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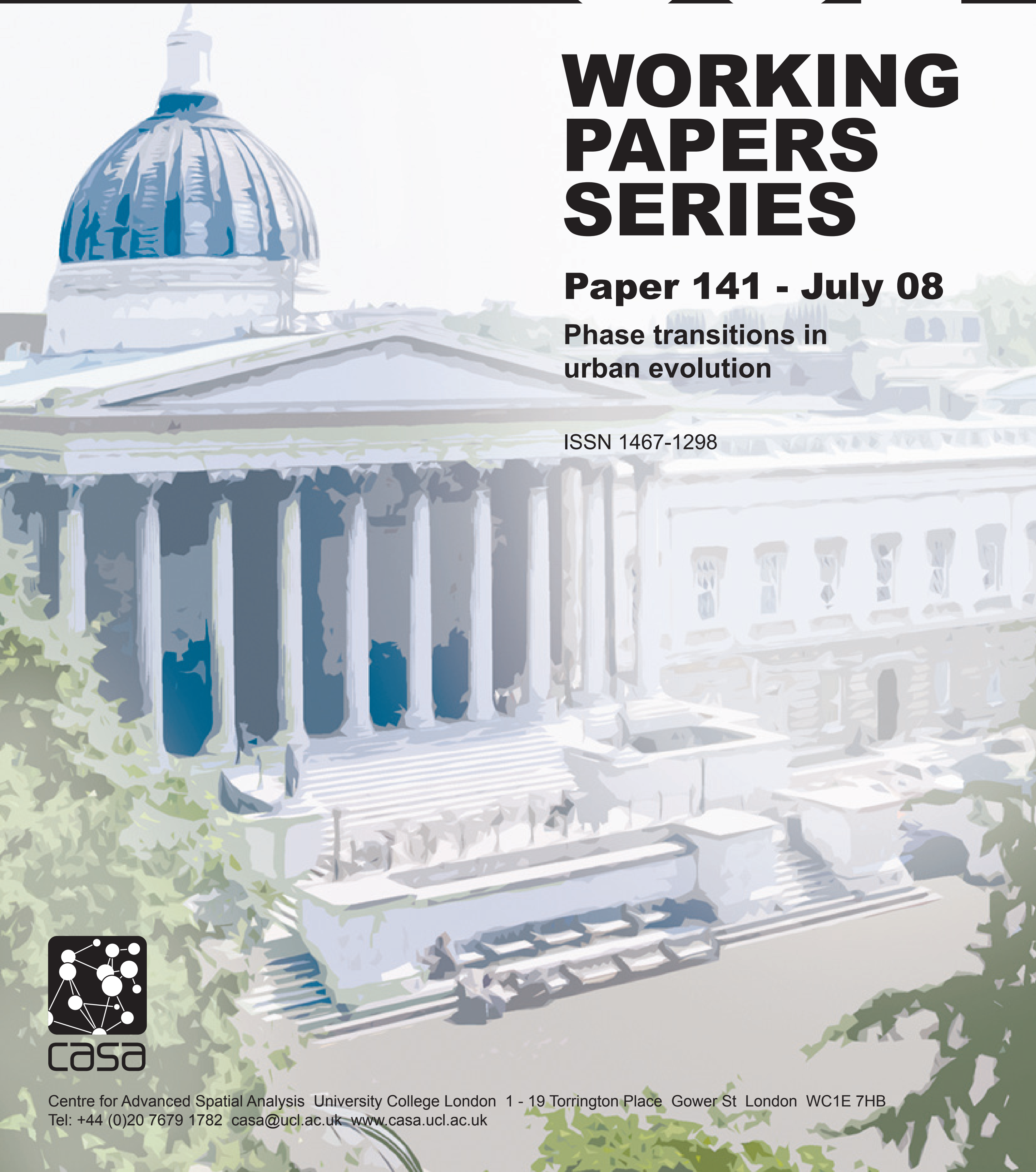

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Phase transitions in urban evolution

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Phase transitions in urban evolution

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Abstract

Phase transitions in statistical mechanics are compared with discrete changes in the evolution of cities and regions. It is argued that these changes do have the character of phase transitions. The range of such transitions in urban and regional evolution is explored. The corresponding mechanisms in urban and regional models are then investigated. The significance of the existence of phase transitions for the planning cities and regions is then considered.

1. Introduction

The task of modelling the evolution of cities is one of the major challenges of the social sciences. It is of particular interest to explore the extent to which there are discrete steps in this evolution. Since some of the modelling methods are mathematically similar to those used in statistical mechanics in physics, and since in that subject, there are discrete changes – phase transitions – it is potentially worthwhile to explore whether urban science can learn from physics in this respect.

We proceed by briefly exploring phase transitions in statistical mechanics along with known discrete changes in urban evolution in section 2 – using the retail model as an archetypal example. In section 3, we present an example: phase changes in a model of a London retail system. In section 4, we look at the range of possible phase transitions in the urban case and in section 5, we briefly review the potential for the application of these ideas in planning.

2. Phase transitions in statistical mechanics and urban modelling: archetypes

The archetypal example in physics is provided by H_2O and its three phases: ice, water and steam. In thermodynamic terms, phase changes are characterised by indeterminacy in either the first or higher order derivatives of the free energy, F. If this occurs in the first derivative it is a first order transition which is also characterised by the release (or absorption) of latent heat. For second and higher order transitions, it is the second derivative that is indeterminate and there is no latent heat.

Phase transitions are usually considered to be between less ordered and more ordered states and a parameter that can characterise this property is known as an *order parameter*. This thermodynamic analysis can be translated into statistical mechanics to facilitate comparison with the urban case (cf. Wilson, 2008-E).

We will use a model of an urban retail system as the archetypal example for initial explorations, illustrating with an example in section 3; and then in section 4, we will broaden our horizons. Can it happen? The obvious example is the transition from 'corner shop' food retailing to supermarkets in the late 1950s and early 1960s (Wilson and Oulton, 1983). This was a very rapid transition almost certainly brought about by the crossing of a threshold associated with increasing incomes and car ownership and therefore an ability to travel increasing distances. This can be analysed with the conventional retail model.

We proceed with a simple aggregated model to illustrate the ideas. Realistic disaggregation does not change the underlying argument. Define S_{ij} as the flow of spending power from residents of i to shops in j; let e_i be spending per head and P_i the population of i. W_j is a measure of the attractiveness of shops in j.

