UCL CENTRE FOR ADVANCED SPATIAL ANALYSIS

UCL

WORKING PAPERS SERIES

Paper 159 - Oct 10

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ISSN 1467-1298

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A general Richardson-Lotka-Volterra-reaction diffusion model

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Abstract

A general model is proposed that embraces the separate models – developed in the different traditions of politics and ecology - of Richardson and Lotka and Volterra. This formulation is shown, with suitable definitions of the 'coefficients' to embrace, among others, the Harris-Wilson dynamic retail model and the Bass model.

1. A general model.

There are three kinds of dynamic model that between them offer a wide range of application: Richardson's (1960) model was originally applied to warfare and has been extensively developed, for example by Epstein (1997); Lotka (1924) and Volterra (1938) independently developed the early version of models which have come to dominate much of ecology; and reaction-diffusion models have wide application in the physical and biological sciences (cf. Turing, 1952, for an approach to morphogenesis) and have been extended into the social sciences, for example by Medda (see for example Medda, Nijkamp and Reitveld, 2009). Bass (1969) showed how to combine two of the models in an application to marketing – with a population of 'adapters' (Richardson) and 'imitators' (Lotka-Volterra). In this paper, we seek to combine all three kinds of model, and to generalise what are usually taken as constant coefficients – so that these can become functions or submodels.

Consider the following model:

$$dQ_j^n/dt = \gamma_j^n[H_j^n(Q_i^m, u_i^m) - \Sigma\lambda_{ij}^{mn}Q_i^m] + \varepsilon_j^n[K_j^n(Q_i^m, u_i^m) - \Sigma_{im}\mu_{ij}^{mn}Q_i^m]Q_j^n + d_j^n\partial^2Q_j^n/\partial x^2 \qquad (1)$$

where Q_j^n is an element of an array labelled by (n, j) and is some variable whose evolution in time we wish to track. To fix ideas, this of this as a species, n, distributed in space through a zone system with zones labelled j (Lotka and Volterra). It could also be a battle force (Richardson) or the size of a retail centre (Harris and Wilson, 1978) – or indeed there are many other interpretations. The diffusion term, $d_j^n\partial^2Q_j^n/\partial x^2$, is represented for convenience as though the spatial representation is continuous. It is difficult to find a notation to write this for discrete zone systems – labelled here by i -, but is straightforward algorithmically and so this term should be taken as standing for the algorithm. The Richardson and Lotka-Volterra models are normally without space and equation (1) builds on the extension in Wilson (2006). The range of models explored here – the different variants of the general model – are inspired by the work of Epstein (1997) though, again, we have added space to what are essentially spaceless models in his work. There are applications in ecology, politics and warfare and urban modelling, as already noted, but also in epidemiology (as this framework can encompass the full range of such models) and, as noted, marketing (Bass, 1969) and, no doubt, other fields. The addition of the diffusion term widens the class of models embraced by this formulation, for example to include morphogenetic models (Turing, 1952, Medda, Nijkamp and Reitveld, 2009).

The first group of terms on the rhs are essentially 'Richardson' terms and the second, Lotka-Volterra. The third term represents the diffusion part of reaction-diffusion equations. The reaction terms would be captured through terms in the first two groups. The difference between the first two groups is the factor Q_j^n. For reference, we can note that simple Richardson and Lotka and Volterra models in this notation are as follows.

The Richardson 'arms race' equations for two countries are:

$$dQ^1/dt = H^1 - \lambda^{11}Q^1 + \lambda^{12}Q^2 \qquad (2)$$

$$dQ^2/dt = H^2 + \lambda^{21}Q^1 - \lambda^{22}Q^2 \qquad (3)$$ [1]

The prey-predator Lotka-Volterra equations are

$$dQ^1/dt = \varepsilon^1[K^1 - \mu^{11}Q^1 + \mu^{12}Q^2]Q^1 \qquad (4)$$

for the predator and

$$dQ^2/dt = \varepsilon^2[K^2 - \mu^{21}Q^1 - \mu^{22}Q^2]Q^2 \qquad (5)$$

for the prey. All the μ coefficients in this case are assumed to be positive. If the positive sign on the μ-coefficient in (4) is changed to a negative one, then the equations take the 'competition for resources' form.

