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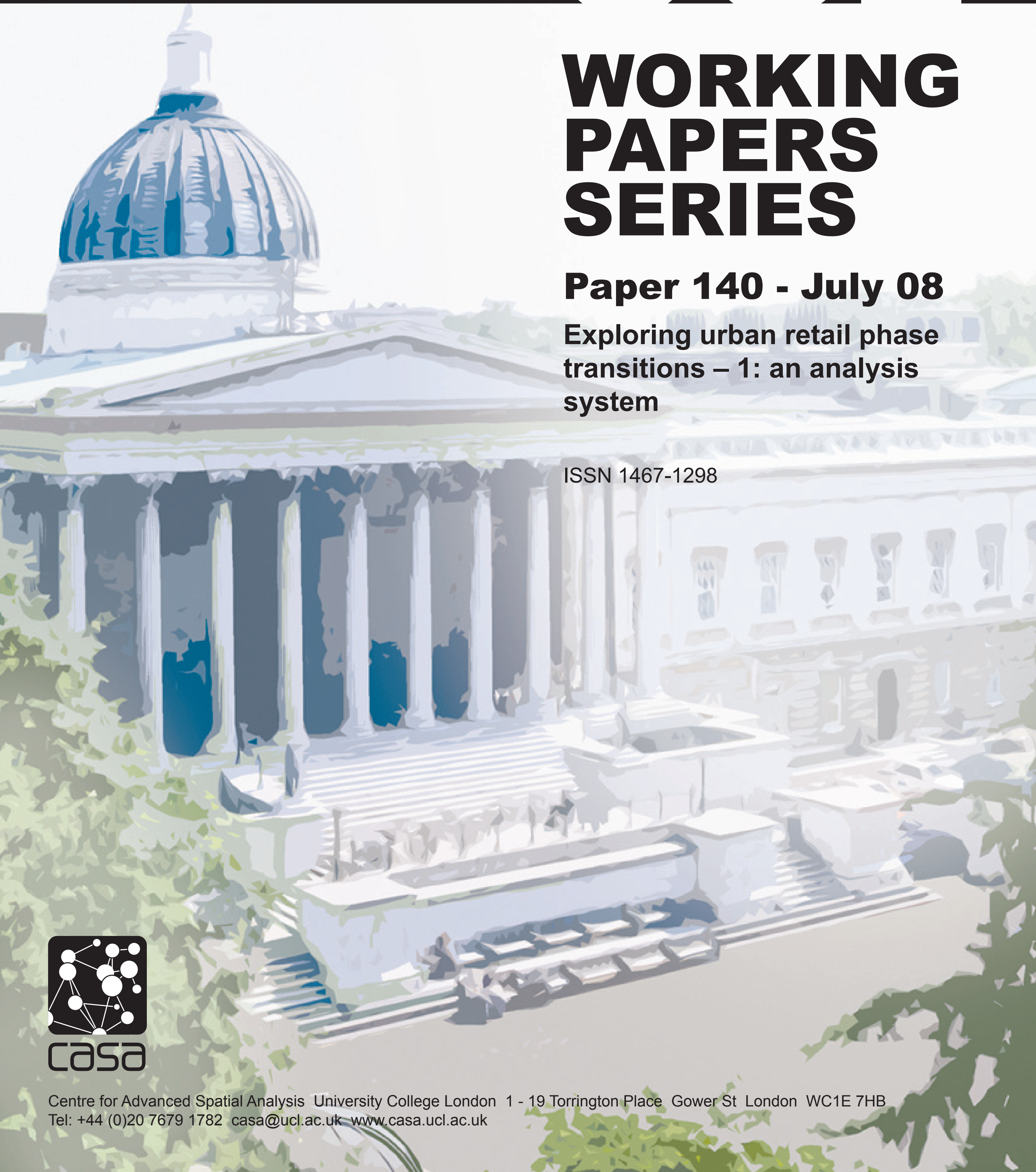

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Exploring urban retail phase transitions – 1: an analysis system

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Abstract

A key area in the analysis of the evolution of urban structure through modelling is identifying phase transitions. At these critical points, the structure changes radically. In planning terms, effective analysis would allow us to work towards or away such transitions depending on whether it was a beneficial or detrimental change. A simple aggregate retail model is used here to illustrate the argument. Such explorations have been carried out in the past. Here we present more powerful visualization methods that facilitate the exploration of phase changes in more depth. This prototype offers a good foundation for the development of more realistic systems in the future.

1. The problem

The aggregate urban retail model can be taken as a simple example to illustrate the task of modeling the evolution of urban systems. The model, with a mechanism for dynamics – system evolution - was articulated by Harris and Wilson (1978). Define S_{ij} as the flow of spending power from residents of i to shops in j; let e_i be spending per head and P_i the population of i. W_j is a measure of the attractiveness of shops in j and in our illustrative model here we take this to be measured by floorspace[1]. The vector $\{W_j\}$ can be taken as a representation of urban structure – the configuration of W_js. If many W_js are non-zero, then this represents a dispersed system. At the other extreme, if only one is non-zero, then that is a very centralised system.

In the following specification, α and β are parameters: high α implies consumers valuing attractiveness (measured by size) highly – a proxy for scale economies, range of choice etc; and low β implies higher ease of travel - consumers more likely to go to a more distant centre to collect their 'size' benefits.

[1] It should be emphasised that these simple assumptions are being made to illustrate the problem. They can be made more realistic in straightforward ways but to do this here would obscure the key points being made.

We will see below that the model representation of system evolution involves nonlinear simultaneous equations that cannot be solved analytically. This makes computer simulation and visualization of outputs essential to achieve an understanding and this is particularly the case in the search for, and analysis of, phase transitions. The purpose of this paper is to offer such an analysis system. Contemporary visualization software enables substantial progress from earlier work (for example, Clarke and Wilson, 1985) and, as we will show, can deepen our understanding of phase transitions.

The usual model of flows (Wilson, 1967, 1970), measured in money terms, from residential zones, i, to shopping centres, j, is

$$S_{ij} = A_i e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) \tag{1}$$

where

$$A_i = 1/ \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik}) \tag{2}$$

to ensure that

$$\Sigma_j S_{ij} = e_i P_I \tag{3}$$

with

$$D_j = \Sigma_i S_{ij} = \Sigma_i [e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) / \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik})] \tag{4}$$

as the total revenue attracted into shopping centre j from all residential zones {i}. A suitable hypothesis for change – the dynamics – is

$$dW_j/dt = \varepsilon (D_j - KW_j) W_j \tag{5}$$

where K is a constant – the cost per unit of floorspace - so that KW_j can be taken as the cost of running the shopping centre in j. This equation then says that if the centre if profitable, it grows; if not, it declines. The parameter ε determines the speed of response to these signals. These equations can be written out in full as

$$dW_j/dt = \varepsilon (\Sigma_i \{e_i P_i W_j^{\alpha} \exp(-\beta c_{ij}) / \Sigma_k W_k^{\alpha} \exp(-\beta c_{ik})\} - KW_j) W_j \tag{6}$$

This shows that we have a system of nonlinear simultaneous differential equations. They can only be solved by simulation and for these purposes, we shift to difference equation form [building on equation (5)]:

$$\Delta W_j(t, t+1) = \varepsilon [D_j(t) - KW_j(t)] W_j(t) \tag{7}$$

for the period (t, t+1). Then

$$W_j(t+1) = W_j(t) + \Delta W_j(t, t+1) \quad (8)$$

The equilibrium position is given by

$$D_j = KW_j \quad (9)$$

which can also be written out in full as

$$\Sigma_i\{e_iP_iW_j^{\alpha}\exp(-\beta c_{ij})/\Sigma_k W_k^{\alpha}\exp(-\beta c_{ik})\} = KW_j \quad (10)$$

showing these are nonlinear simultaneous equations in the $\{W_j\}$.

It is well known that for non linear systems

- solutions are dependent on the initial conditions – 'path dependence';
- there are phase transitions: that is, there are critical values of the parameters – such as α and β, but in fact any exogenous parameter or variable – at which the structure changes suddenly.

We have also remarked that the equations can only be 'solved' by simulation methods. What is needed therefore is an analysis system with the following properties:

- single model runs with a wide range of outputs – including measures that characterise the structures generated – the 'order parameters'[2];
- the model can be run over a range of α and β values to generate a 'results grid'
- explorations of varying initial conditions;
- phase transitions in the results grid can be identified and investigated closely;
- evolution of the system over time can be studied.

2. Model simulation: generating equilibrium patterns.

The equilibrium pattern – the solutions to equation (9) [or (10)] - can be obtained by iterating the equations (7) and treating the time steps t to t+1 as iterative steps n to n+1 hence solving the equations (9) at a fixed time. We need to specify a set of initial conditions – starting values for the $\{W_j\}$ and the $\{e_iP_i\}$ – and of course the $\{W_j\}$ will not be equilibrium values. When (and indeed, if) the iteration converges, the equations (9) will be satisfied and hence, by summing both sides over j, we will have

$$\sum_j D_j = \sum_j KW_j \quad (11)$$

[2] This is the term used in physics – as phase transitions are usually from a less ordered to a more ordered state, or vice versa.

If K is fixed, this means that the total floorspace will have been adjusted by the iterative process to ensure that (10) is satisfied. Denote this by *equilibrium model 1*. If, however, for comparative purposes, we want to find equilibrium $\{W_j\}$ for which

$$\sum_j W_j(t+1) = \sum_j W_j(t) \tag{12}$$

that is, keeping the total floorspace, ($\sum_j W_j$), constant, we can do this by amending equation (8) by adding a normalising factor λ to ensure this. Call this *equilibrium model 2*. (8) becomes

$$W_j(t+1) = \lambda[W_j(t) + \Delta W_j(t, t+1)] \tag{13}$$

Summing both sides over j:

$$\sum_j W_j(t+1) = \lambda(t+1)\sum_j[W_j(t) + \Delta W_j(t, t+1)] \tag{14}$$

so that

$$\lambda(t+1) = \sum_j W_j(t+1)/\sum_j[W_j(t) + \Delta W_j(t, t+1)] \tag{15}$$

We will assume below that we will use model 2 for illustrative purposes in exploring phase transitions between equilibrium states.

3. Investigating phase transitions.

Consider the task of finding critical parameter values. It is argued in Wilson (1981), pp. 128 et seq., that there will be a curve of critical values in (α, β) space (and it will be interesting to try to generate these curves explicitly). We can follow the argument of Wilson (1981) to explore what might be happening at a particular centre in relation to criticality. It can be shown that the heart of the structural criticality problem is whether, at the location of a particular centre, conditions permit a non-zero value. If there are many locations where it is permitted, then this will be a distributed retail system; and vice versa. Another task for the analysis system, therefore, is to simulate that analysis.

It is difficult to isolate what is happening in a particular retail centre because as the equations show, each W_j-equation shows a W_j-dependence on $\{W_k\}$, $k \neq j$[3]. To make progress, we have to make an heroic assumption: that we can plot D_j against W assuming that all the $\{W_k\}$, $k \neq j$ are fixed[4]. Equation (4) can be written

$$D_j(W_j) = W_j^{\alpha}\Sigma_i[e_iP_i\exp(-\beta c_{ij})/ \{\Sigma_{k\neq j}W_k^{\alpha}\exp(-\beta c_{ik})+W_j^{\alpha}\exp(-\beta c_{ij})\}] \tag{16}$$

to show explicitly the dependence of D_j as a function of W_j assuming all the other W_k are fixed.