The vector $\{W_j\}$ can then be taken as a representation of urban structure – the configuration of W_js. If many W_js are non-zero, then this represents a dispersed system. At the other extreme, if only one is non-zero, then that is a very centralised system. There is clearly, potentially, a measure of order in this specification of structure. The obvious order parameter would be $N(W_j>0)$ – the number of centres which are non-zero. In a fully dispersed system, then $N(W_j>0)$ would be equal to the number of possible centres and would be large; while in a very centralised system, $N(W_j>0)$ would be 1. We will see later that it is sometimes better to take $N(W_j>M)$ for some constant M greater than 0 as a better measure of structural change.

A spatial interaction model can be built by maximizing an entropy function in the usual way to give:

$$S_{ij} = A_i e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) \qquad (1)$$

where

$$A_i = 1/\ \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik}) \qquad (2)$$

to ensure that

$$\Sigma_j S_{ij} = e_i P_i \qquad (3)$$

and

$$\Sigma_{ij} S_{ij} \log W_j = X \qquad (4)$$

where $\log W_j$ is a measure of size benefits and X an estimate of the total.

We also have

$$\Sigma_{ij} S_{ij} c_{ij} = C \qquad (5)$$

α and β are parameters – the Lagrangian multipliers - associated with equations (4) and (5). Because the matrix is only constrained the origin end, we can calculate the total flows into destinations as

$$D_j = \Sigma_i S_{ij} = \Sigma_i e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) / \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik}) \qquad (6)$$

The model is essentially based on a microcanonical ensemble with double labels (i, j) for 'energy' states instead of the single i-labels in the classical gas model. A_i is the inverse of the statistical mechanics partition function – but at a zonal level. C can be taken to represent total 'energy' in some sense and the c_{ij}s, individual energy states. c_{ij} is a measure of impedance, some kind of transport cost. If there were contributions of an opposite sign, they might be interpreted as 'utility' and we will see such an example shortly. It is generally recognised that to make the models work, c_{ij} should be taken as a generalised cost, a weighted sum of elements like travel time and money cost. To take the thermodynamic analogy further, we do need a unit and, to fix ideas, we will take 'money' as that unit. These will then be the units of 'energy' in the system. (For simplicity, we will henceforth drop the quotation marks and let them be understood when concepts are being used through analogies.) Given that the units are defined, then the β parameter, and the definition of a suitable Boltzmann constant, k, will enable us to define 'temperature' through

$$\beta = 1/kT \qquad (7)$$

We are accustomed to estimating β through model calibration. An interesting question is how we define k as a 'universal urban constant' which would then enable us to estimate the 'transport temperature' of a city. Note that if c_{ij} has the dimensions of money, then β has the dimensions of $(\text{money})^{-1}$, then from (7), kT would have the dimensions of money. If k is to be a universal constant, then T would have the dimensions of money.

Note that W_j^{α} can be written

$$W_j^{\alpha} = \exp(\alpha \log W_j) \qquad (8)$$

and the core equations can be written

$$S_{ij} = A_i e_i P_i \exp(\alpha \log W_j - \beta c_{ij}) \qquad (9)$$

where

$$A_i = 1/\ \Sigma_k \exp(\alpha \log W_k - \beta c_{ik}) \quad (10)$$

Thus, $\alpha \log W_k$ is the promised example of a positive contribution to energy: it shows explicitly that $\alpha \log W_j$ can be taken as a measure of the utility of an individual going to a shopping centre of size W_j but at a transport cost, or disutility, represented by $-\beta c_{ik}$. (If c_{ij} is measured in money units, then it would be better to divide through by β and take the utility measure as $(\alpha/\beta)\log W_k$.) The significance of this in the thermodynamic context is that α can be seen (via another Boltzmann constant, k') as a different kind of temperature:

$$\alpha = 1/\ k'T' \quad (11)$$

It was originally shown in Wilson (1970), following Jaynes (1957), that this argument can be generalised to any number of constraints and hence any number of temperatures. It can easily be shown, as in Physics, that if two systems are brought together with different temperatures, then they will move to an equilibrium position at an intermediate temperature through flows of heat from the hotter to the colder body. This also means, therefore, that in this case, there can be flows of different kinds of heat. In this case, the flow of heat means that more people 'choose' destinations in the 'cooler' region. In the context of this paper, the two temperatures are different drivers of possible phase changes.

Can there be phase changes in spatial interaction systems? This seems intuitively unlikely for the spatial interaction models themselves: smooth and fast shifts to a new equilibrium following any change is the likely outcome. If the model is made more realistic – and more complicated – by adding different transport modes, then the position could be different. There could then be phase changes that result in a major switch between modes at some critical parameter values (cf. Wilson, 1976). However, there is the possibility of significant phase changes in a structural model for $\{W_j\}$ and it is to this that we now turn.

A suitable hypothesis for representing the dynamics is (Harris and Wilson, 1978):

$$dW_j/dt = \varepsilon\ (D_j - KW_j)W_j \quad (12)$$

where K is a constant such that KW_j can be taken as the (notional) cost of running the shopping centre in j. This equation then says that if the centre if profitable, it grows; if not, it declines. The parameter ε determines the speed of response to these signals.

The equilibrium position is given by

$$D_j = KW_j \quad (13)$$

which can be written out in full as

$$\Sigma i\{e_i P_i W_j^{\alpha} \exp(-\beta c_{ij})/\Sigma_k W_k^{\alpha} \exp(-\beta c_{ik})\} = KW_j \quad (14)$$

and these are clearly nonlinear simultaneous equations in the $\{W_j\}$.

It is possible to characterise the kinds of configurations that can arise for different regions of α and β space: for larger α and lower β, there are a smaller number of larger centres; and vice versa[1] - as characterized in broad terms by the order parameter, $N(W_j>0)$, defined earlier. This can be interpreted to an extent for a particular zone, say j, by fixing all the W_k, k not equal to j. [A key challenge is to solve this problem with all the W_js varying simultaneously – see Wilson (1988)]. There are many procedures for solving the equations (14) iteratively but we constantly need to bear in mind the sensitivity to the initial conditions – the path dependence – and the possibility of multiple solutions. What we might learn from a statistical mechanics formulation is whether among the multiple solutions, there is one that is very much the most probable. [See Wilson (2008-A) for a broad review and how this methodology is more widely applicable.]

