Vlasov Equation, Violent Relaxation and some comments on the nature of Entropy

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(ApJ 846, 125, 2017; ApJ 870, 128, 2019; ApJ 872, 20, 2019)

Boltzmann's Kinetic theory

- Boltzmann equation:

  df/dt = Γ[f]

  f(x,v,t): phase-space density for a typical/test particle
- df/dt: time reversible
- Γ[f]: collisional/relaxation term – irreversible

  introduces the arrow of time
- self - gravitational N-body systems → 'collisionless' ($t_{coll} > t_{cr}$): Γ[f] = 0
- can be described by the Vlasov-Poisson equation:

  df/dt = 0 → time – reversible!

- violent relaxation: clearly non-reversible & assumed described by Vlasov-Poisson - "Fundamental paradox of stellar dynamics" (Ogorodnikov 1965)

testing Vlasov

- transport equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \phi}{\partial \mathbf{r}}.\frac{\partial f}{\partial \mathbf{v}} = \Gamma[f]$$

- Shannon entropy:

$$S = -\int f \ln f d\mathbf{r} d\mathbf{v}$$

- entropy evolution:

$$\frac{dS}{dt} = -\int (1 + \ln f)\Gamma[f] d\mathbf{r} d\mathbf{v}$$

- collisionless system:

$$\Gamma[f] = 0 \longrightarrow \frac{dS}{dt} = 0$$

- let's test Vlasov by looking the entropy!

- N-body simulations with NBODY6, NBODY2, GADGET-2
  N = 10^4, 10^5, 10^6
- ICs: uniform sphere, Q_0=T/|W|= 0.5

- entropy estimation:

$$\hat{S} = -\frac{1}{N}\sum_i \ln \hat{f}_i$$

f is estimated in the phase-space by 3 methods: adaptive kernel, k-NN and EnBiD

(Sharma & Steinmetz 2006)

testing Vlasov

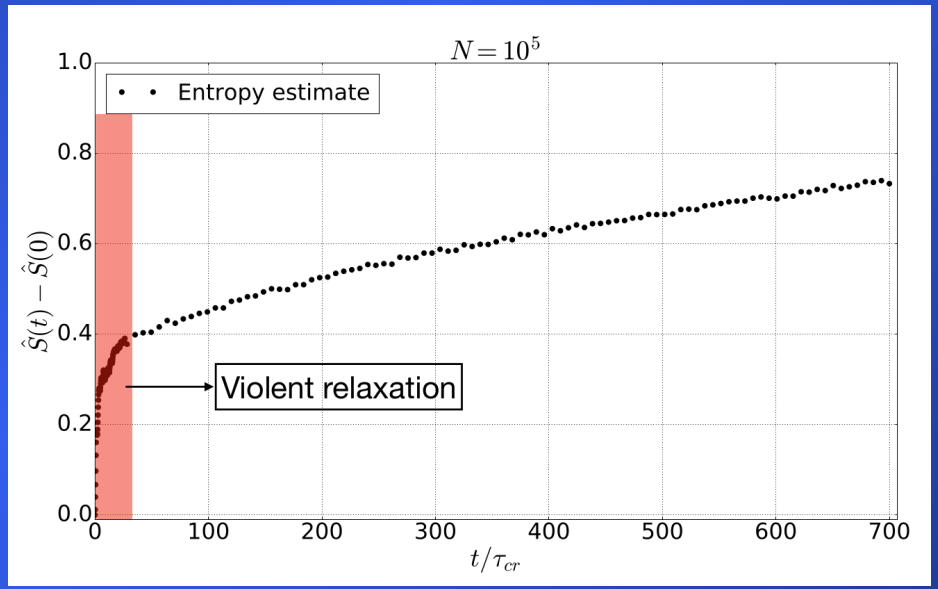
- overall increase of entropy
- entropy oscillations during violent relaxation:

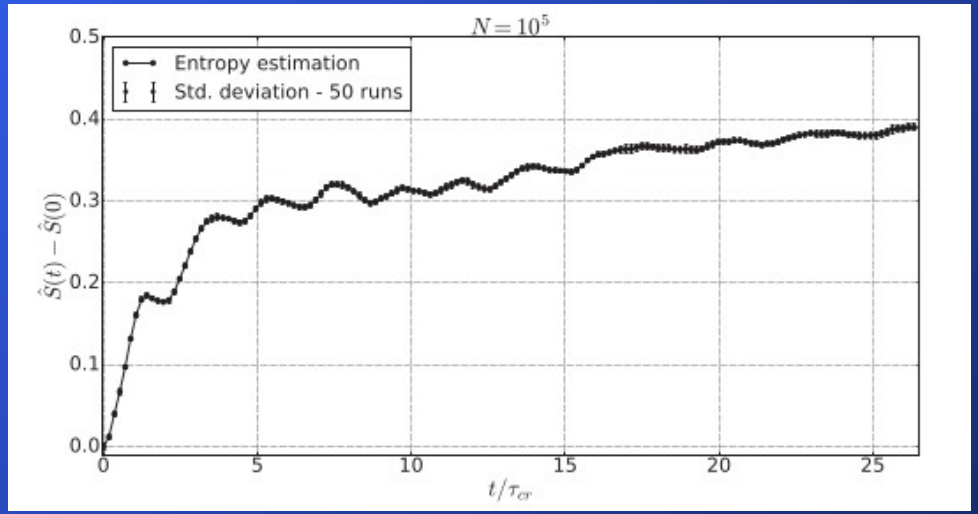
  H theorem: monotonic increase of S if the system is described by the Boltzmann equation → violation!

- oscillations: result of convertion of kinetic to potential energy and vice-versa

- Prigogine & Severne (1966), Jaynes (1971): behaviour expected in systems with appreciable potential energy – they present correlations, memory

Vlasov doesn't work for VR!



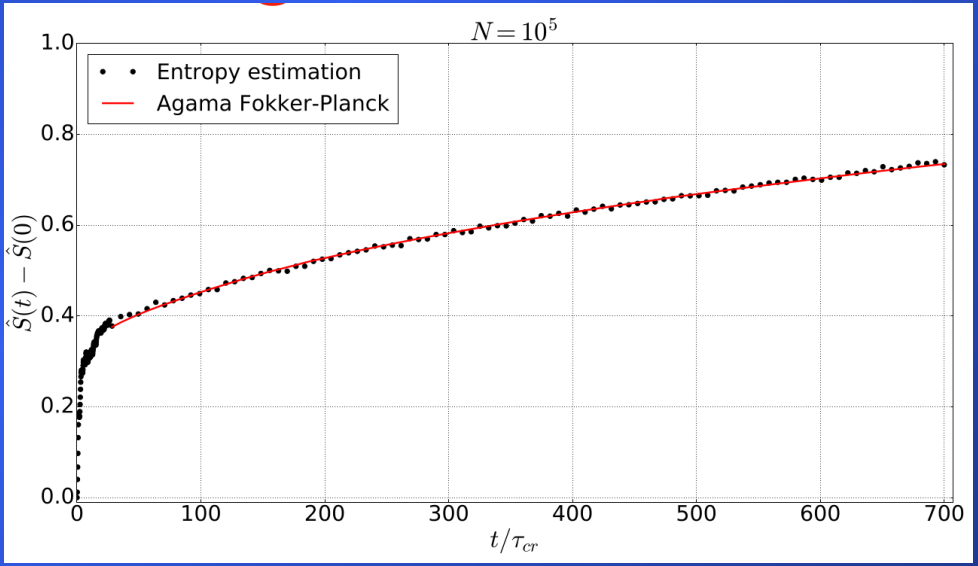


long-term evolution

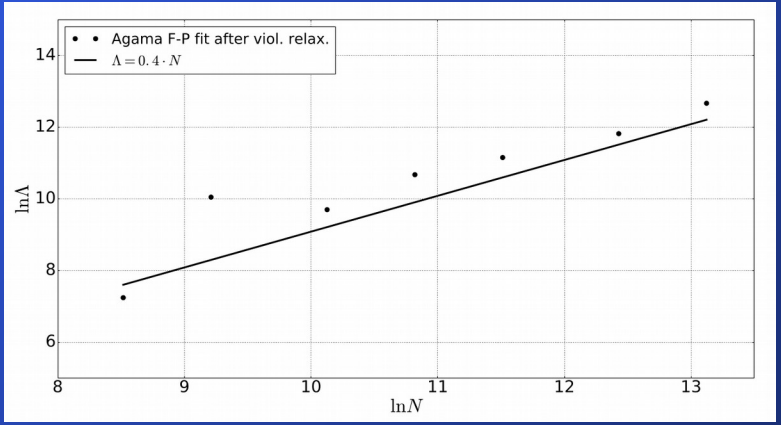
- (Orbit-averaged) Fokker-Planck: $\frac{df}{dt} = \Gamma_{FP}[f]$
- Weak encounters, static potential, $f = f(E)$

$\Gamma_{FP}[f(E)] \propto \ln \Lambda ...$ Theory: $\ln \Lambda \approx \ln(0.4 \cdot N)$

(free parameter)



Integration with AGAMA (Vasiliev 2019)



two phenomena, two time-scales:

- violent relaxation

  fast - t_{cr}

  inconsistent with Vlasov

- collisional relaxation

  slow – t_{coll}

  consistent with Fokker-Planck

What causes irreversibility?