[3] These difficulties were articulated in Wilson (1988)
[4] In practice, they will all be varying simultaneously

The cost is KW_j which is, of course, a straight line when plotted. At equilibrium, the $D(W_j)$ curve and the KW_j line will intersect. It was shown analytically in Wilson (1981) that there are three cases – shown in Figure 1. It is interesting within the analysis system to attempt to simulate these cases for some zones where it has already been shown that there is a system phase change in the neighbourhood.

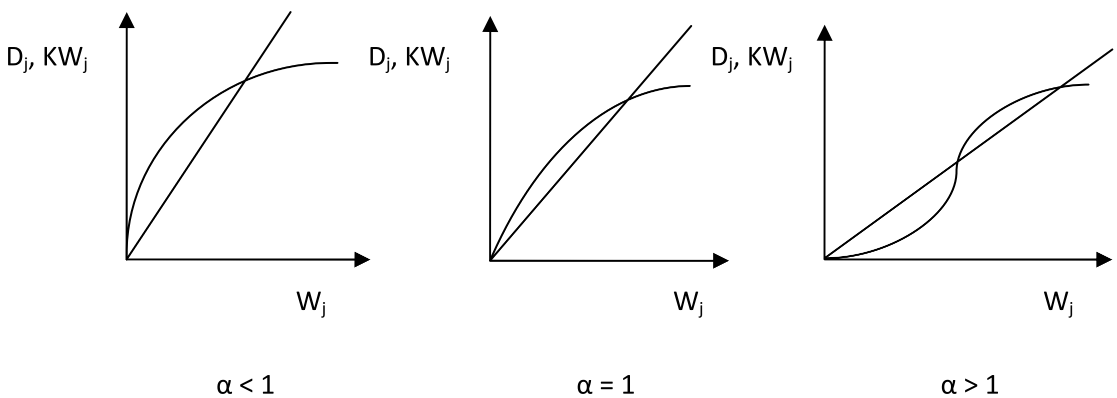


Figure 1. **(a)** **(b)** **(c)**

In Figure 1(a), $\alpha < 1$, in 1(b), $\alpha = 1$ and in 1(c), $\alpha > 1$. In 1(a), the gradient of the $D_j(W_j)$ curve is infinite at the origin and so there is always an intersection with the cost line and that can be shown to be stable. Thus for $\alpha < 1$, we would expect a dispersed system. In the 1(c) case the cost line either intersects the curve twice (excluding the origin) or only at the origin. In the former case, the upper intersection is stable and a non-zero W_j is possible; in the latter case, W_j will be zero. More centralized patterns will have many such zeros. $\alpha = 1$ is a special case. The D_j curve has a finite gradient at the origin and the possibilities of intersection generating a stable point are like the 1(c) case. We should therefore always expect a phase transition at $\alpha = 1$.

4. System outputs.

4.1. Single model runs

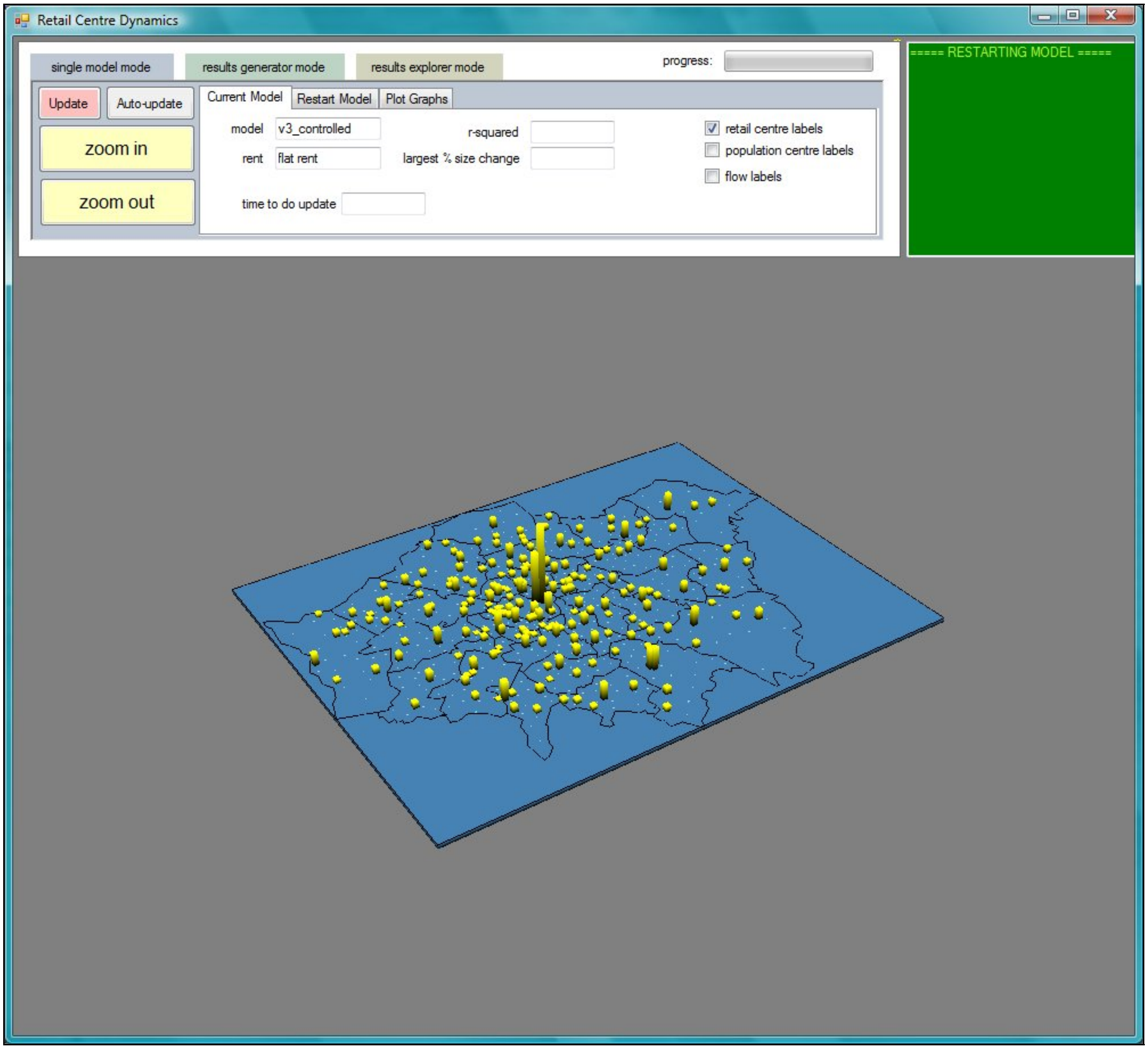


Figure 2. Single model run mode

The single model mode provides the following features:

- o 3D graphic presentation of structure with detail available from drop-down menus
- o Animation of $\{W_j\}$ structure evolving through the iterative procedure
- o Visual indicators of total money flow in and size change for each retail centre

- At each iteration a readout of
    - R^2 comparison to starting $\{W_j\}$ values
    - Largest percentage size change in a retail centre
- Background shapefile
- The facility to restart model with new parameters:
    - Alpha
    - Beta
    - Epsilon
- Alternative floor space models:
    - *equilibrium model 1*
    - equilibrium model 2 (which we use for all the results below)
- Alternative rent models
    - flat rate rents
    - declining rents from the centre
- Size of retail centres
- Log of all outputs to a CSV file
- Visualisation of flow from each population centre to each retail centre
- Retail centre detail
    - Name
    - Floorspace
    - Rent
    - Total money flow in
- Population centre detail
    - Name
    - Spending Power
- Graph plotting
    - zonal analyses (based on Figure 1)
    - rank-size distribution
    - network analyses through significant link counts
    - histograms
    - rank size
    - $\{W_j\}$ histogram

4.2. The results grid

Multiple model run mode generates a results grid with the following features:

- Run a set of models within a user specified (α, β) grid to equilibrium

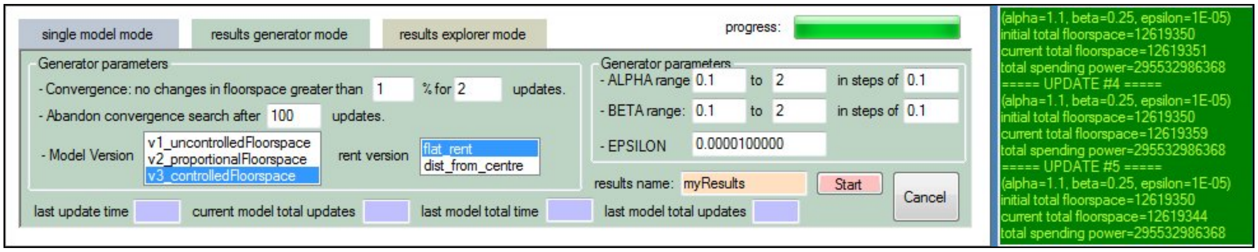


Figure 3. Results generator mode

The user specifies the following parameters:

- Alpha: start / stepping / end
- Beta: start / stepping / end
- Epsilon
- Maximum updates per model
- Equilibrium conditions: maximum percentage floorspace change for N updates
- Floor space model:
  - *equilibrium model 1*
  - equilibrium model 2
- Rent model
  - flat rate rents
  - declining rents from the centre

The results are saved to a file and can be loaded in and viewed using the results explorer mode explained in the next section.

4.3. The results explorer

The user can load a pre-generated (α, β) grid of models that have run to equilibrium or a linear sequence representing system evolution through time.