The zonal interpretation is shown in Figure 1. The left and right hand sides of equation (13) are plotted separately and of course, the intersections are the possible equilibrium points. If $\alpha \leq 1$, there is always a possible equilibrium point, but if $\alpha > 1$, there are three possible cases: only zero as an equilibrium; one additional non-zero stable state; and the limiting case that joins the two. The β value also determines the position of the equilibria. This analysis shows a number of properties that are typical of nonlinear dynamical systems: multiple (system) equilibria and strong path dependence. It also shows that as the parameters α and β (and indeed any other exogenous variables) change slowly, there is the possibility of a sudden change in a zone's state – from development being possible to development not being possible, or vice versa. These kinds of change can be characterised as phase transitions – in this case at a zonal level, but clearly there will be system wide changes of this kind as well. It will turn out that there is the essence of a very powerful tool here for identifying complex phase transitions.

[1] Clarke and Wilson (1985), Wilson (1984)

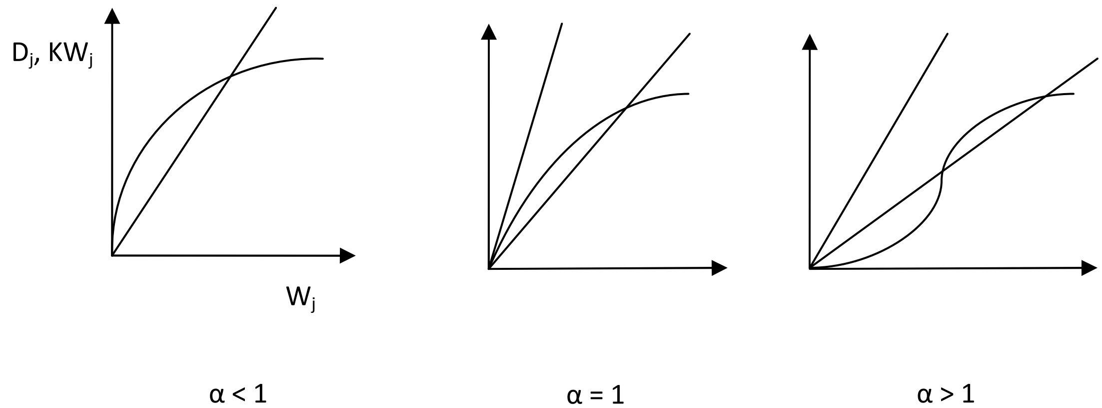


Figure 1. (a) (b) (c)

What we know from the analysis of Figure 1 is that at a zonal level, there are critical values of α and β, for example, beyond which only $W_j = 0$ is a stable solution for that zone. So we know that there are critical points at a zonal level at which, for example, there can be a jump from a finite W_j to a zero W_j (or vice versa). (This will be illustrated below, following Dearden and Wilson, 2008.)This implies there is a set of α and β at which there will be critical changes somewhere in the system. This is particularly interesting when we compare this situation to that in statistical mechanics. There, we are usually looking for critical temperatures for the whole system at which there is a phase transition. Here, there will be many more system phase transitions, but in each case consisting of a zonal transition (which then affects the system as a whole – since if a W_j jumps to zero, then other W_ks will jump upwards – or vice versa). It would be interesting to see whether the set of critical αs and βs form a continuous curve – and if we add 'K' as a parameter, then we would be looking for a critical, possibly continuous, surface.

In fact, this analysis shows that almost any change in model exogenous variables can in principle bring about a phase transition. For example, any change in the $\{e_iP_i\}$: then we are looking for a many-dimensional critical surface. It will also be interesting to see whether there are other systems that exhibit this kind of phase change: ecosystems, given the argument of Wilson (2006) about their structural similarity to urban systems, would be a possibility. We could possibly take the argument a stage further and build on the fact that equilibrium solutions in nonlinear models are path dependent: we would expect to find phase transitions along some paths but not on others for the same model with different initial conditions.

Recall that this analysis is dependent, for a particular W_j, on the set $\{W_k\}$, $k \neq j$, being constant. It is almost certainly a good enough approximation to offer insight, but it is still necessary, as noted earlier, to address the problem of simultaneous variation. The system problem is to predict equilibrium values for the whole set $\{W_j\}$ and the trajectories through time, recognising the points at which phase changes take place. This analysis does, of course, indicate discrete changes, and in the example in section 3, we will explore whether these discrete changes can be interpreted as phase changes. We will find the retail equivalent of free energy and look for the kind of indeterminacy that characterizes phase changes in physics.

We noted above that the retail model can be seen in statistical mechanics terms as being derived from a microcanonical ensemble. We now take the analogy further. We begin by interpreting the model in thermodynamic terms, initially exploring the retail analogue of 'work'. In our case, there are two kinds of change through work being done on the system (or heat flowing). In terms of the transport elements of the model, this can be a δC change or a δc_{ij} change. The former is a whole system change that means, for example, there is a greater resource available for individuals to spend on transport – and this will decrease β and hence increase the temperature; the latter would probably be produced by a network change – say the investment in a new link. Even with fixed C, if this leads to a reduction in cost, we would expect it to generate an increase in temperature. There will be an analogous argument for changes in δW_j or X and hence in α, but in the first instance, we restrict ourselves to transport changes.

This analysis enables us to interpret the principal laws of thermodynamics in the retail context. 'Work done' on the system will be manifested through either δC or δc_{ij} changes. Essentially, what the second law tell us is that there will be some 'waste' through the equivalent of heat loss. In thermodynamics, the core relationship for a change in energy, U, is

$$dU = dQ + \sum_i X_i dx_i \tag{15}$$

where Q represents heat, the X_i are the 'forces' doing work and the x_i are 'coordinates' of the system changed by the work done. We have available to us in the retail case a temperature through the parameter β (actually, $1/kT$, an inverse temperature). A first step is to explore whether there is an x_i which is the equivalent of a volume, V. The volume of a gas is the size of its container. In this case, for simplicity at this stage, we can take the area, A, of the city as a measure of size[2]. In thermodynamics, the free energy is taken as a function of T and V – in the urban case, this could be β and A: $F(\beta, A)$, say. We can then explore the idea of a state equation and it seems reasonable to start with Boyle's law since people in cities are being modelled on the same basis as an ideal classical gas. This suggests:

$$PA = NRT \tag{16}$$

[2] We should explore whether we can determine a measure of A from the topology of the $\{c_{ij}\}$.

where N is the total population and R is a constant. In terms of β, this becomes

$$P = NR/\beta kA \qquad (17)$$

where we have taken A to the other side of the equation[3]. There are, of course, two constants, R and k, in this equation which cannot be obtained in the same way as in Physics, but let us assume for the moment that they could. Then, equation (17) gives us a definition of an urban 'pressure'. It has the right properties intuitively: it increases if A or β decreases or N increases (in each case, other variables held constant). Suppose we proceed to work with the free energy and the partition function:

$$F = -[N/\beta]\log Z \qquad (18)$$

In our case, we only have a zonal partition function

$$Z_i = \Sigma_k \exp(\alpha \log W_k - \beta c_{ik}) \qquad (19)$$

[as the inverse of A_i in equation (2)]. We might conjecture that the system partition function is

$$Z = \Sigma_{ik} \exp(\alpha \log W_k - \beta c_{ik}) \qquad (19A)$$

In this approach to modelling $\{W_j\}$, the equilibrium solution is some kind of optimum.

