$ increases.

An inspection of figures 2 and 3 together with their supporting tables 6 and 9 suggests that the slope is unity meaning that the Minimum Mutual Information and Maximum Entropy rents differ only by a constant thus:

$$\begin{aligned} Minimum\,Information\,Origin\,rent &= Maximum\,Entropy\,Origin\,rent + 16.75 \\ Minimum\,Information\,Destination\,rent &= Maximum\,Entropy\,Destination\,rent + 15.68 \end{aligned} \tag{6}$$

Since it is sometimes difficult to determine the correct sign convention for the rents it is useful to have a check. This is possible using the equation of state of the model given by:

$$\frac{1}{\beta} S = U + R - G \tag{7}$$

where S is entropy, U is mean trip cost, R is expected rent(origin plus destination) and G is the Gibbs free energy corresponding to the consumer surplus $-\frac{1}{\beta} lnZ$ where Z is the log sum or partition function.

The two different constants are both greater than the corresponding most negative rents of the Maximum Entropy model of -11.7 and -10.1 as shown in tables 6 and 9.

4.4 The Uniform Prior Model

Comparison of figures 4 and 5 for the UP model with those for the MMI model,(figures 2 and 3) shows that the UP rents are exactly the same as the MMI rents. This is confirmed in table 11 suggesting that the rents may be invariant to a wide range of biproportional distribution pre-multipliers reinforcing the validity of their use in terms of model consistency.