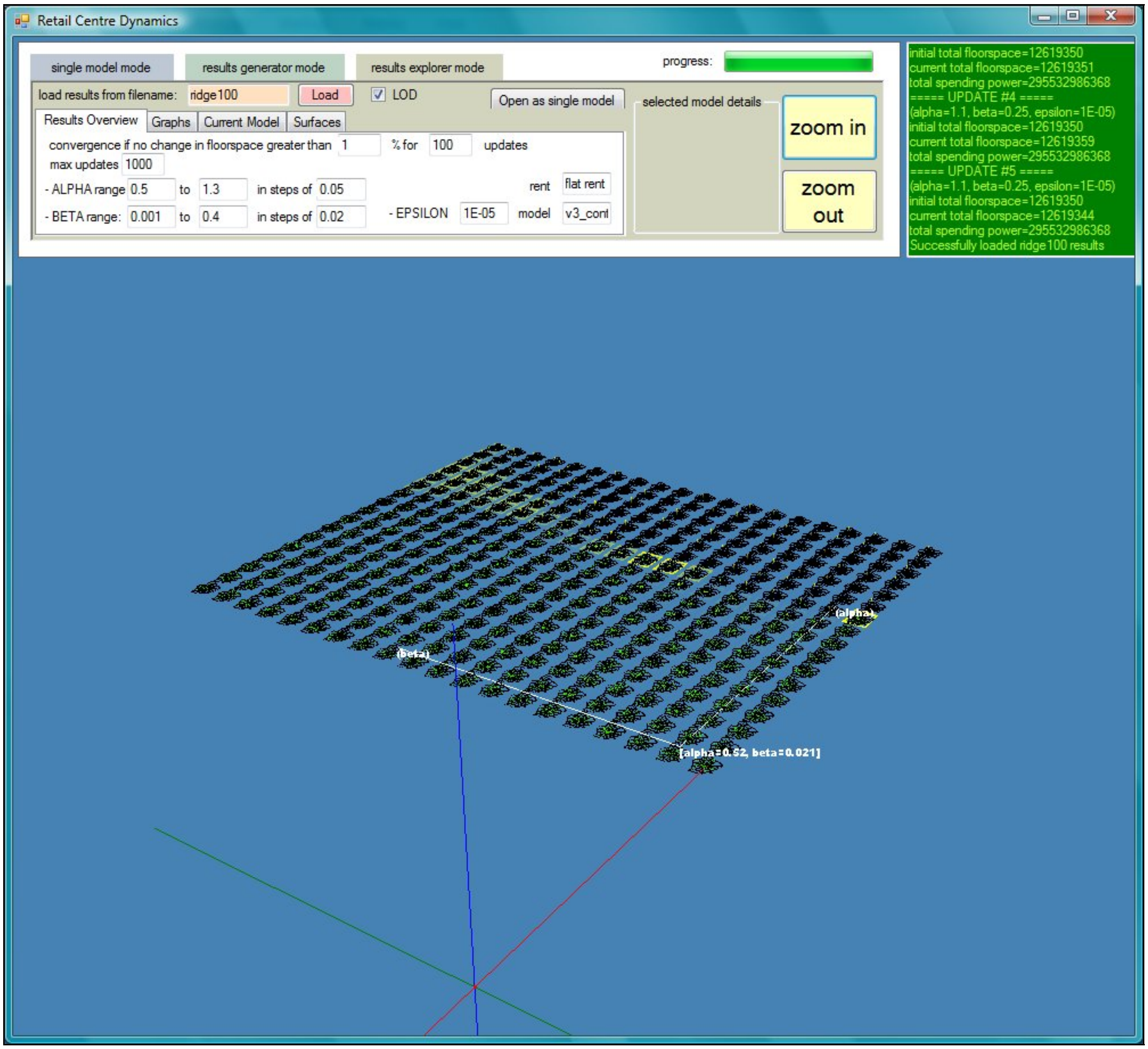


Figure 4. Results explorer mode

This mode is useful for phase change explorations.

The following features are available:

- 3D graphics representing the $\{W_j\}$ structure at equilibrium for each model on the grid
- Visual indication of R^2 value comparison with the starting W_j values for each model's equilibrium state
- 3D Parameter surface generation for:
  - order parameter: total retail centres > user specified threshold
  - R^2
- User can select a model on the grid for more detail:
  - Last Largest percentage size change
  - R^2
  - State: converged or not converged
  - Total updates before reaching equilibrium state
  - Option to open any model on the grid in Single model run mode to see it in more detail.
  - Select any retail centre in any model to view its name and Wj value at equilibrium

5. The data

Rather than use purely hypothetical data as has been common in the past, the simulations in the next section are based on London data[5].

The population centre data was derived from a combination of ward-level population data from the 2001 census and CACI Paycheck which provided income data at postcode level. For each ward we found an average income level from all the post codes inside that ward (except those with zero population) and then multiplied this together with the resident population for the ward to give a spending power. The ward centre points were used as the location of each population centre.

The retail centre data comes from the Town Centres Project 2002, which provides statistics for each town centre in London. We used the name, easting, northing and retail floor space fields.

[5] Clarke and Wilson (1985) did a version of this analysis – generating the (α, β) grid for a hypothetical system.

6. Some results

6.1. Single model run

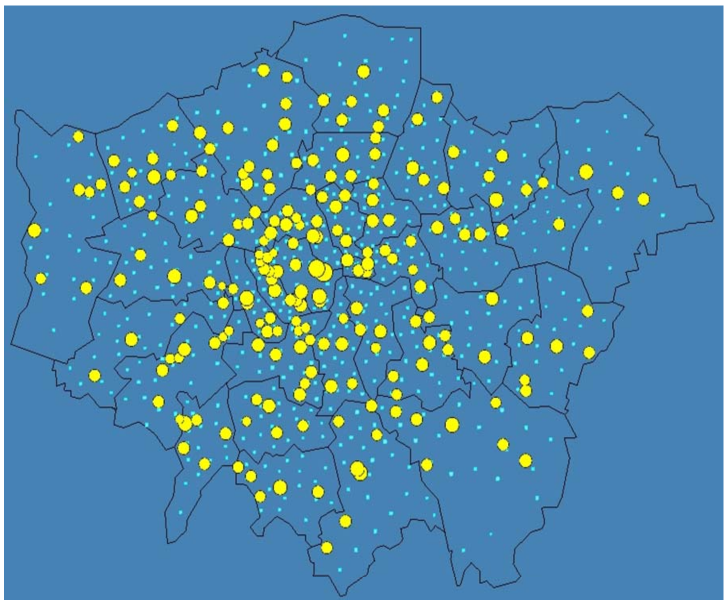


Figure 5. The initial conditions for the London test

The model was run for the London data described in section 3 and shown in Figure 5. Comparable output of the model run itself is shown in Figure 6 and the extent to which there is more detail available is shown in Figure 7 and Figure 8.

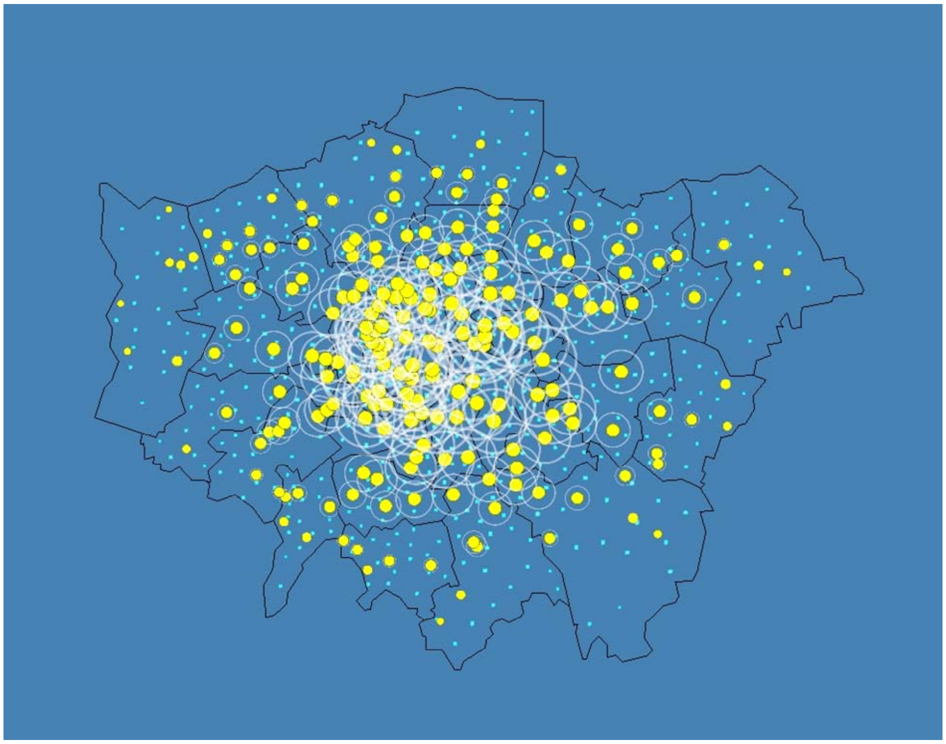


Figure 6 (a). Model output

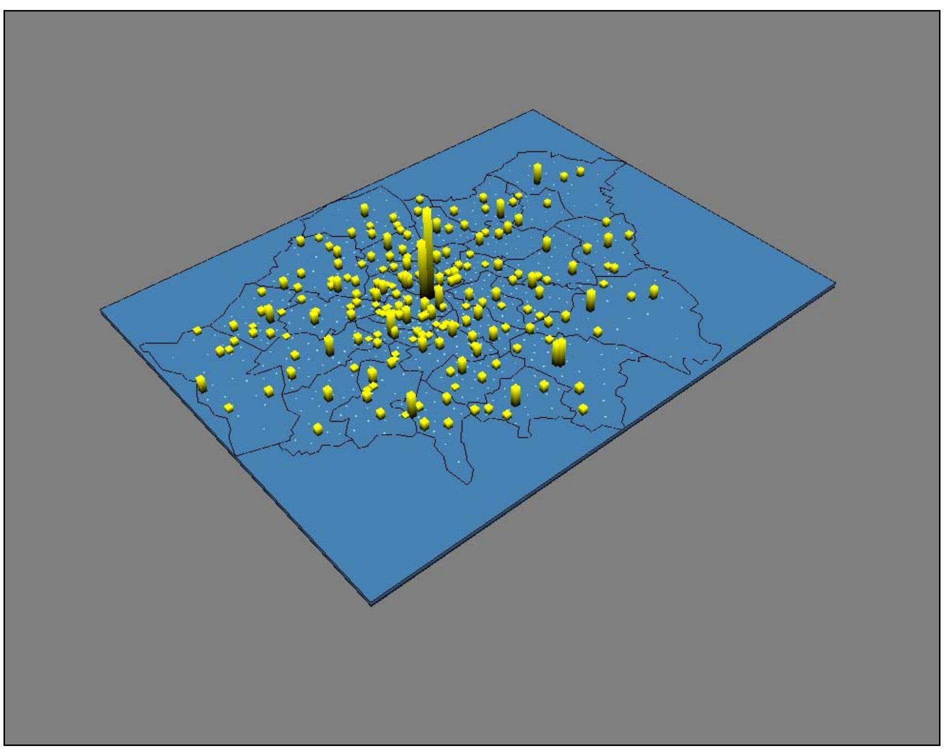


Figure 6(b) Model output in 3-D

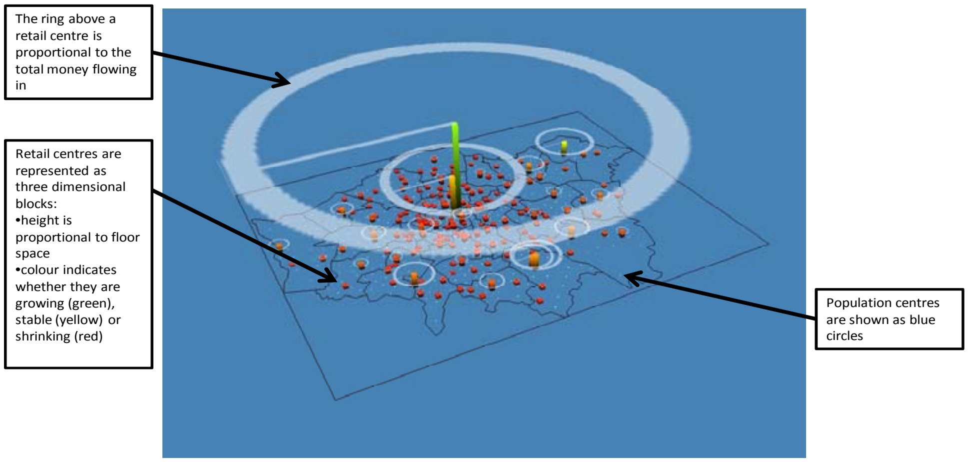


Figure 7. Revenue and dynamics

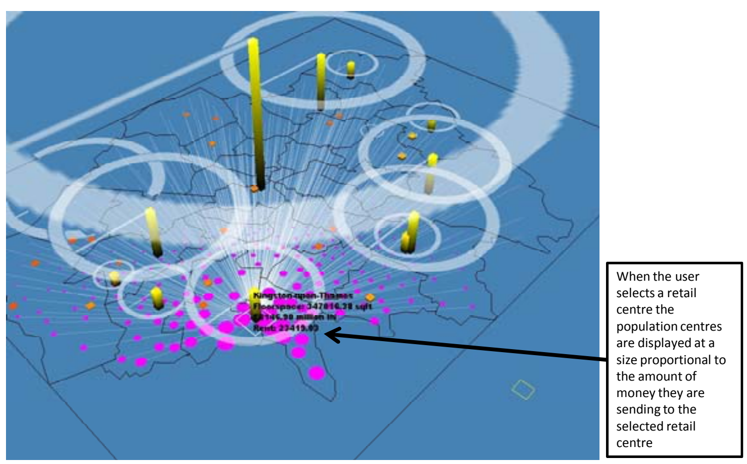


Figure 8. Flows to a centre.