The next step is to combine equations (18) and (19A) by substituting for logZ:

$$F = -[N/\beta]\log \Sigma_{ik} \exp(\alpha \log W_k - \beta c_{ik}) \qquad (20)$$

Then

$$\partial F/\partial \alpha = -[N/\alpha\beta][\ \Sigma_k \log W_k]/\ [\Sigma_{ik}\{\log W_k - [\beta/\alpha]c_{ik}\}] \qquad (21)$$

which leaves the intriguing possibility that $\partial F/\partial \alpha$ is indeterminate (or simply infinite?) when

$$[\Sigma_k\{\log W_k - [\beta/\alpha]c_{ik}\}] = 0 \qquad (22)$$

If we follow physics, it will be interesting to explore the behaviour of the order parameter, $N(W_j>0)$, near a critical point (β). Can it then be derived as a derivative of F, with a singularity at the phase transition and a corresponding power relationship? We should also consider classification. In a first order transition, it is the first derivatives of F that are indeterminate and there will be latent heat. What would be the 'latent heat' equivalent in urban systems? And can we find second order transitions?

[3] Note that P appears to have the dimensions of 'density'x'money'.

We note in the thermodynamics case, that changes in temperature, T (or in our case, β) can bring about phase changes – as in the case of water freezing or boiling. Equation (17) poses the question of whether changes in A or P – urban densities? – can bring about changes in our order parameter, $N(W_j>0)$? Changes in temperature will be explored below when we investigate critical values of β.

It is important to recognise that the illustrative analysis presented here depends on the particular functional forms that have been assumed for revenue and cost/production functions. The forms of these functions determine (to an extent) the discrete phase transitions. It is relatively straightforward to explore alternative assumptions or mechanisms. An alternative is offered in Wilson (2008-F) and this leads to alternative mechanisms for phase transitions. This alternative is based on the analogy that relates the spatial interaction model and the transportation problem of linear programming. This takes us nearer to a statistical mechanics 'lattice' model for the $\{W_j\}$ as shown in in Wilson (2008-E).

In the thermodynamics/statistical mechanics case, a phase change involves the whole system in some sense: as the temperature passes through a critical point, the liquid water freezes for example. In the retail case, we have shown that there will be a discrete change in a zone at critical points in any of α, β or K. We also know that if there is a discrete change in one zone, there will be corresponding changes in other zones that will also be discrete. This suggests that in the retail system, there is a much richer set of discrete changes that can be characterised as phase changes. We have also identified – cf. Figure 1 – that $\alpha = 1$ is likely, but not necessarily, to be a critical point since for all lower values of α, all the W_js must be non-zero while for $\alpha \geq 1$, there is the possibility of a subset of $\{W_j\}$ being zero.

3. An example: retail centres in London.

To illustrate the argument, we construct an aggregate retail model using London data. This is not intended to be realistic but simply at this stage to show what is involved in the process of trying to find and interpret retail/urban phase transitions. The data base is shown in Figure 2. It consists of 633 residential zone and 223 retail centres. The model represented by equations (1), (2) and (6) in terms of flows and revenue attracted and equations (14) are solved for the equilibrium $\{W_j\}$. The methods are presented in Dearden and Wilson (2008). The results of a model run are shown visually in Figures 3 and 4. Figure 5 shows a grid of (α, β) values with a model run for each element of the grid. One element, a model run displayed in 3-D form, is shown in Figure 6.

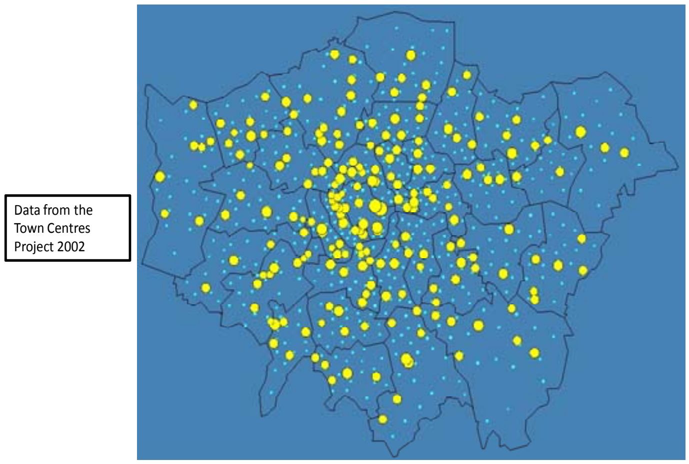


Figure 2. The initial conditions for the London illustration

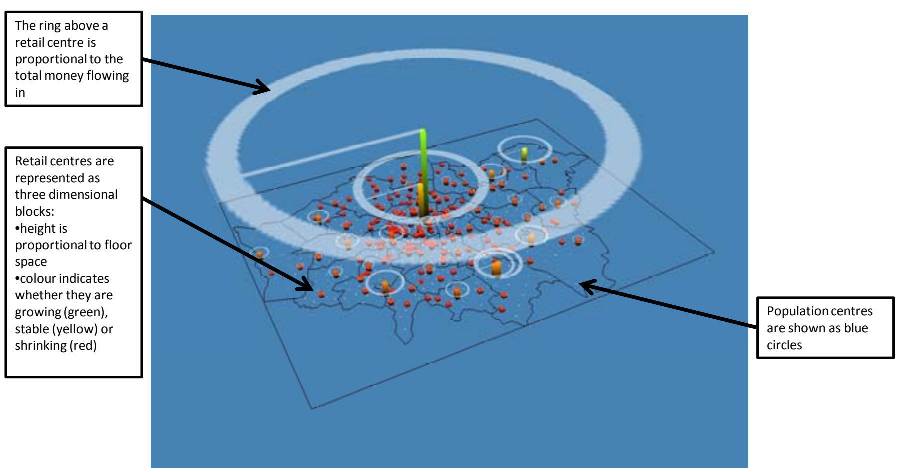


Figure 3. Model outputs.