- chaotic mixing?

  Merritt & Valluri 1996, Prigogine 1999

- time-varying potential?

  Lynden-Bell 1967

- large N and phase mixing?

  Lebowitz 1993; 1999

tool: orbit integration (with AGAMA) in fixed external potential

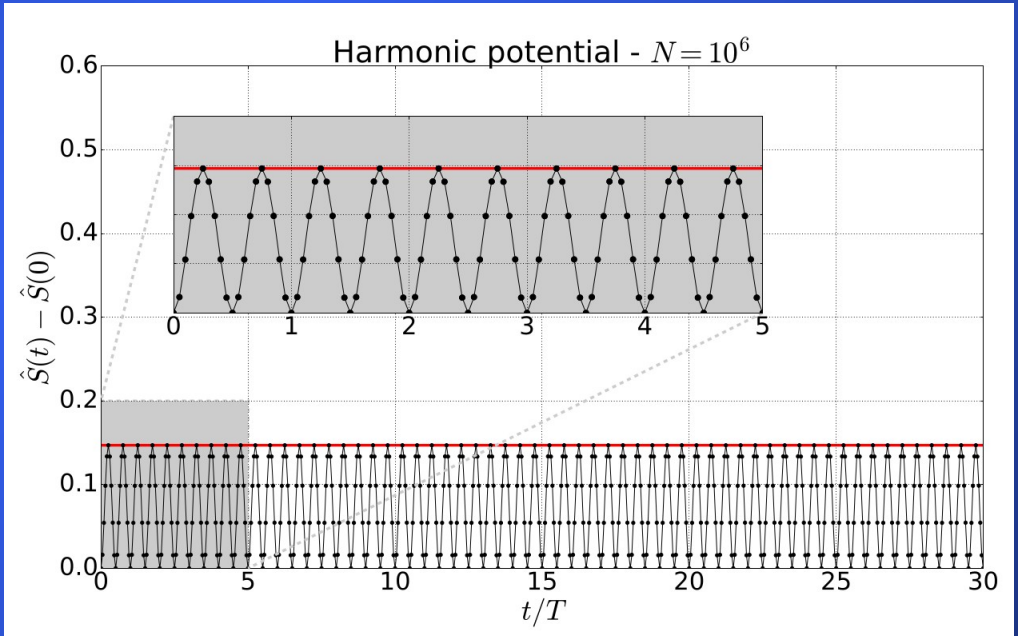
harmonic potential

- potential:

$$\Phi(r) = \tfrac{1}{2}\,\Omega^2\, r^2$$

- all particles have the same frequency
- macroscopic <u>reversibility</u>
- IC: sample of a Plummer model

- entropy changes due to “mechanical” phase-mixing



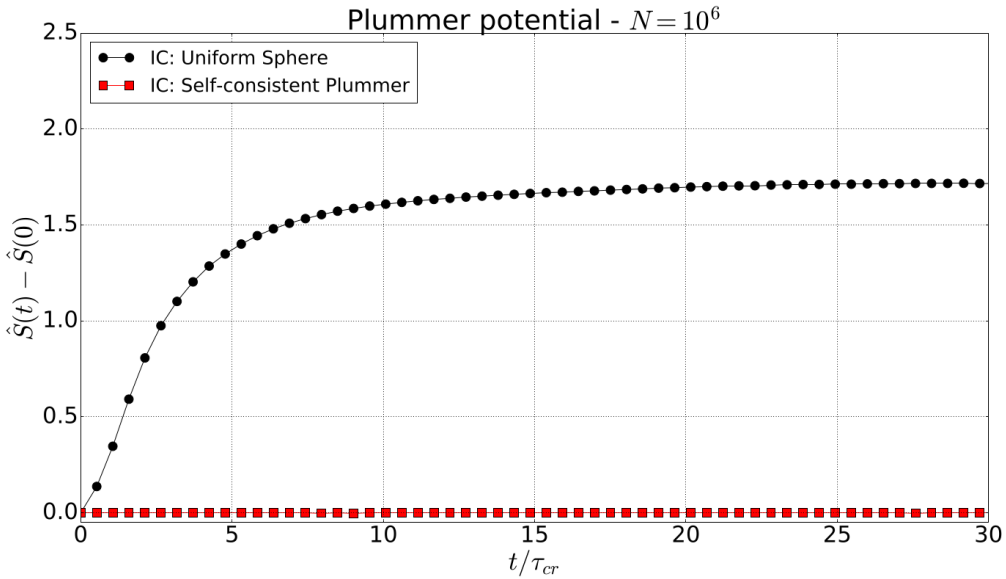
Plummer potential

- potential:

$$\phi(r) = -\frac{GM/a}{\sqrt{1+(r/a)^2}}$$

- IC: uniform sphere
- IC: self-consistent Plummer sample

$$f(E) \propto (-E)^{7/2}$$

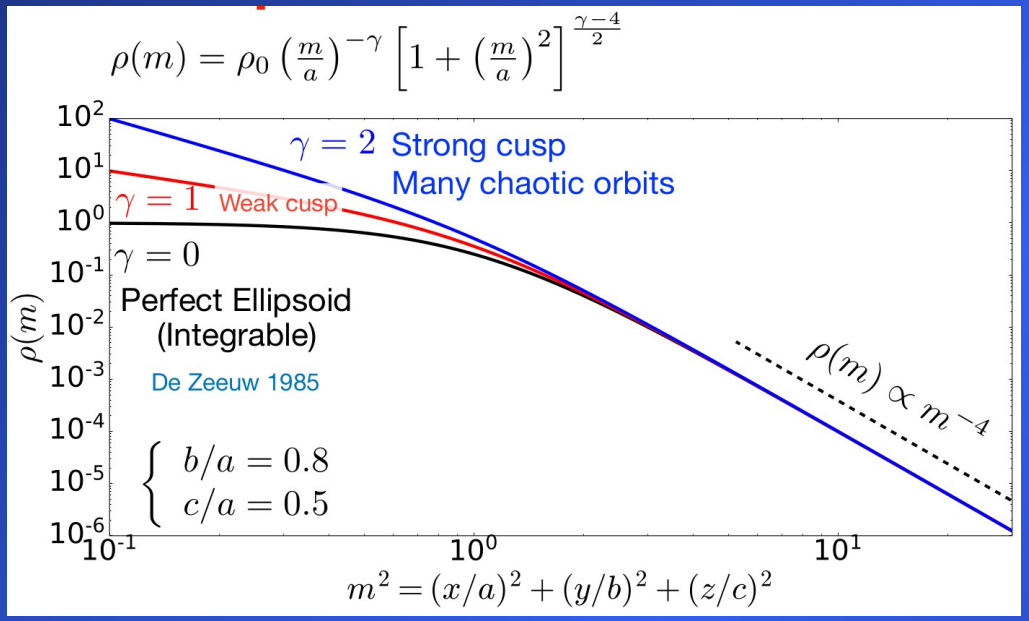


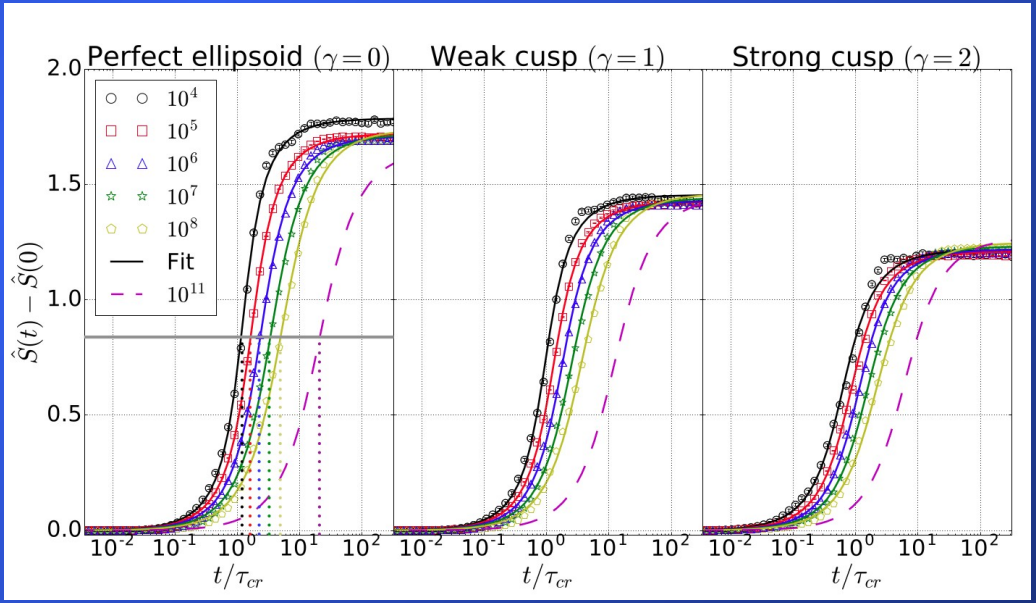
- No spurious entropy production
- Agreement with 2nd law; similar to N-body sims.