So far, we have assumed a flat rent across the city. An obvious alternative assumption is to show rents declining from the city centre as shown in Figure 9 and it is straightforward to run the model in this mode though for the remainder of these illustrative results, we retain the assumption of a constant K.

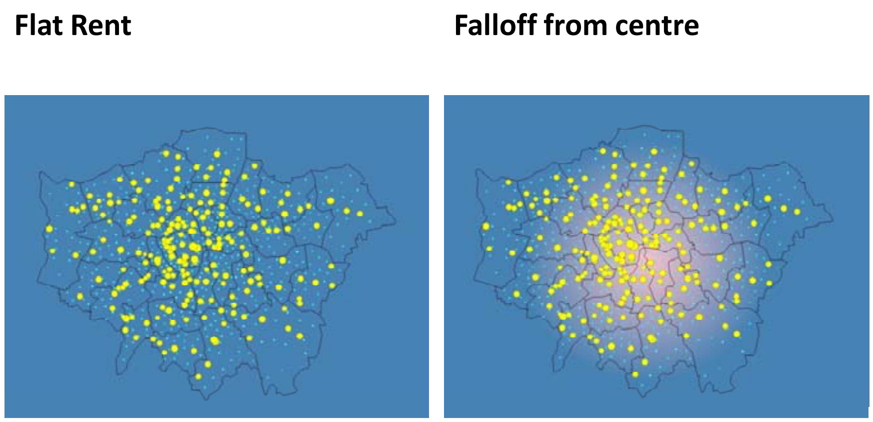


Figure 9.

Define N(W) to be the number of centres within a size band, W. A histogram of N(W) vs W in size bands is shown in Figure 10 (a) and a corresponding log-log plot in Figure 10 (b).

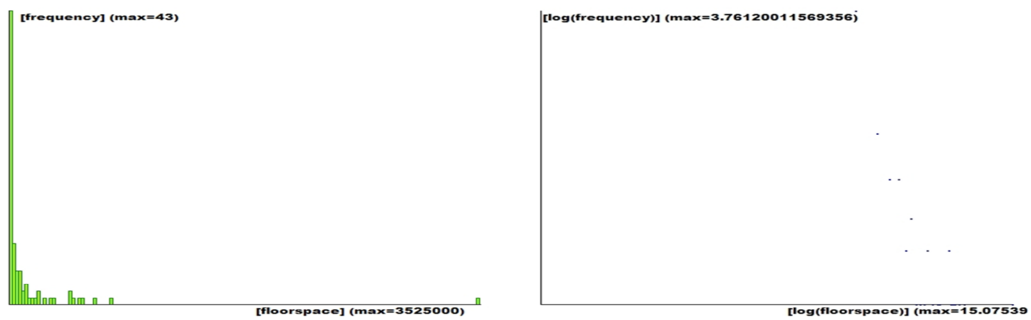


Figure 10 **(a)** **(b)**

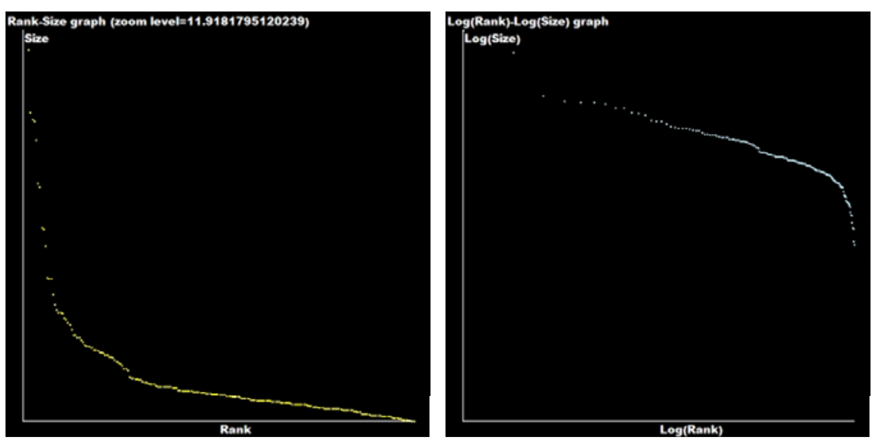


Figure 11. **(a)** **(b)**

In Figure 11(a), we plot $\{W_j\}$ by rank with the corresponding log-log plot in 10(b). This is a characteristic log normal distribution.

In Figure 12, we offer a rudimentary network analysis by counting the links at each node on which the flow exceeds a certain size. This is a characteristic scale-free plot.

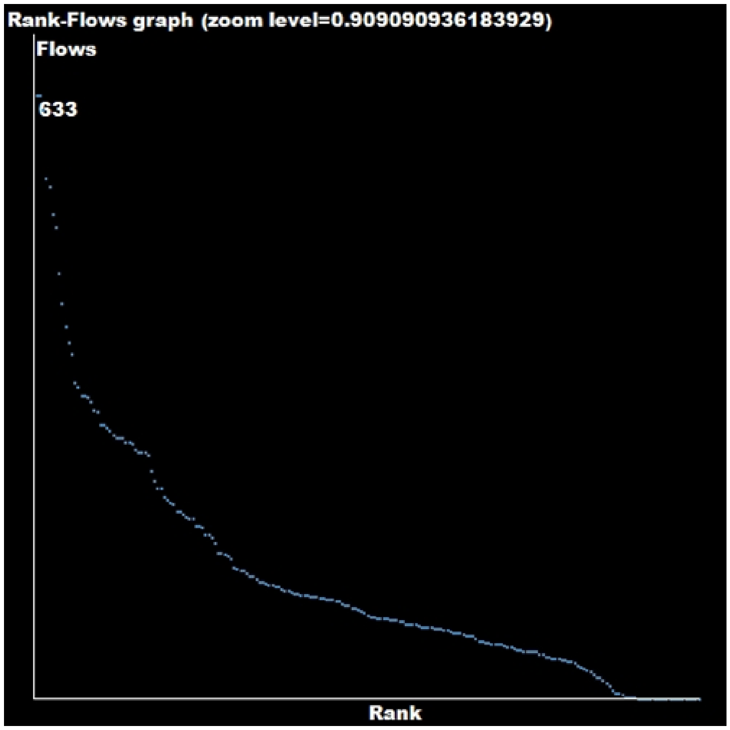


Figure 12.

6.2. The (α, β) results grid: phase change explorations

The next step is to begin to explore a series of runs for different parameter values and in Figure 13, we show an (α, β) grid.

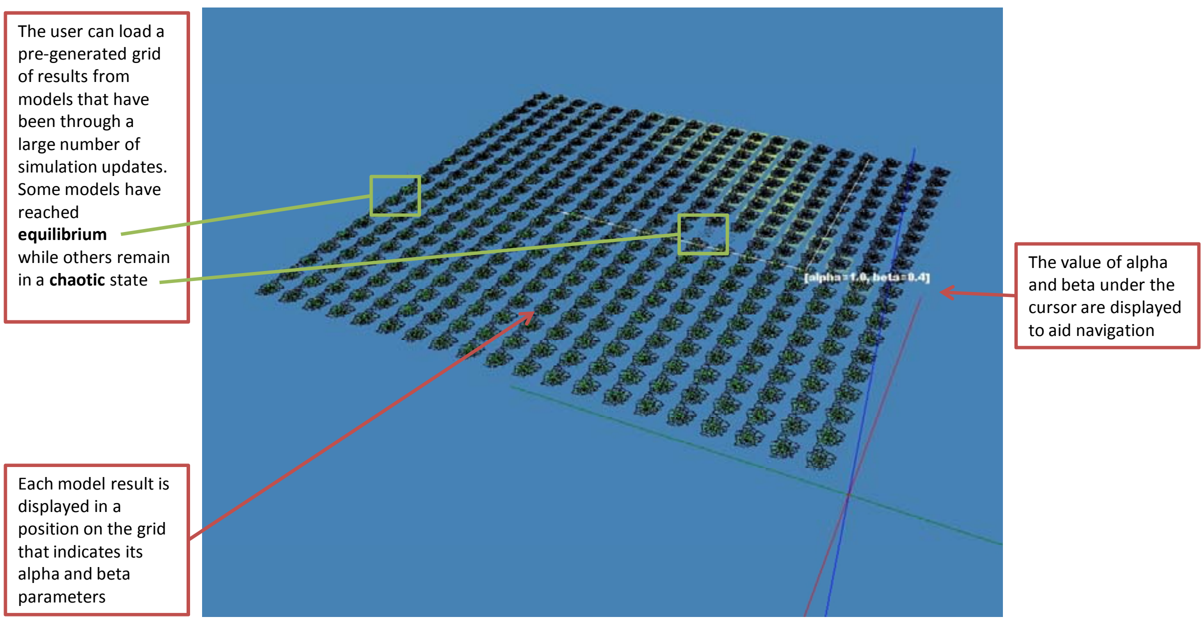


Figure 13. The grid.

As an order parameter, we take $N(W_j>x)$ – the number of centres with floorspace greater than some parameter, x. This is then plotted as a surface as shown in Figure 14. We initially set x= 1,000,000 sq ft. The ringed area in Figure 14 might represent a phase change and so we investigate this further.