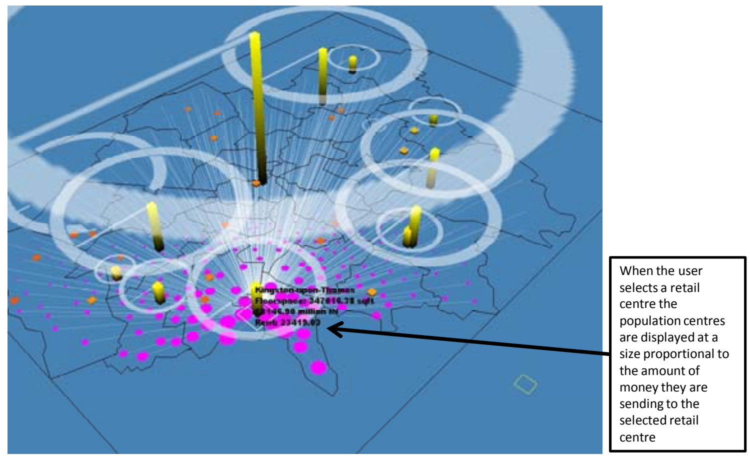


Figure 4. Model outputs showing flows

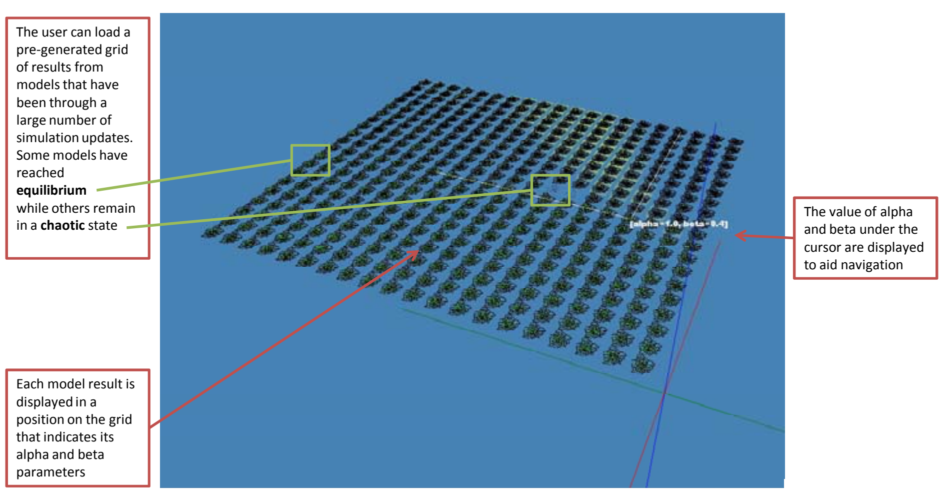


Figure 5. An (α, β) grid.

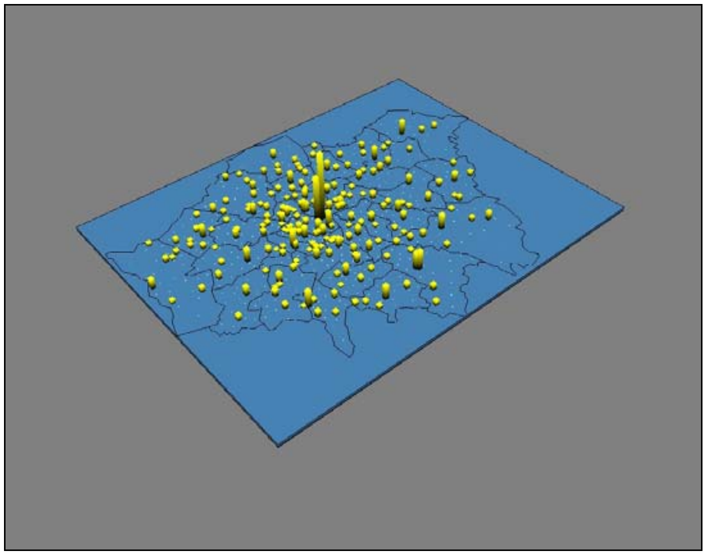


Figure 6. Retail centres for one run on the grid

The next step is to investigate possible phase changes. For illustrative purposes we concentrate as implied by the (α, β) grid in Figure 5, on phase transitions in (α, β)-space. We consider a possible phase change across $\alpha = 1$ as in the ringed area in Figure 7.

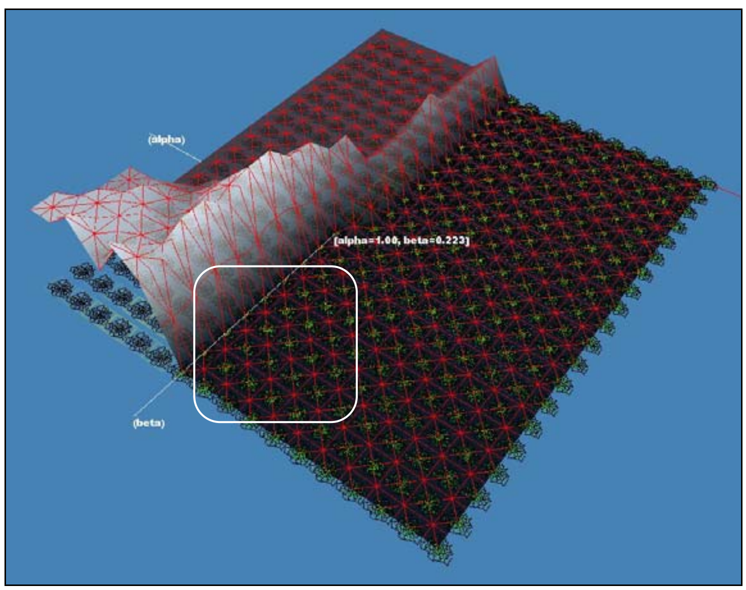


Figure 7. The search for a phase change around $\alpha = 1$.

Zooming in closer we can identify two result maps, one on either side of the possible phase change (Figure 8).