In the general formulation, the H and K arrays are shown as functions both of the full Q-array and external variables, {u}. The λ, μ and d coefficients are usually taken as constants in the standard models but in a general formulation can also be thought of as functions. In particular, as in the ecological models in Wilson (2006), the can be elements of spatial interaction models.

There are essentially eight distinct cases:

- $\gamma \neq 0$, $\varepsilon = 0$, $d = 0$ (Richardson, Harris and Wilson)
- $\gamma = 0$, $\varepsilon \neq 0$, $d = 0$ (Lotka-Volterra)
- $\gamma \neq 0$, $\varepsilon \neq 0$, $d = 0$ (Bass)
- $d \neq 0$, and some λ and μ coefficient non-zero (reaction-diffusion)
- $\gamma \neq 0$, $\varepsilon \neq 0$, $d \neq 0$ (all processes operating simultaneously)

[1] NB Note that the superscripts, as in Q^2, are not 'powers'!

The last two involve diffusion terms and hence must have a spatial system; the first three can be with a spatial system or without – hence the eight cases: (3x2) + 2.

It is worth rearranging the terms in case 2, generalised Lotka-Volterra, to make clear the 'carrying capacity in this formalism.

$$dQ_j^n/dt = \varepsilon_j^n[\{K_j^n(Q_i^m, u_i^m) - \Sigma_{im}\mu_{ij}^{mn}Q_i^m + Q_j^n\} - Q_j^n]Q_j^n + d_j^n\partial^2Q_j^n/\partial x^2(1) \quad (6)$$

By adding and subtracting a Q_j^n term as shown,

$$\{K_j^n(Q_i^m, u_i^m) - \Sigma_{im}\mu_{ij}^{mn}Q_i^m + Q_j^n\} \quad (7)$$

can then be seen to be the carrying capacity. (The additional Q_j^n term could, if convenient, be absorbed into the μ_{jj}^{mn} coefficients without loss of generality.)

2. Some particular cases

2.1. Introduction.

The variety of cases beyond the eight noted above can be considerably extended by considering different combinations of λ and μ coefficients which represent different systems of interest – different signs and different functional forms. Explorations of these cases are interesting in their own right but also offer the possibility that an application in one area, in this formalism, might suggest new models to be tested in others. This has already been explored for example in adding space to Lotka-Volterra models and comparing the result to urban models (Wilson, 2006) and these are pursued briefly below. Epstein (1997) pursues a very rich set of models which illustrate the variety that can be generated and an interesting question is the extent to which space, as used in (1) with a discrete zone system, can be added to his models, and we pursue one example of this in some detail. We investigate in turn

- the retail model
- the prey-predator model with space
- an Epstein-Lotka-Volterra model with space

2.2. The retail model

In the formalism of (1), the retail model of Harris and Wilson (1978) can be written

$$dQ_j/dt = \lambda[K_j - \textstyle\sum_i\mu_{ij}Q_i] \quad (8)$$

By reorganising the coefficients (and taking the K_js to be zero), this can be written in the usual notation as

$$dW_j/dt = \lambda[\Sigma_i\mu_{ij}W_i - KW_j] \quad (9)$$

where we would take

$\mu_{ij} = e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) / W_i \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik})$
(10)

to generate the usual spatial interaction model. (To retail some consistency with (1) we have replaced the usual ε by λ.) The presence of all the W's in each μ_{ij} generates 'interaction' between the W_js. This illustrates how a coefficient can become a sub-model-based function and indeed it is through this function that the nonlinearities come into play in what would otherwise be a linear Richardson model.

This can be interpreted as a one species model with space, or a multi-species model, with the W-population of each zone j being defined as a species. In the latter case, it can be seen as a 'competition for resources' model with retailers $\{W_j\}$ competing for consumers with spending powers $\{e_i P_i\}$. It could also be considered as a prey-predator model if the populations $\{e_i P_i\}$ are considered as the prey.

2.3. The prey-predator model with space.

We can add space to equations (4) and (5) to produce a two version L-V version of the general model.

$dQ_j^1/dt = \varepsilon_j^1[K_j^1 - \Sigma_i(\mu_{ij}^{11}Qi1 + \mu_{ij}^{12}Qi^2))]Q^1$
(11)

for the predator and

$dQ_j^2/dt = \varepsilon_j^2[K_j^2 - \Sigma_i(\mu_{ij}^{21}Q_i^1 - \mu_{ij}^{22}Q_i^2)]Q^2$
(12)

for the prey. It is then necessary to specify the spatial interaction terms, $\{\mu_{ij}^{mn}\}$ as in Wilson (2006).