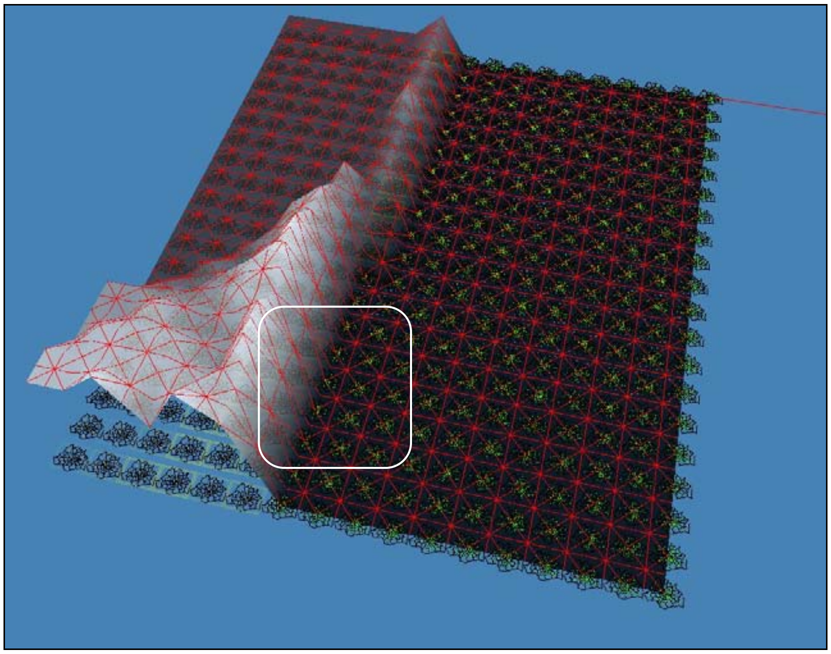


Figure 14. The grid with the order parameter plotted as a surface.

Zooming in closer we can identify two result maps, one on either side of the possible phase change – see Figure 15 and then the two maps that follow in Figures 16 and 17.

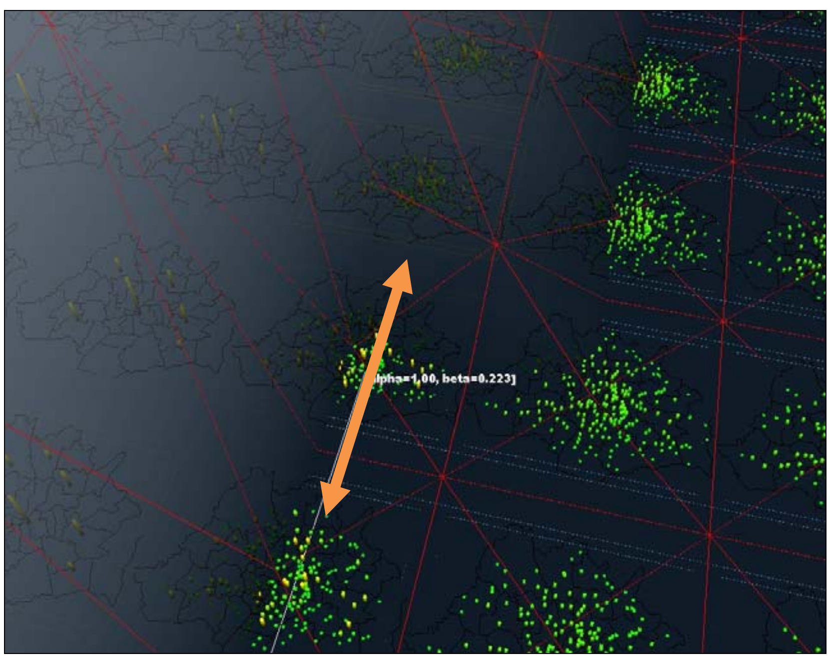


Figure 15.

At equilibrium, we generate Figures 16 and 17.

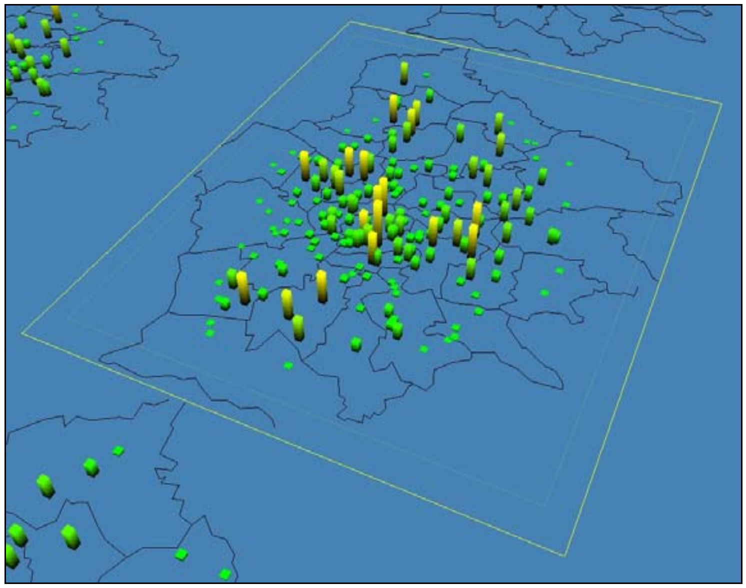


Figure 16. Alpha=1.00, beta=0.221

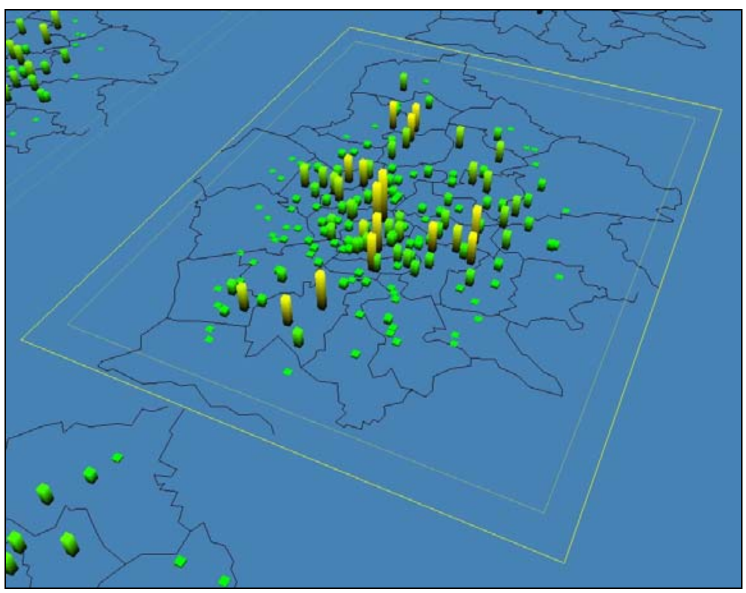


Figure 17. Alpha=1.00, beta=0.201, converged after 548 updates

In this case, there are no obvious differences. After comparing the two maps in details, we note that:

- *Central London* changes from **529,061 sq ft** in Map 1 to **1,017,853 sq** ft in Map 2

- *Twickenham* changes from **58,590 sq ft** in Map 1 to **2,054 sq** ft in Map 2
- *Teddington* changes from **14.70 sq ft** in Map 1 to **0.51 sq** ft in Map 2
- *Kensington High Street* changes from **744,763 sq ft** in Map 1 to **555,566 sq** ft in Map 2
- *Wood Green* changes from **323,584 sq ft** in Map 1 to **186,729 sq** ft in Map 2

When we examine the zonal maps, we can say that while there are some large changes, these may not be phase changes of the Figure 1 type. It remains a matter for future investigation as to whether they are phase changes of another type. (The Central London change comes about because in this simple version of the model, 'Central' absorbs 'West Ednd'. In later versions, a constraint can be added to prevent this happening.)

If we look at the two histograms of the W_j-distributions, they are broadly the same shape (see Figure 18).

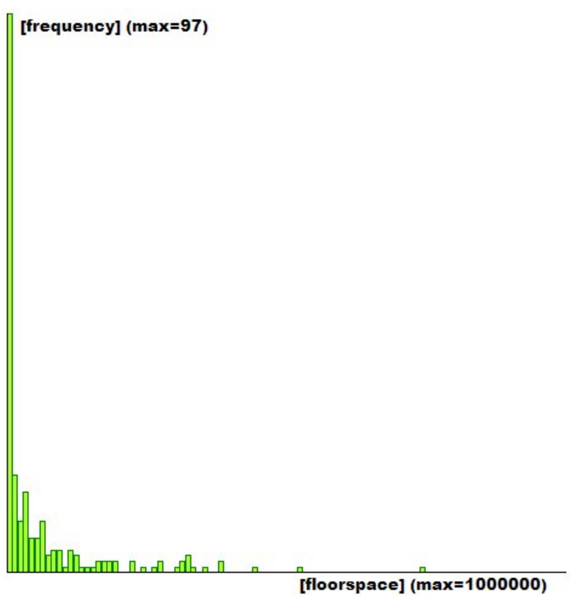


Figure 18. (a) W_j Histogram for Figure 15

(100 bins of 10,000)

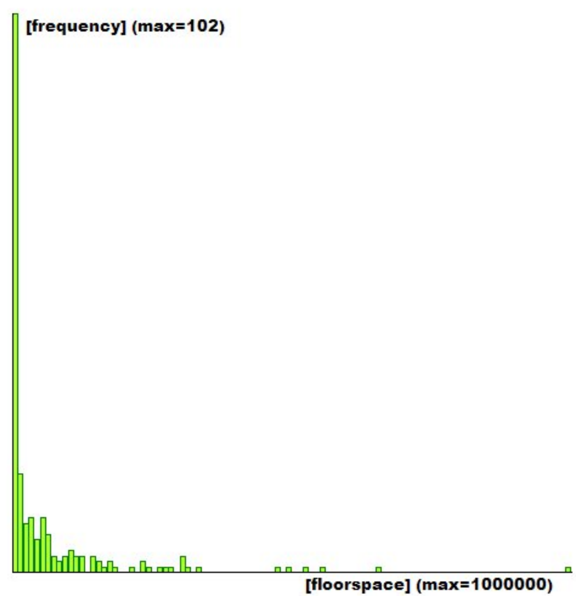


Figure 18 (b). W_j Histogram for Figure 16

(100 bins of 10,000

For a second exploration, we examine a section of the grid crossing α = 1 where we have greater expectation from the theoretical analysis (as well as the appearance of the grids), of finding a phase change. We generate a surface showing the order parameter with threshold set at 1,000,000 sq ft. The ringed area should be a good place to look for a phase change (see Figure 19).

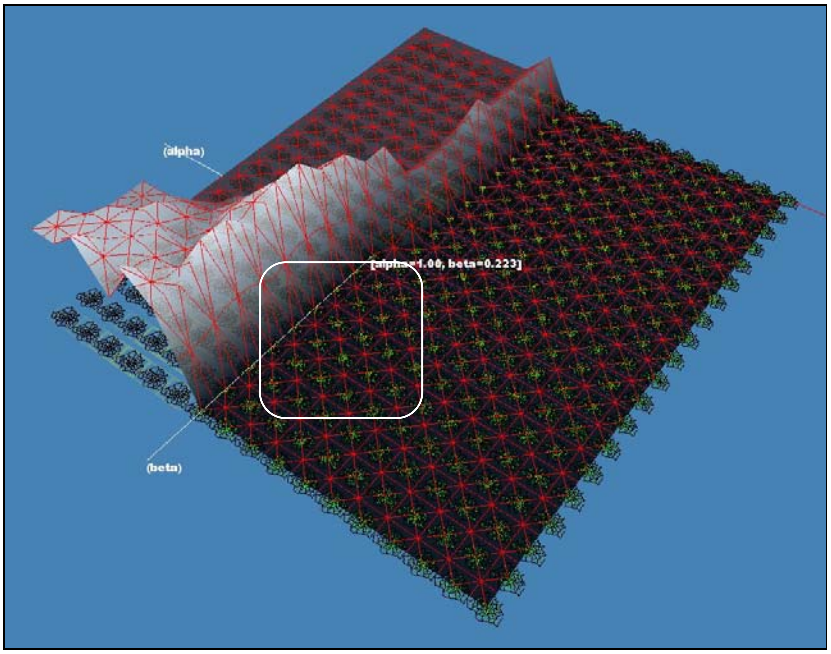


Figure 19.