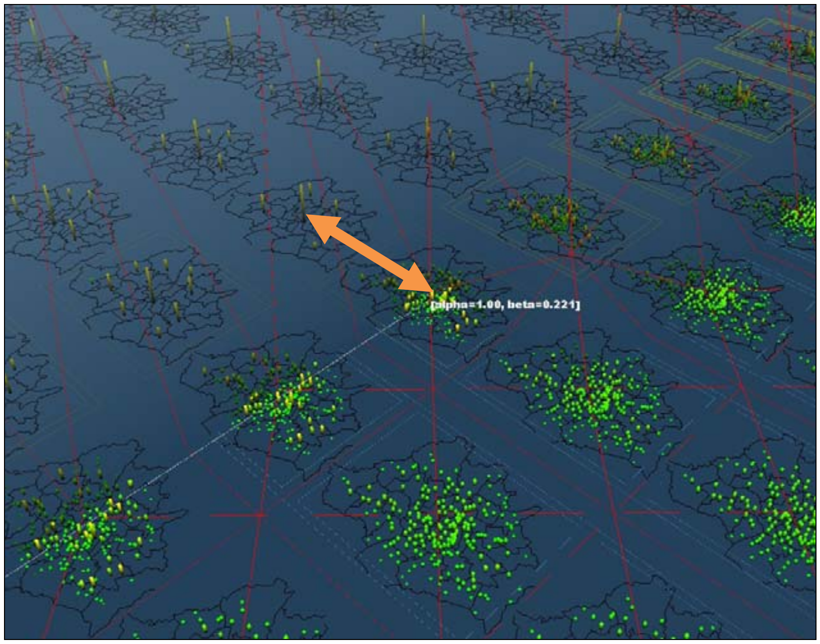


Figure 8. The area of possible phase change in more detail.

The two results are shown in Figures 9 and 10.

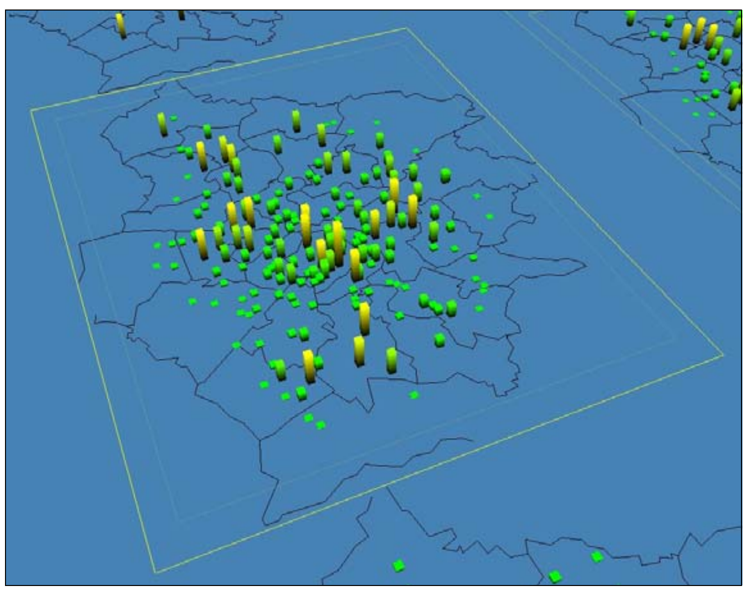


Figure 9. Alpha=1.00, beta=0.221

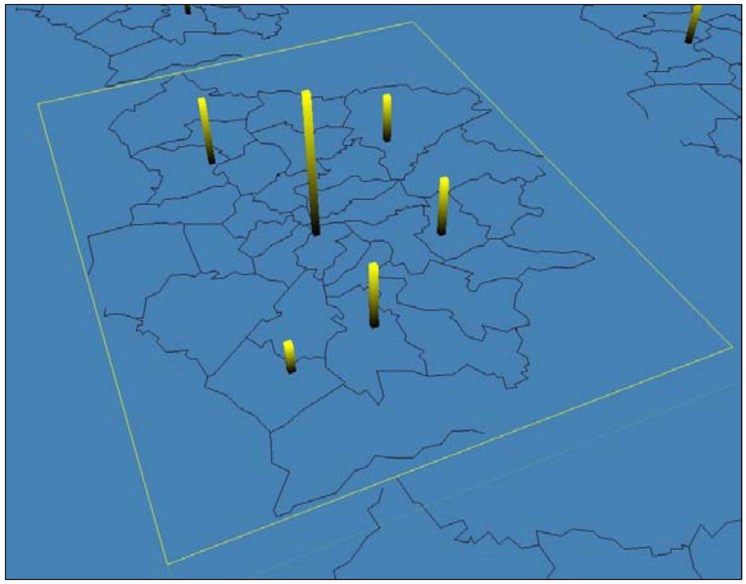
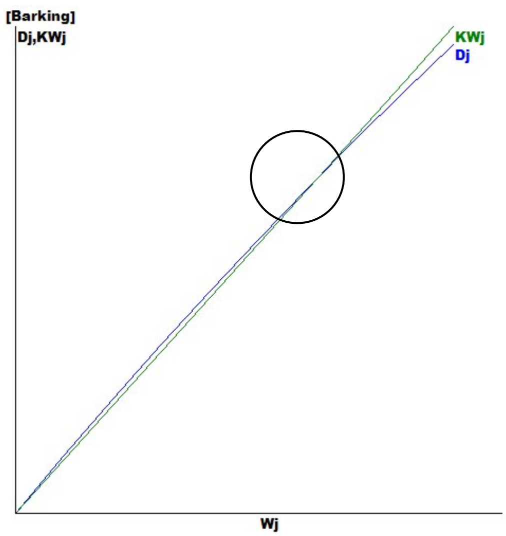


Figure 10. Alpha= 1.05, beta= 0.221

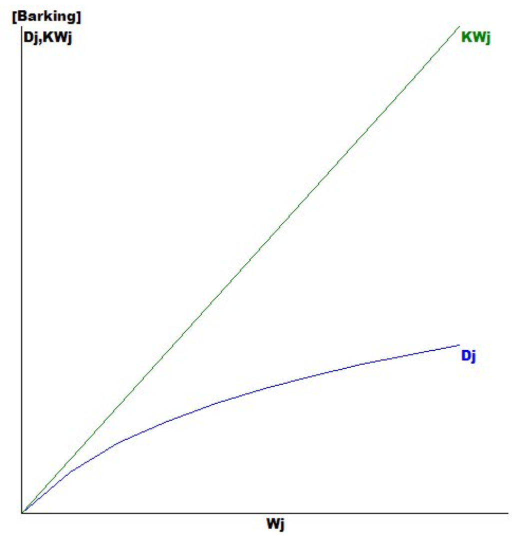
Only five retail centres survive in Figure 10. Most retail centres drop to zero floor space. One of these is Barking and we use this to illustrate the zonal analyses of Figure 1. The plots for this case are shown in Figure 11 and this clearly represents a phase change: Barking is present in Figure 9, but not in Figure 10 The zonal plots for in Figure 11 show why: (a) has a stable intersection, (b) does not.