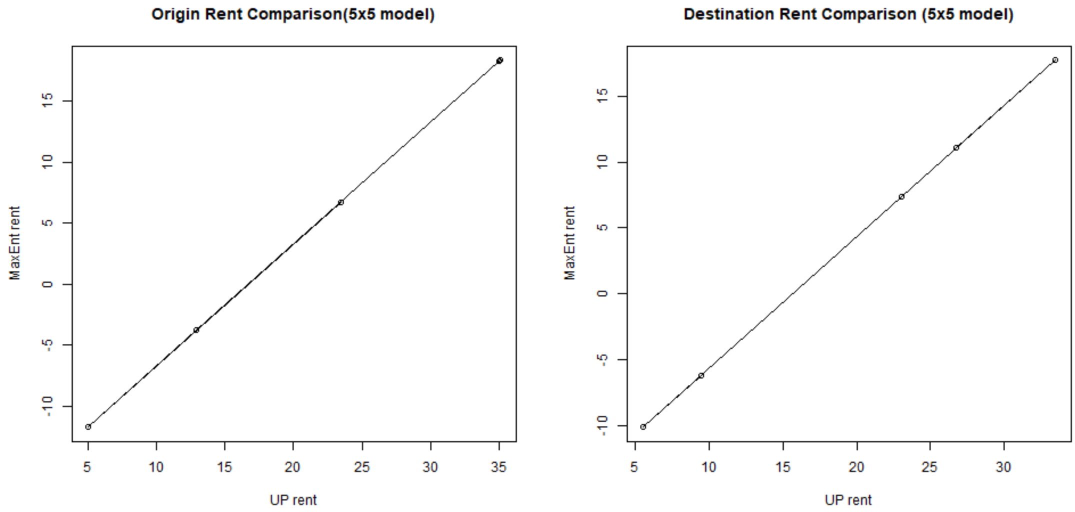

Figure 4: Uniform Prior Origin and Destination Comparison

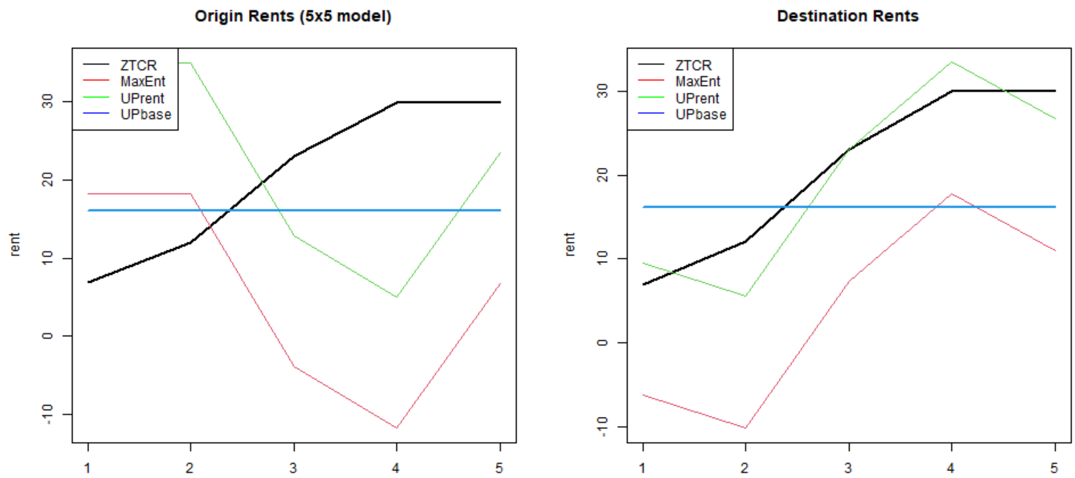

Figure 5: Uniform Prior Origin and Destination Profile Comparison

Table 11: UP and MMI Rents

	Z1	Z2	Z3	Z4	Z5
UP Origin Rent	35.07964	35.021226	12.93312	5.008697	23.47246
MMI Origin Rent	35.07964	35.021226	12.93312	5.008697	23.47246
UP Destination Rent	9.45356	5.550701	23.05261	33.424534	26.75386
MMI Destination Rent	9.45356	5.550701	23.05261	33.424534	26.75386

5 Full Scale Test

In this test we use the Arcadia model (Batty, 2009) for the London Metropolitan Area. It has 1767 zones but over 80% of the 1767x1767 interchanges are zero. We calibrate the Maximum Entropy model using the maximum likelihood method of Hyman (1969). The resultant value of β is 0.13975 and this is also used for the Minimum Mutual Information model. We test the ME and MMI models only as the UP model duplicates the latter.

The resulting model has a β value of 0.139748 and a mean trip cost of 18.9350504 These values are used in both the Maximum Entropy model and the Minimum Mutual Information model.

5.1 The Arcadian Maximum Entropy and Minimum Mutual Information Models

In this section we reproduce for comparison the analyses of the 5x5 model in section 4.

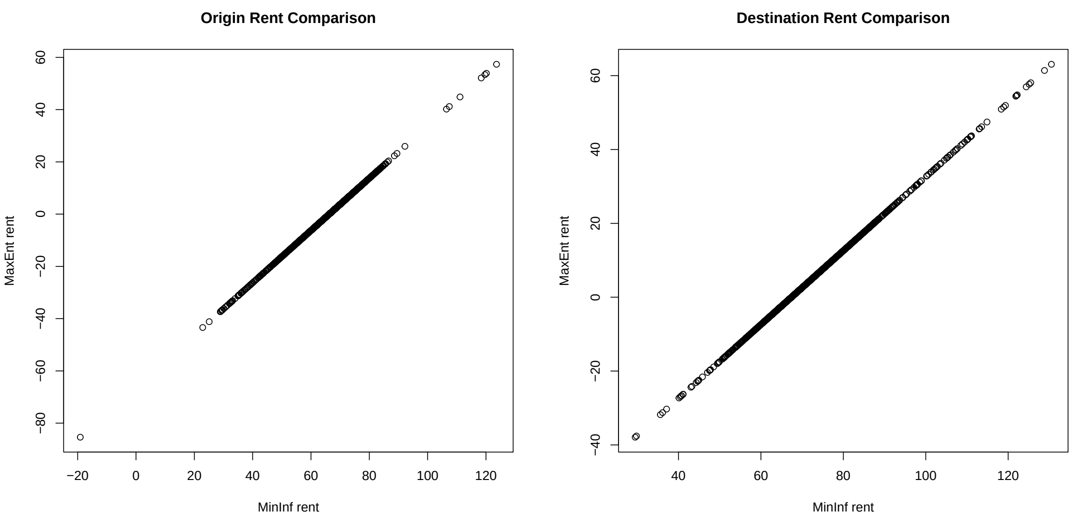

Figure 6: Arcadia Origin and Destination Comparison

Figure 6 shows a similar pattern to its equivalent in the 5x5 model (Figure 2. They do, however, show a similarity in that the origin patterns are like the destination patterns.

The average rents in Table 12are simple means rather than the trip weighted expectations of the rents used below in Figure 8.

Figure 7 shows that the Minimum Mutual Information and Maximum Entropy rents are parallel like those in Figure 3.