Zooming in closer – see Figure 20 - we can identify two result maps, one on either side of the possible phase change – Figures 21 and 22.

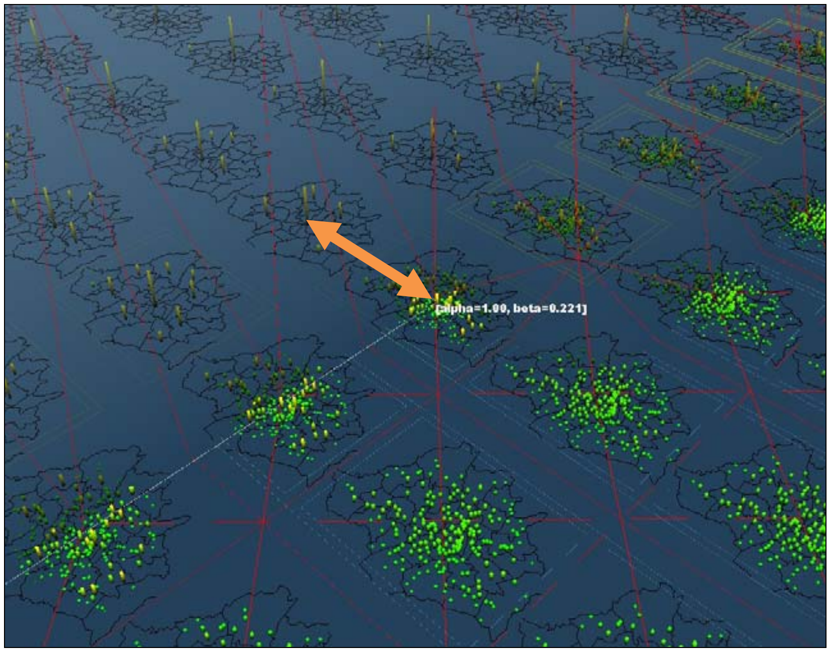


Figure 20.

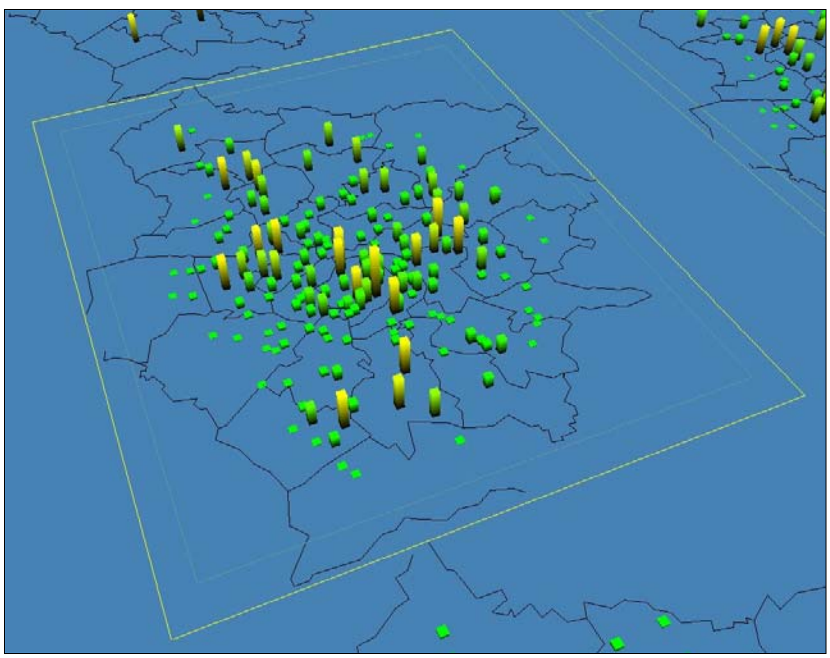


Figure 21. Alpha=1.00, beta=0.221

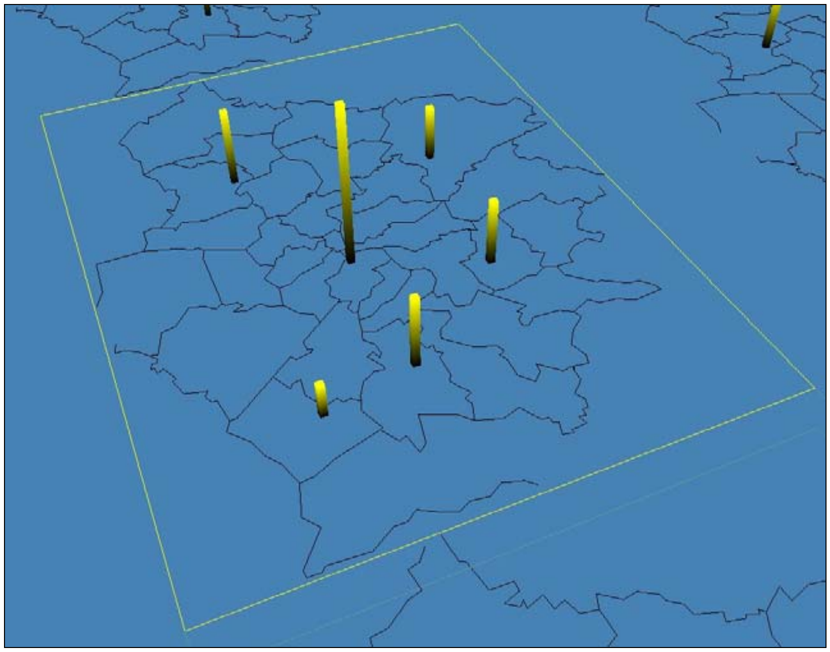


Figure 22. Alpha= 1.05, beta= 0.221

There are now obvious differences between the two maps. The only retail centres that survive in Figure 22 are:

- **Central London**: 7,036,082.66 sq ft
- **Ilford**: 1,722,372.37 sq ft
- **Bromley**: 901313.99 sq ft
- **Wimbledon**: 1,227,640.95 sq ft
- **Ealing**: 1,426,393.72 sq ft
- **Harrow**: 305,546.30 sq ft

Most retail centres drop to zero floor space in Figure 22 including Barking. This is illustrated by the image in Figure 23.

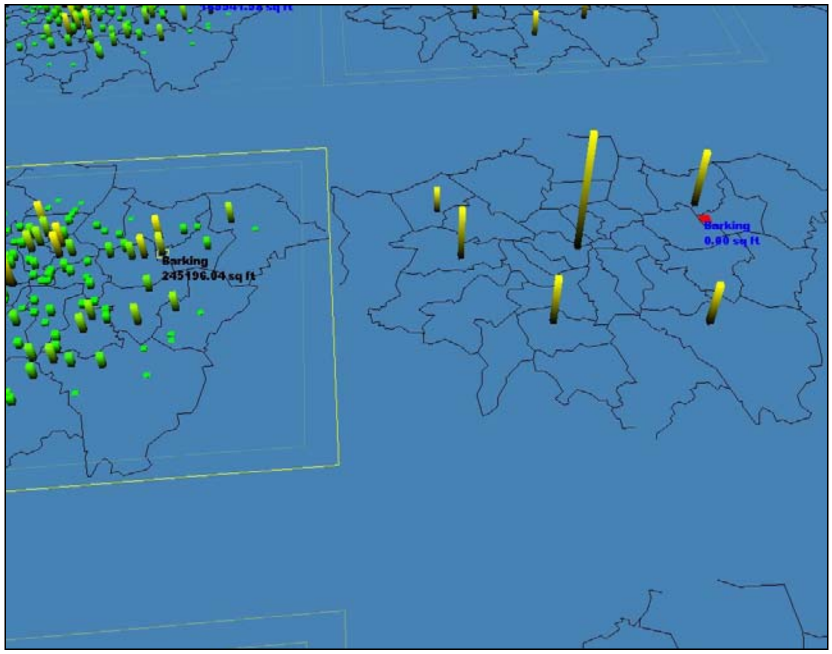
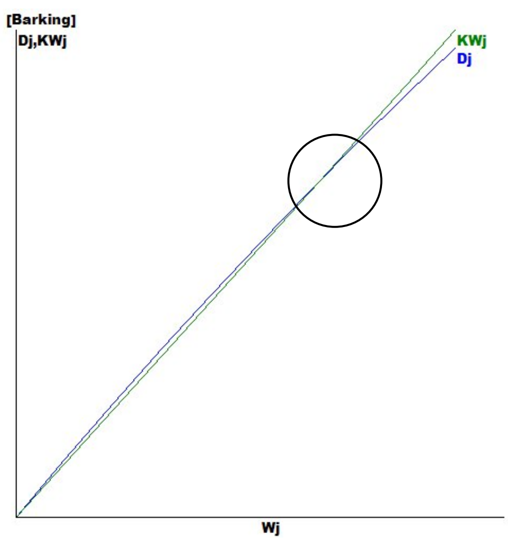


Figure 23.

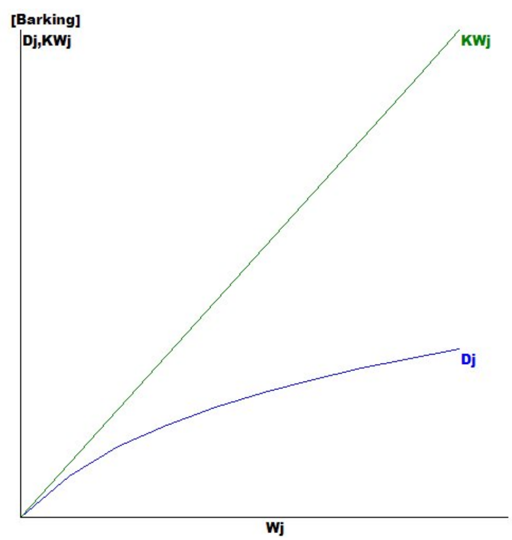
We can then examine the zonal D_j-W_j plots for Barking in Figure 24.



The graph goes up to 400,000 sq ft, so the lines intersect at about 250,000 sq ft.

Figure 24 (a)

D_j-W_j plot for Barking from Figure 21



The graph goes up to 2,000,000 sq ft. There is no intersection.

Figure 24 (b)

D_j-W_j plot for Barking from Figure 22

This shows that the results follow the theoretical analysis of Figure 1 with an intersection of line and curve in Figure 24 (a) and no intersection in Figure 24 (b). The histograms in Figure 25 also show striking differences. There is an interesting observation that comes to light through this computer plot being available: when α is close to 1, as in these runs, the D_j curve runs very close to the KW_j line. This can easily be checked by reference to equation (16).

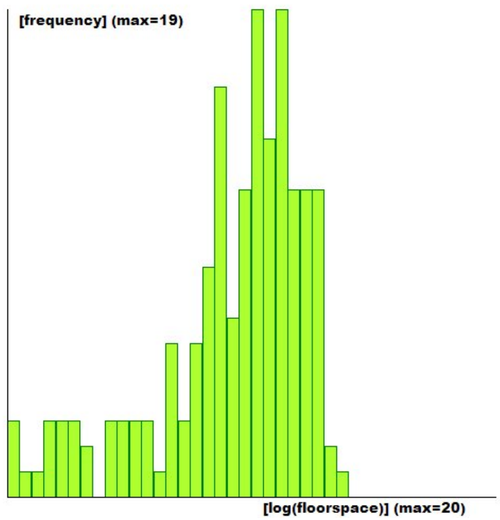


Figure 25(a).