D_j-KW_j Plots for Barking in Map 1



The graph goes up to 400,000 sq ft, so the lines intersect at about 250,000 sq ft.

D_j-KW_j Plots for Barking in Map 2



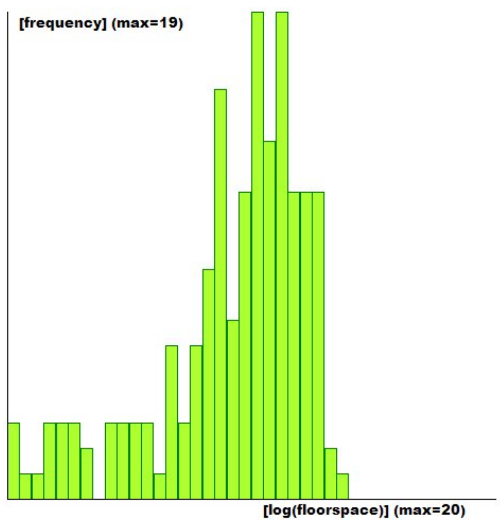
The graph goes up to 2,000,000 sq ft. There is no intersection.

Figure 11 (a) (b)

The histograms are shown for comparison in Figure 12 (with the horizontal scale converted to logs to illustrate the differences more clearly). This, of course, also illustrates a significant change in the order parameter. It should be emphasised that this is a wholly artificial illustration but shows what we should be looking for in real cases.

Log(Wj) Histogram for Map 1

(40 bins of 0.5)



Log(Wj) Histogram for Map 2

(40 bins of 0.5)

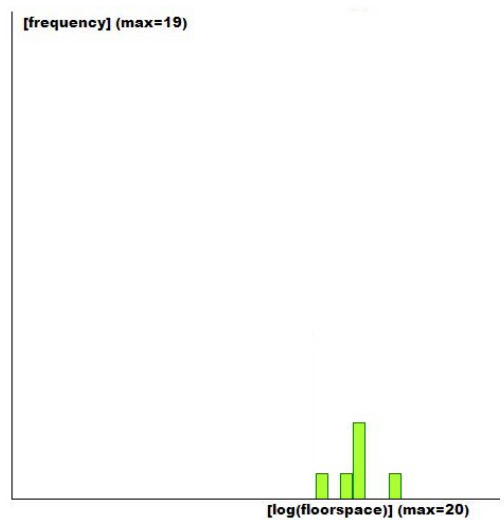


Figure 12. (a) (b)

This all demonstrates that the apparatus can be assembled, using powerful visualisation methods, to investigate and identify phase transitions. We next explore how they can occur much more widely in urban systems.

4. The range of phase transitions in urban evolution

We can review the engines of change as the basis for seeking phase transitions as distinct from other kinds of discrete change – such as the introduction of new technologies. Economic growth will have to be dealt with at least in part exogenously because of otherwise unpredictable technological change. Essentially, we will derive from here changes in , for example, incomes, and we have already seen that such changes can bring about phase transitions in retailing – through the β parameter alone, but also in other ways. Population change will also be handled exogenously in relation to the structural submodels. Technological change, as we have already seen, is essentially exogenous and will be input through cost and production functions. This will have impacts on all the submodels – housing and employment as well as retail for instance. Infrastructure investment – in areas such as transport, telecoms, buildings, utilities – will be the remaining area of mainly exogenous input – but this time, in principle, in a planned way. Essentially this will be the connection between model-based futures analysis and planning and we will consider in section 5 the extent to which we can, through investment, plan to encourage desirable phase transitions or to avoid undesirable ones.

It is in the models that represent the different markets that we are more likely to find phase changes analogous to those we have identified in the retail system. We can add to retailing (which we include in the list for completeness):

- the retail market
- the housing market
- the labour market
- mechanisms for the delivery of public services; and in areas like health and education, there are private markets also
- the business market
- interdependence: the land market

In the housing case, phenomena such as suburbanization, the 'tipping points' of population relocation – middle class flight from central areas for example – and gentrification are all ripe territory for exploring phase transitions. In the labour market, the shift to services and the location of employment in suburban locations will also impact on the housing market. This is the sense in which all these market submodels are interdependent. Public services are in general not 'markets' in the usual sense, but in major sectors such as health and education there are issues of optimum size of facilities in different kinds of areas – a modern version of central place theory. The development of capacity in the economy – labour, buildings, offices, equipment etc – represents another kind of market. Many of these submodels then integrate through the competition for land and the land market – with rents and densities to be determined – and phase transitions are possible in both cases. (In the retail example, for instance, we were in effect taking facility densities as the order parameter.)

One way to organise the argument presented informally above is to review the list of key variables that we would find in a comprehensive model – in this case following Wilson (2008-B). Consider:

- P_i^m, the number of type m people in zone i
- H_i^k, the number of type k houses in zone i
- V_j^n, the capacity of type-n consumer services in j; F_j^{nm}, the take-up (on a suitable measure – school places e.g.) by m-type people
- W_j^n, the capacity of type-n retail facilities in j; D_j^{nm}, the take-up of these facilities by m-type people

- Q_j^n, the capacity of the n^{th} sector in zone j; X_j^n, the product of goods in sector n in j
- G_j^n, government spend in n in j

We distinguish the public services sector (such as education and health) and the retail sector from the general set of economic sectors to facilitate model construction and policy interpretation later. That is, we are distinguishing the V and W sectors from the other X-sectors (and the population sector, P, in so far as it is being treated as a producer of labour).

The main interaction variables are[4]:

- Y_{ij}^{mVn}, Y_{ij}^{mWn}, Y_{ij}^{mQn}, the flows to work in sectors V, W and Q from population group m in zone i to sector n in zone j for V, W and Q respectively;
- N_{ij}^{mnk}, the allocation of type m people who work in sector n in j to type k houses in i;
- U_{ij}^{mn}, the flow of type m people in zone i to consumer services of type n in j;
- S_{ij}^{mn}, the flow to retail facilities of type n;
- J_{ij}^{mn}, the flow of goods sector m in i to the consumer services sectors n in j;
- K_{ij}^{mn}, the flow of goods from sector m in i to the retail services sectors n in j;
- M_{ij}^{mn}, the flow of goods from sector m in i to sector n in j.

So far, we have considered cities more or less in isolation. It has been argued in another paper (Wilson, 2008-C) that there are up to four layers of aggregation that we should consider:

(1) nations within the international system;
(2) regions within a country;
(3) cities within a region;
(4) intra-urban structure.