Table 12: Mean Costs

	Beta	Trip Cost	Orent	Drent
MaxEnt	0.139748	18.93574	-1.487205	3.355748
MMI	0.139748	18.93574	49.972205	48.629073

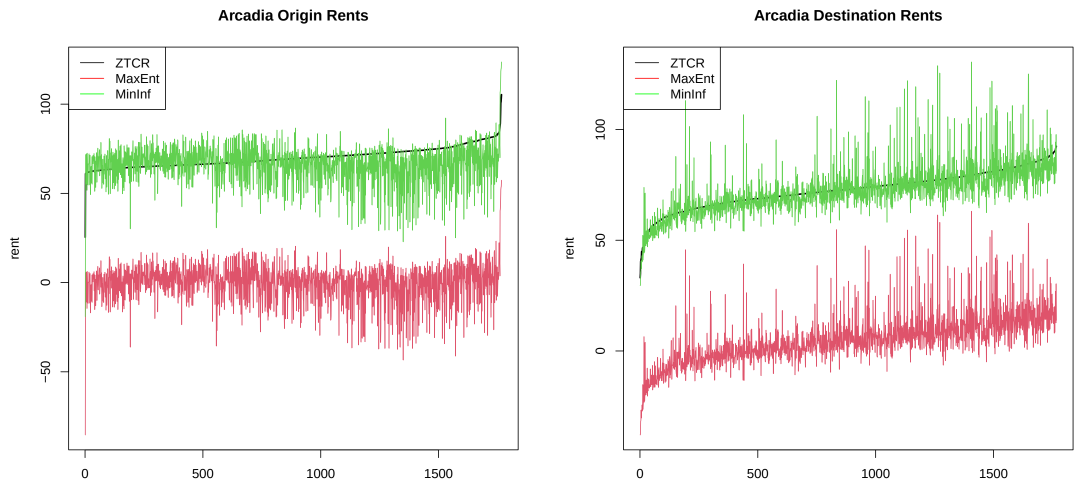

Figure 7: Arcadia Origin and Destination Profile Comparison

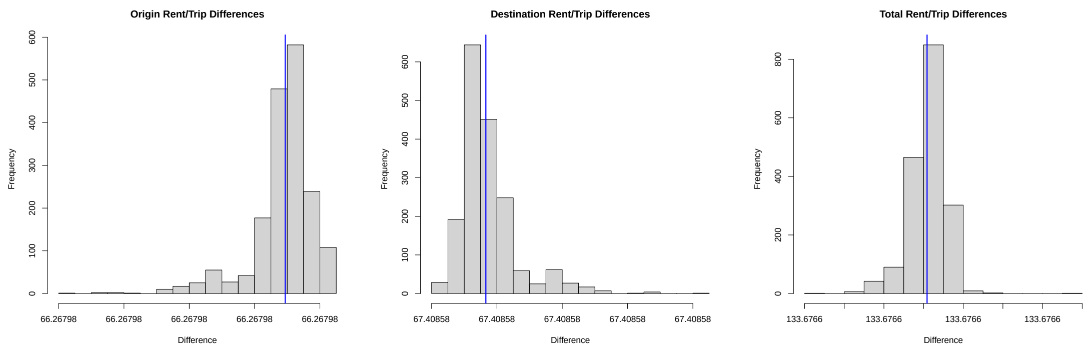

Figure 8: Arcadia MaxEnt and MMI Rent Difference Distributions

Table 13: Arcadia Energy Balance

	Entropy	Mean Trip Cost	Rent	Free Energy
MaxEnt	86.70323	18.93574	-5.560129	-73.32763
MMI	86.70323	18.93574	98.601278	30.83378

Figure 8 shows the distribution of the differences in total zonal rent values(e.g Oi x origin rent(i)/trip)) about their mean (vertical line)which corresponds to the uplift. It should be noted that the rent per trip is a constant but the expected rents are subject to random variation and this is true for the differentials which are identical for all origins and for all destinations The distributions show a similar skewed shape reflecting the zone size in terms of origins or destinations. The origin rent uplift is 51.4594097 and that for destinations 52.7019974 so we may write:

$$\begin{aligned} Minimum\, Information\, Origin\, rent &= Maximum\, Entropy\, Origin\, rent + 51.46 \\ Minimum\, Information\, Destination\, rent &= Maximum\, Entropy\, Destination\, rent + 52.70 \end{aligned} \tag{8}$$