Log(Wj) Histogram for Figure 20

(40 bins of 0.5)

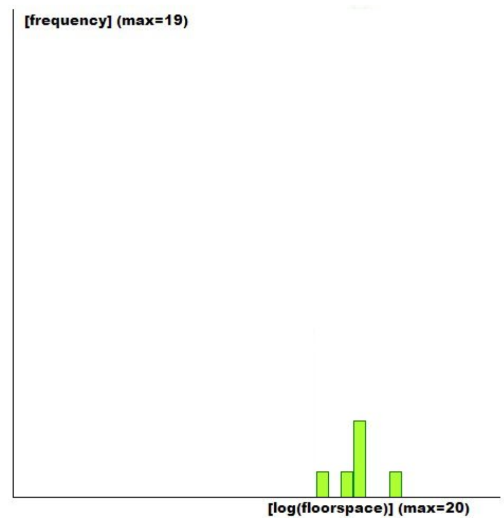


Figure 25(b).

Log(Wj) Histogram for Figure 21

(40 bins of 0.5)

6.3. Evolution through time

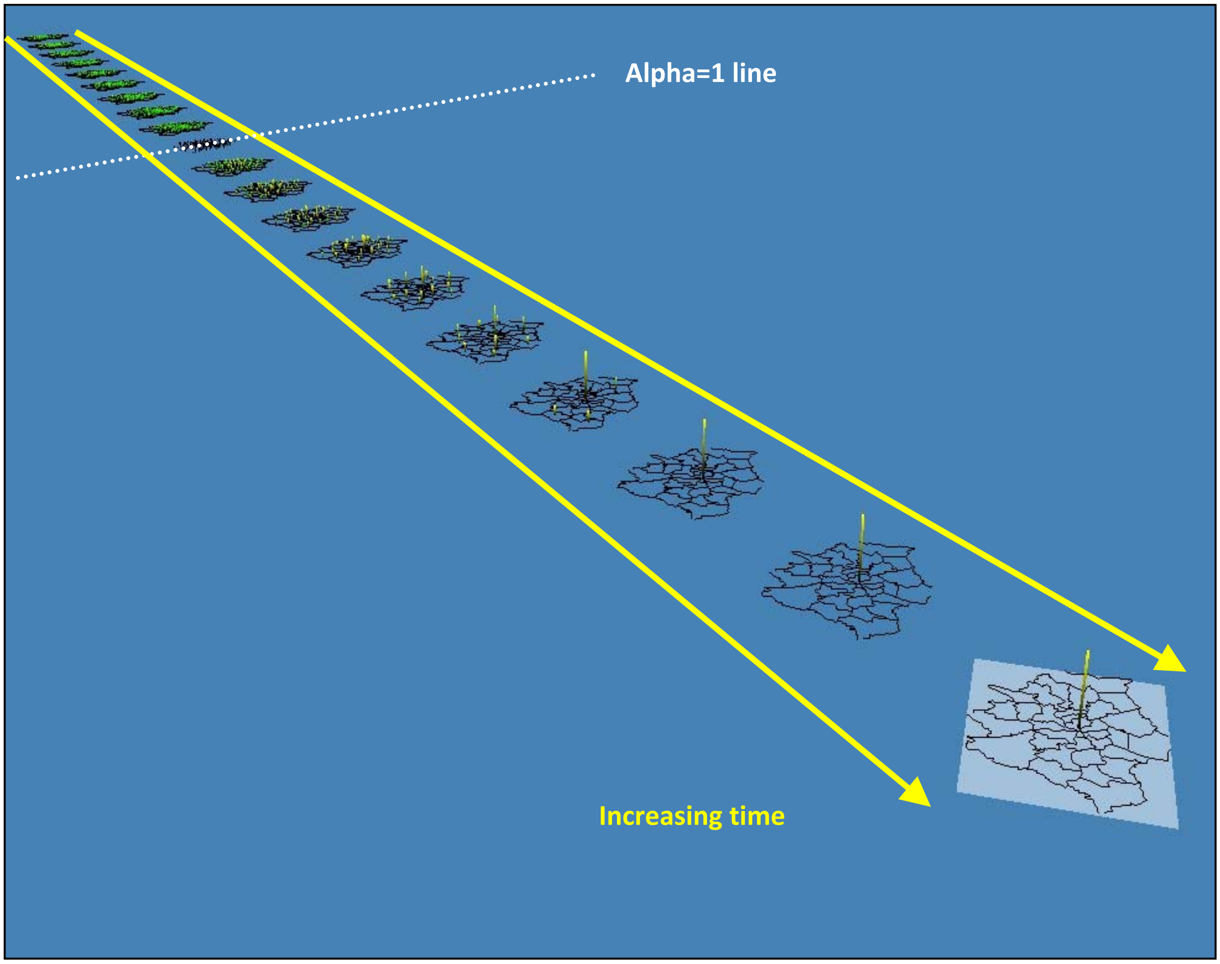


Figure 26. A timeline through the (α, β) grid

By taking a diagonal slice through the (α, β) grid, we can simulate the evolution of the system in time, and an artificial illustration is offered in Figure 26. This shows a shift to a small number of outlets in the first step, and then a gradual change through to one centre. The most significant change comes at $\alpha=1$.

6.4 Varying initial conditions

We can vary the starting W_j structure in the model using two parameters:

delta: controls the distribution of floor space in the retail centres relative to the centre of the map. A positive value gives large W values close to the centre using the following rule:

$$W_j = (distance\_from\_map\_edge)^{delta}$$

and a negative value gives large values close to the edge:

$$W_j = (distance\_from\_map\_centre)^{delta}$$

A zero value gives an equal distribution of floorspace.

tau: controls the number of retail centres that have non-zero floorspace. How those centres are chosen depends on the sign of the value: a positive value selects the largest centres first while a negative value selects the smallest centres first. For example: a value of -20 would select the 20 smallest centres while a value of +20 gives the 20 largest centres.

Delta and Tau can be used together or separately and always take the London data as their starting point before modifying the Wj values. If used together delta is applied first.

We can plot the starting structures that are generated using the above method on a (tau, delta) grid as shown in Figure 27.

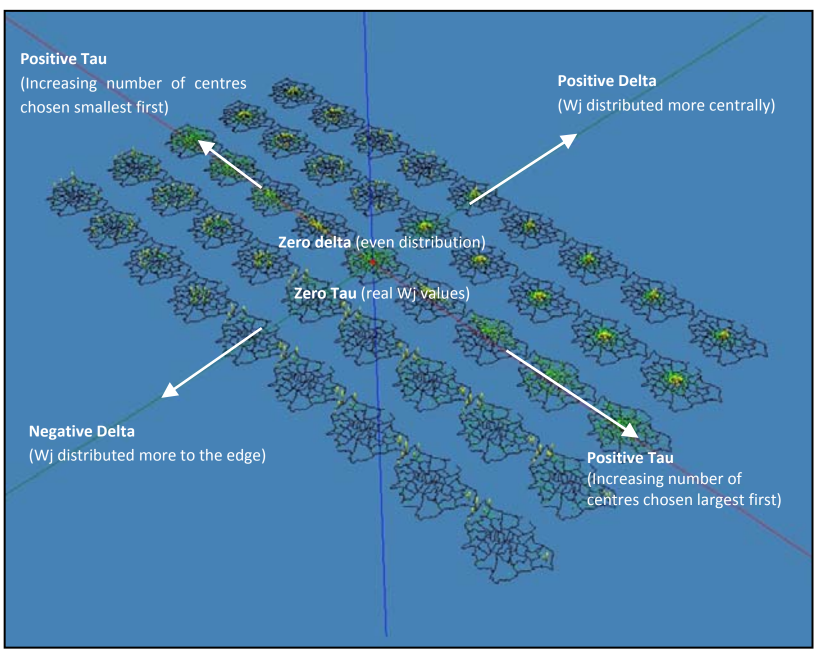


Figure 27. varying starting W_j structure

We can then choose constant values for alpha and beta and generate a (delta, tau) grid that shows how the equilibrium state varies as these two parameters change. **Error! Reference source not found.** shows the results for alpha = 1.1, beta = 0.25.

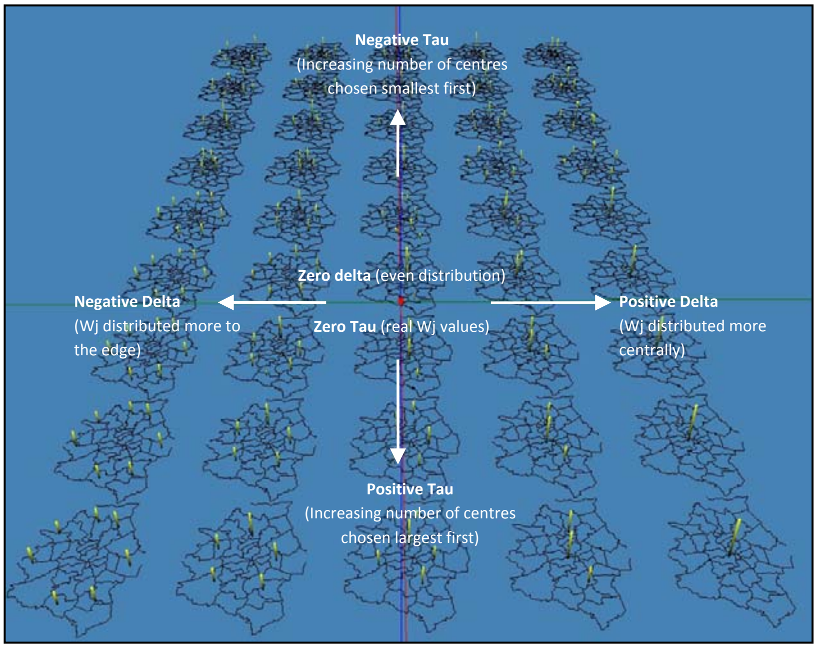


Figure 28.

We can also introduce a third variable to generate a three dimensional grid of models. Figure 29 varies delta and tau on X and Y while varying alpha on Z. The difficulty comes in interpreting the results because the multiple layers get in each other's way and so it is less clear than the two dimensional grids shown previously.