In the retail example, we have considered possible phase changes in relation to α, β and K. This broader formulation shows that, in any submodel, any variables that are exogenous to that model can, when they change, trigger phase transitions. In the retail case, for instance, changes in one or more e_is or P_is can bring about such changes.

A further advantage in adding this kind of formality is that we can consider a structural vector {P, H, V, W, X, L, p, c, G} to be the city's DNA (cf. Wilson, 2008-D). It is then particularly interesting to consider how we could modify future structures through investment

[4] We will note later the possibility of indices such as m and n themselves being lists

– either by seeking to avoid undesirable phase changes or to encourage movement to desirable ones. For example:

- there is probably a relationship between residential densities and the ability of a public transport system to attract the bulk of the trips. There will be a phase change at a critical density – or more likely, a surface of densities.
- How do we ensure affordable housing?
- How do we attract employment? This is related to the 'layers' question referred to above.
- What is an efficient size distribution of schools?

5. The next steps and implications for planning.

A progression of further analytical work follows from this argument.

- Using the retail system as archetype, fully understand the range of possible phase changes.
- It would be particularly interesting to follow through the argument around equations (16) and (17) and see if it was possible to construct a measure of 'area' – 'A' in the equations – in terms of the topology represented by the matrix $\{c_{ij}\}$ and to follow that argument through in relation to possible phase transitions.
- Pursue equivalent changes in each urban submodel – as sketched out in section 4.
- Explore what this means for a comprehensive model – a system of submodels.
- Explore the argument at different scales: particularly looking at systems of cities within an input-output framework with interaction models for the trade flows.
- Consider shifting the focus of model applications to explore the extent to which the kinds of major issues mentioned at the end of the previous section can be related to phase changes in corresponding models. This implies something of a change of focus in model use.

Finally and briefly, we can consider the implications for the uses of urban models in planning. Our understanding of the nature of nonlinear models has enabled us to recognise that they cannot be used in a simple way for long-term forecasting. However, increased understanding of phase transitions could lead to a new element of model-based planning: seeking to avoid undesirable phase transitions or investing to bring about desirable ones. Intriguingly, Friston and Stephan (2006) develop a 'free energy' model of the brain and argue that free energy is mininised – the equivalent of entropy being maximised – to avoid phase transitions. It is not a huge step to envisage a planning system as a 'brain' in this kind of framework!

More generally, we should note that this kind of analysis implies that there are new uses for models in planning – and which to an extent could compensate for the fact that the nonlinearity of the models inhibits their use in long-term forecasting. Essentially, model-based analysis could be used to avoid undesirable phase changes and to encourage desirable ones. A particular kind of transport investment that would be registered as $\{c_{ij}\}$ changes for example, could bring about could bring about a desirable phase transition. More generally still, we could use the 'DNA' argument to explore the possibilities of transition given the structural starting point. This takes the initial conditions of the structural variables as limiting the space of development possibilities. This is in itself valuable policy and planning information, but we could also pose the question: what should be changed in this structure to bring about a desired transition?

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References

Clarke, G. P., Clarke, M. and Wilson, A. G. (1986) Multiple bifurcation effects with a logistic attractiveness function in the supply side of a service system, *Systemi Urbani, 7*, pp. 43-76.

Clarke, G. P. and Wilson, A. G. (1986) Combining theoretical and empirical research in retail location analysis, Working Paper 468, School of Geography, University of Leeds.

Clarke, M. and Wilson, A. G. (1985) The dynamics of urban spatial structure: the progress of a research programme, *Transactions, Institute of British Geographers*, *NS 10*, 427-451.

Dearden and Wilson (2008) An analysis system for exploring urban retail phase transitions, Working paper, Centre for Advanced Spatial Analysis, University College London.

Fotheringham, A. S. (1985) Spatial competition and agglomeration in urban modeling, Environment and Planning, A, 17, pp. 213-230.

Friston, K. and Stephan, K. E. (2007) Free energy and the brain, *Synthèse, 159*, 417-458.

Harris, B. and Wilson, A. G. (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial-interaction models, *Environment and Planning, A, 10*, 371-88.

Lombardo, S. R (1986) New developments of a dynamic urban retail model with reference to consumers' mobility and costs for developers, in Griffith, D. A. and Haining, R. J. (eds.) *Transformations through space and time*, Martinus Nijhoff, Dordrecht.

Wilson, A. G. (1967) A statistical theory of spatial distribution models, *Transportation Research, 1,* 253-69

Wilson, A. G. (1970) Entropy in urban and regional modelling, Pion, London.

Wilson, A. G. (1976) Catastrophe theory and urban modelling: an application to modal choice, *Environment and Planning, A, 8*, pp 351-6, 1976.

Wilson, A. G. (1981) *Catastrophe theory and bifurcation: applications to urban and regional systems*, Croom Helm, London; University of California Press, Berkeley.

Wilson, A. G. (1983) A generalised and unified approach to the modeling of service supply structures, Working Paper 352, School of Geography, University of Leeds.

Wilson, A. G. (1988) Configurational analysis and urban and regional theory, in *Sistemi Urbani, 10*, 51-62.

Wilson, A. G. (2006) Ecological and urban systems' models: some explorations of similarities in the context of complexity theory, *Environment and Planning*, *A*, 38, pp. 633-646.

Wilson, A. G. (2008-A) Boltzmann, Lotka and Volterra and spatial structural evolution: an integrated methodology for some dynamical systems, *Journal of the Royal Society, Interface*, *5*, pp. 865-871, *doi:10.1098/rsif.2007.1288*.

Wilson, A. G. (2008-B) Urban and regional dynamics – 1: A core model for the analysis of urban dynamics, Working paper 128, Centre for Advanced Spatial Analysis, University College London.

Wilson, A. G. (2008-C) Urban and regional dynamics – 2: an hierarchical model for interacting regions, Working Paper 129, Centre for Advanced Spatial Analysis, University College London.

Wilson, A. G. (2008-D) Urban and regional dynamics – 3: Urban and regional 'DNA': the basis for constructing as typology of areas, Working paper 130, Centre for Advanced Spatial Analysis, University College London.

Wilson, A. G. (2008-E) The 'thermodynamics' of the city, Working Paper, Centre for Advanced Spatial Analysis, University College London.

Wilson, A. G. (2008-F) An imperfect market model of urban development, forthcoming Working paper, Centre for Advanced Spatial Analysis, University College London.

Wilson, A. G. and Oulton, M. J. (1983) The corner shop to supermarket transition in retailing: the beginnings of empirical evidence, *Environment and Planning, A, 15*, pp. 265-274.