This pair of equations may be compared with those in equation (6)

We see that in both cases in table 13 the difference of rent minus free energy is 67.7675. This matches the difference of entropy (in energy terms) minus mean trip cost and confirms equation (7) in the case of the Arcadia model. We see that although at equilibrium, the two parts of equation (7) are separable, dynamically considered the two parts must keep their differences equal

6 Discussion

The comparison between the exact 5x5 model and the Arcadia model allows us to expose those effects which are endogenous characteristics of the model and those which emanate from random or systematic errors in the data. Of course the fact that the 5x5 model is exact does not mean it is necessarily good as it may be mispecified for the purpose in hand. We can however, say from the exact model relationships between Maximum Entropy rents and Minimum Mutual Information rents that the Minimum Mutual Information model succeeds in ensuring rent positivity and we can gain insight into why if we explore the equivalence between the two models thus:

$$p_{ij} = \frac{e^{-\beta c_{ij}} e^{-\lambda_i^E} e^{-\lambda_j^E}}{Z^E} = \frac{p_i p_j e^{-\beta c_{ij}} e^{-\lambda_i^I} e^{-\lambda_j^I}}{Z^I} \tag{9}$$

so

$$ln(p_i) + ln(p_j) - \lambda_i^I - \lambda_j^I - lnZ^I = -\lambda_i^E - \lambda_j^E - lnZ^E \tag{10}$$

and

$$-ln(p_i) - ln(p_j) + \lambda_i^I + \lambda_j^I - \lambda_i^E - \lambda_j^E = -lnZ^I + lnZ^E \tag{11}$$

giving,after dividing by β multiplying through by p_{ij} and summing over ij

$$origin\, rent^I - origin\, rent^E + destination\, rent^I - destination\, rent^E = CS^E - CS^I \tag{12}$$

In this equation CS is the logsum expression for the consumer surplus i.e. $CS = -\frac{1}{\beta}Z$ where $Z = \sum_i \sum_j p_i p_j e^{-\beta c_{ij} - \lambda_i - \lambda_j}$ in the case of the Minimum Mutual Information model and $Z = \sum_i \sum_j e^{-\beta cij - \lambda_i - \lambda_j}$

in the case of the Maximum Entropy model. The equation shows us that the total uplift in expected rent is equivalent to the change in consumer surplus such that the increase in rents is at the expense of an equivalent decrease in consumer surplus. For the 5x5 model a similar pattern is shown in table 10. The economic interpretation of the rents and the statistical interpretation of MMI suggests that we should prefer the MMI formulation over the alternatives we have examined. This analysis follows quite closely the simple gas model of thermodynamics where it may be argued that the minimum information quantity is rather more fundamental than the entropy (Jaynes, 1968). It is also identified as forming the basis of a non-equilibrium thermodynamics by Koopman (Good, 1963). Apart from the identification of the MMI with the Kullback-Leibler distance of the observed matrix from the perfect competition matrix we may also identify this dimensionless divergence with the area of Harberger's triangle (Morphet, 2015). The Mutual Information may be interpreted as the expected Bayes-Turing log-factor (Good, 1963) and as such it represents the weight of evidence of the model over that of the null hypothesis which in our case is the assumption of zero trip costs.

The question of the extent to which unearned rents accruing to landlords are equivalent to benefits applying to consumers is debateable but its resolution will have some impact on the viability of proposed transport infrastructure improvements (see Mishan (1959)). The identification of rents by zone affords the possibility of a partial spatial equity analysis of benefits in addition t the equity considerations around the change in consumer surplus identified above. Of course, with zero transport costs all benefits are rent. The presence of rents greater than those suggested by perfect competition may be a measure of agglomeration benefits accruing to some areas. This will need to be explored in a spatial analysis but such excess benefits should not be double counted with rents. It should also be noted that such excess benefits will accrue to origins as well as destinations. A thick labour market benefits both employee and employer. The fact that the perfect competition baseline works will be of little comfort to proponents of perfect competition even as a datum or yardstick since the uniform prior and indeed any biproportional baseline work equally well. The proof(see Appendix) merely requires two unprescribed probability distributions as pre-multipliers in the model. In the calibration and running of the model, there is no such latitude and the model must conform to the standard constraints of the origins and destinations as usual.