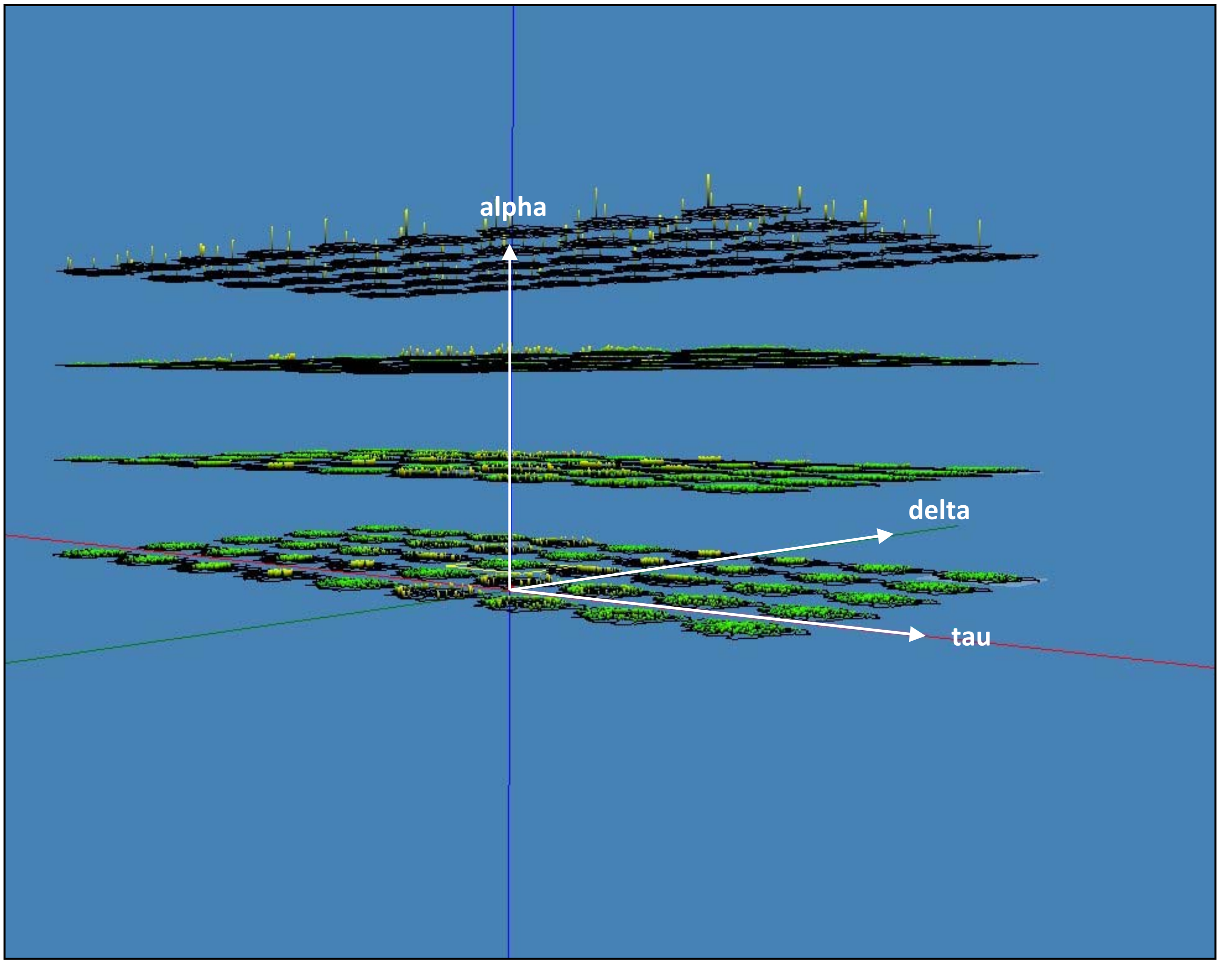


Figure 29. Three dimensional (tau, delta, alpha) grid

6.5 Adding a hypothetical retail centre

A more realistic way to vary the initial conditions of the model is to modify the London data by adding a new retail centre to the map. To keep the model consistent we scale all the retail centres in the model after this addition to maintain the same total floor space. We can vary the position and size of the new centre to see how this affects the equilibrium state of the model.

Figure 30 shows a grid of models where a new centre has been added to the starting W_j structure. The easting and northing of the centre are varied over the model runs in the x and y axes while the total floor space of the new centre is varied over the z axis.

The parameters used to generate this grid were:

- Alpha=1.1

- Beta=0.25
- Easting ranges (in metres) from 517500 to 547500 in steps of 10000
- Northing ranges from 172100 to 182100 in steps of 10000
- Floor space ranges (in square feet) from 0 to 2,000,000 in steps of 1,000,000

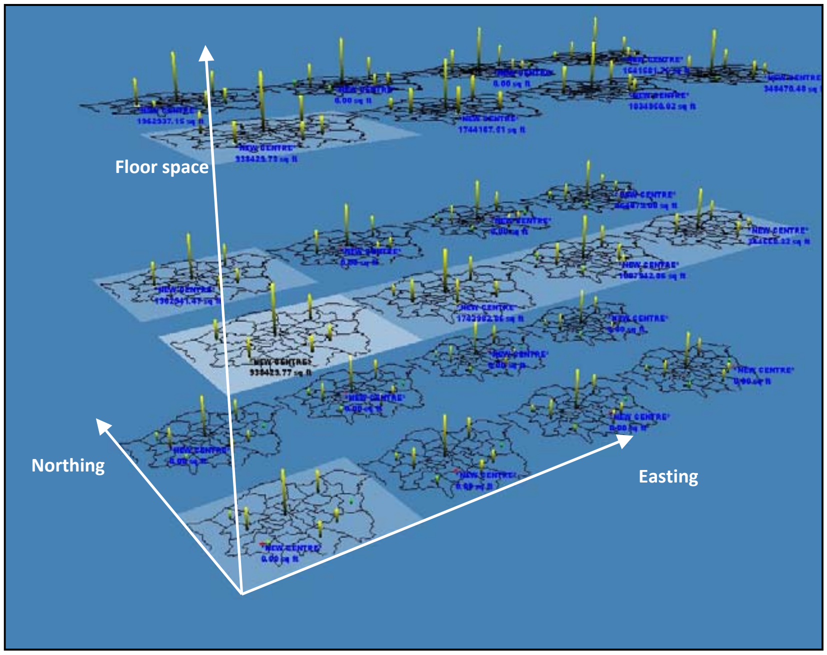


Figure 30. Three dimensional (easting, northing, floor space) grid for a hypothetical retail centre

Generally the new centre survives to equilibrium when located on the outskirts of London but has zero size when placed near central London.

We can zoom into the grid and compare two models more closely to try and identify a phase change. Taking the case when the new centre is located in south west London we look at the change that occurs as the new centre is introduced (i.e. its size changes from 0 to 1,000,000 sq ft). The new centre obviously takes floor space away from other retail centres and we can see that three centres are pushed to zero size, including:

- **Kingston upon Thames**: from 181,713.44 sq ft
- **Ealing**: from 1,705,415.87 sq ft
- **Bexleyheath**: from 45,304.02 sq ft

Harrow seems to benefit from the new centre as it increases in size from 229,562.08 sq ft to 1,034,782.25 sq ft. These changes are shown in Figure 31. This is obviously all very artificial, but illustrates in principle the kinds of investigations that can be carried out with the system.

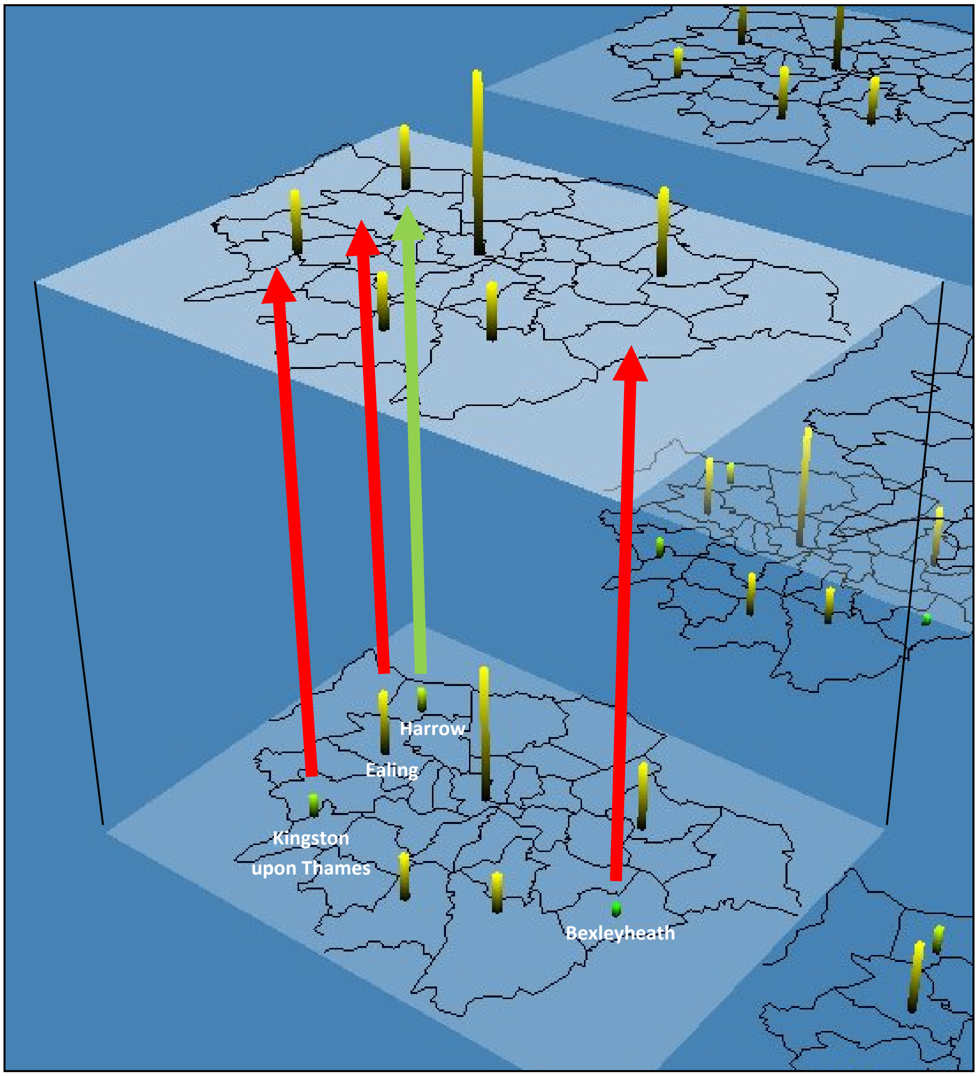


Figure 31.

7. Concluding comments.

We have demonstrated a rich visualisation system which has enabled us to identify phase transitions in retail model runs based on semi-realistic data. This paves the way for the development of more realistic tests with the retail model, with other kinds of urban model, and indeed with other related systems that can be modelled using this kind of methodology (cf. Wilson, 2008). The adjustment of single zones in the initial conditions also shows how the system could be used i9n a variety of planning applications.

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References

Clarke, M. and Wilson, A. G. (1985) The dynamics of urban spatial structure: the progress of a research programme, *Transactions, Institute of British Geographers*, *NS 10*, 427-451.

Harris, B. and Wilson, A. G. (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial-interaction models, *Environment and Planning, A, 10*, 371-88.

Wilson, A. G. (1967) A statistical theory of spatial distribution models, *Transportation Research, 1,* 253-69

Wilson, A. G. (1970) *Entropy in urban and regional modelling*, Pion, London.

Wilson, A. G. (1981) *Catastrophe theory and bifurcation: applications to urban and regional systems*, Croom Helm, London; University of California Press, Berkeley.

Wilson, A. G. (1988) Configurational analysis and urban and regional theory, *Sistemi Urbani, 10*, 51-62.

Wilson, A. G. (2008) Boltzmann, Lotka and Volterra and spatial structural evolution: an integrated methodology for some dynamical systems, *Journal of the Royal Society, Interface, 5*, pp. 865-871.