In the next subsection, we explore a 'warfare' version of this model.

2.4. An Epstein-Lotka-Volterra model with space.

Epstein (1997, p. 24 et seq) develops an interesting variant of one of Lanchester's (1916) warfare models. He has 'red' and 'blue' forces in a combat zone and, using an obvious notation

$dR/dt = -bRB + \alpha R(1 - R/K)$
(13)

$dB/dt = -rBR + \beta B(1-B/L)$
(14)

The coefficient b in (13) represents the rate at which B forces can inflict casualties on R (with an additional factor, R, to represent 'density of target'; and vice versa for (14). The second term in (13) represents the capacity for R to add reinforcements up to some carrying capacity, K, of the combat zone for R; similarly for the second term in (14). At equilibrium

$$B = L - rLR/\beta \quad (15)$$

and

$$R = K - bKB/\alpha \quad (16)$$

Epstein shows that there are four kinds of dynamical behaviour according to the relative sizes of the coefficients. He also points out that these equations are equivalent to those developed by Gause (1934) as a 'competition for resources' version of the Lotka-Volterra model.

The next step is to add space to this model. B and R become $\{B_i\}$ and $\{R_i\}$ respectively. In the spirit of equation (1), we can then write

$$dR_i/dt = -bR_i\Sigma_j\lambda_{ij}B_j + \gamma R_i(1 - R_i/K_i) \quad (17)$$

and

$$dB_i/dt = -rB_i\Sigma_j\mu_{ij}R_j + \varepsilon B_i(1-B_i/L_i) \quad (18)$$

where we use γ and ε from equation (1) instead of α and β. In equation (17), the first term on the rhs now represents the results of B-attacks from all zones $\{i\}$ attenuated by the coefficients $\{\lambda_{ij}\}$ which would no doubt have a spatial interaction element.

The equilibrium equations in this case can be shown to be

$$R_i = K_i - bK_i(\Sigma_j\lambda_{ij}B_j)/\gamma \quad (19)$$

$$B_i = L_i - rL_i(\Sigma_j\mu_{ij}R_j)/\varepsilon \quad (20)$$

These are now all interlinked and could only be solved iteratively.

This spatial representation could be applied at various scales – ranging from $\{i\}$ being a set of countries at war to being subdivisions of a combat zone in a battle.

As always, the solutions to equations (17) and (18) will be path dependent and will depend critically on the initial conditions. In this case, the initial conditions will represent the commanders' strategies for the deployment of forces. A model run could then be seen as a test of these strategies; or, after one or a small number of iterations, an opportunity to change strategies – the 'genetic planning' argument applied in a military context (cf. Wilson, 2010).

3. Concluding comments: further extensions to be explored.

These explorations could be extended in a variety of directions and a number of examples are offered by way of conclusion.

(1) It would be interesting to extend the range of 'species' in an urban model. We have already indicated how retail developers and the consuming population could be two species. It would be possible to take a Lowry (1964) model as an archetypal comprehensive urban model and to have species that represented population, $\{P_i\}$, housing, $\{H_i\}$, retail and non-retail employment, $\{E^R_i\}$, $\{E^{NR}_i\}$, and retail attractiveness, $\{W_j\}$.

(2) An example should be developed which makes use of the diffusion terms – particularly if this could incorporate two interacting diffusion processes to generate morphogenesis (Turing, 1952). This could be done in principle by extending Medda's work from one continuous dimension to two dimensions with a zone system.

(3)The Epstein-Lotka-Volterra example suggests two possible extensions. First, the developing strategies in a path dependent dynamic framework could be put into gaming mode – as has been done for the retail sector in Dearden and Wilson (2010). Secondly, It is clearly necessary to make this model more realistic – and other models – by ensuring that various constraints are satisfied through the dynamic process. It is interesting in this respect that the original Lowry model had to have ad hoc constraints on zonal land areas introduced and there is likely to be an equivalent here. This would lead to an extension of the model formulation in equation (1).

(4) Epstein shows how to carry out stability analyses for his non-spatial models which enables him to characterise modes of dynamic behaviour. This could in principle be carried out for the general model and for the examples presented in section 2 above.

These are all offered as 'exercises for the reader'!!

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