The ability to model rents, which have a financial value means that an observable cash measure can be compared with the globally estimated value of time. One might expect such a comparison to find spatial variations in the value of time which would be of use in more accurate analysis and understanding of movements. There may also be a divergence between the global values of time and those inferred from the rents whose values are measured in the same units as trip cost and accordingly include measures of the value of time. Aligning the two values would improve the integrity and consistency of the model. The rent function for the ME model is similar to the accessibility function of Martinez (1995) and the adjustment of this to the MMI measure may offer benefits similar to those found in the rent analysis.

The rents identified in the model are von Thünen rents and represent the value of land in terms which depend upon its accessibility. Measured house prices reflect more than just land price although in urban areas this may be the largest component. Nevertheless in comparing modelled rents with actual prices the actual prices will need modification to take into account construction costs and some local environmental variables.Much of the data now exists for this kind of analysis to be tackled at a fine grained scale (Chi et al., 2021). The calculated rents offer a good opportunity to undertake their spatial analysis and their impact on project evaluation. They also open up the possibility of a relatively cheap way of updating distribution models between surveys since rent data is comparatively abundant and regularly updated. These potential applications will be the subject of further study.

7 Conclusions

The Minimum Mutual Information model offers a good prospect of avoiding negative rents and uses a non-arbitrary baseline derived from the model. This method also makes the model reproducible and will assist in intercity comparisons. The rent in the Maximum Entropy model is given by $\frac{\lambda_j}{\beta}$ whereas that in the Minimum Mutual Information model is given by$-\frac{lnp_j-\lambda'_j}{\beta}$. In the second case the λ' value is effectively a correction to that rent value corresponding to a zero trip cost. The use of a non arbitrary datum gives greater reassurance in the use of the derived rents in the determination of the size and distribution of costs and benefits. Where generalised cost is measured in terms of time they also open up an approach to determining the value of time from observed rents. The use of rents in calibration has already been explored(Morphet and Shabrina, 2020) but will be strengthened by the use of this method. We see that the method rules out negative values but an analysis of the lowest values and their causes may be as important as the analysis of high values.

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A Appendix

A.1 The positivity of the rents

The rents in the MMI model are given by ,in the case of origins, $-\frac{1}{\beta}(ln(p_i) - \lambda_i)$. We see immediately that $-ln(p_i)$ is positive as p_i is less than unity. In the Maximum Entropy model the rents are given by $-\frac{1}{\beta}(\lambda_i)$ where if $e^{-\lambda_i}$ is greater than one the rent is negative. a balancing factor greater than one inflates the corresponding trip matrix estimates and suggests a subsidy is being paid to maintain the balance of the trip matrix to its marginal origins and destinations. This does not undermine the Maximum Entropy model in terms of trip estimates as it gives the same estimates as the MMI model where the rents are always positive. In the Minimum Information model the balancing factors operate in the other direction so a balancing factor less than one operates to reduce $-ln(p_i)$ which can be taken as a prior. We now show why this is so.

We consider two cases , first when the balancing factor is less than 1 and second when it is greater.

A.1.1 Balancing factor greater than unity

In this case we may write:-

$$e^{-\lambda_i} > 1 \tag{13}$$

$$\therefore \lambda_i < 0 \tag{14}$$

$$\therefore rent_i = -\frac{1}{\beta}(ln(p_i) - \lambda_i) > 0 \tag{15}$$

A.1.2 Balancing factor less than unity

In this case we may write:-

$$e^{-\lambda_i} < 1 \tag{16}$$

$$\therefore \lambda_i > 0 \tag{17}$$

$$\therefore rent_i = \frac{1}{\beta}(-ln(p_i) - \lambda_i) > 0 \Leftrightarrow \lambda_i < -\ln(p_i) \tag{18}$$

Thus we see that in the first case the rent is positive and in the second it is indeterminate. We can, however, approach the second case differently by writing :-

$$e^{-\lambda_i} < 1 \; and \; \lambda_i > 0 \; \therefore \; \frac{p_i}{e^{-\lambda_i}} > 1 \; \therefore \; ln(p_i) - \lambda_i < 0 \; \therefore \; rent_i = \frac{1}{\beta}(-ln(p_i) + \lambda_i) > 0 \tag{19}$$

Thus, in both cases the rents are positive. The argument applies for any prior probability and in particular it applies to the uniform prior where $p_i = p_j = \frac{1}{n} \; \forall \; i, j$.