- Entropy increase due to “mechanical” phase-mixing (no chaos)

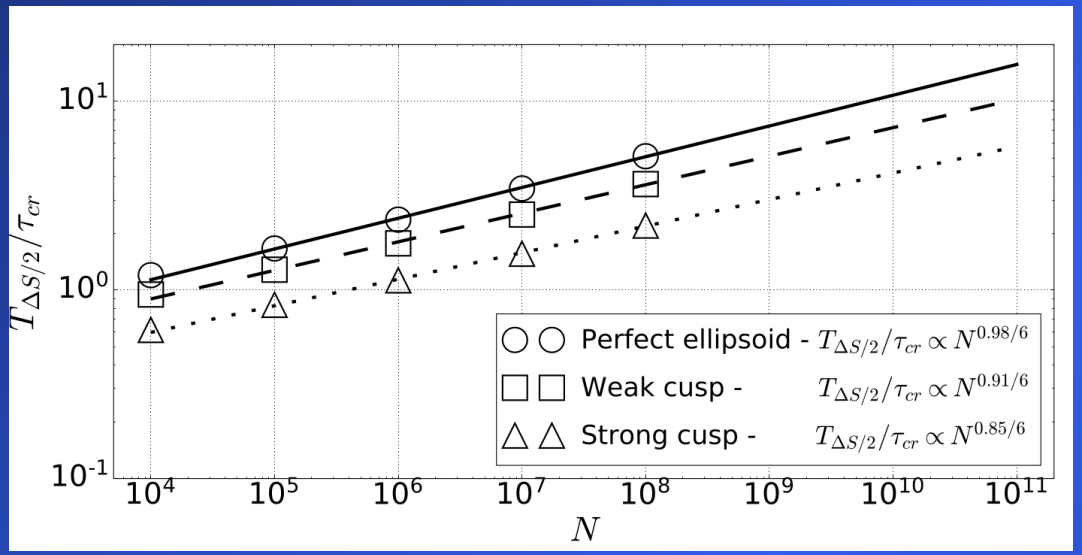
chaotic mixing

- ellipsoidal model





Chaos and N-dependence



- chaos accelerates entropy production

- N-dependence:

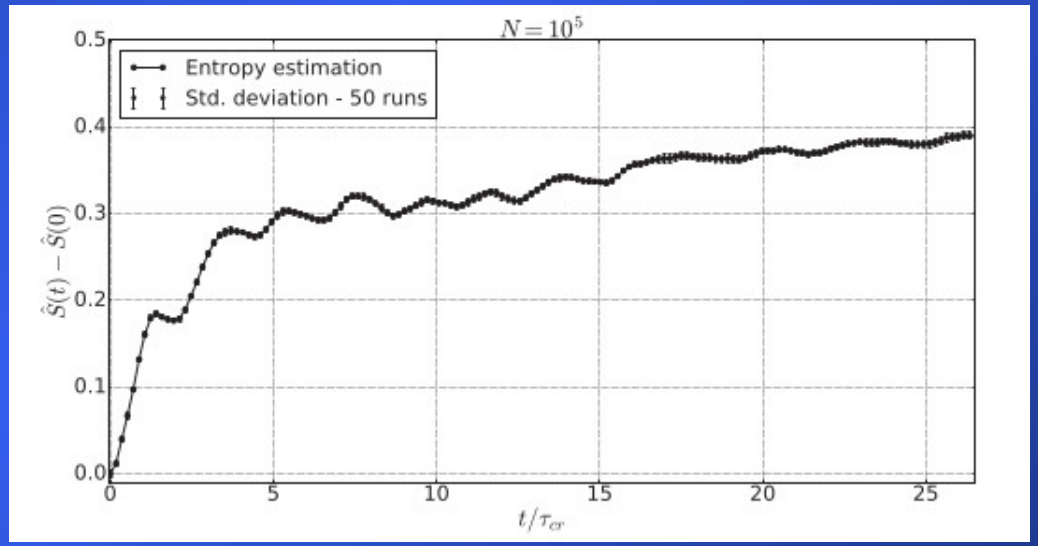
$$\frac{T_{\Delta S/2}}{\tau_{cr}} \approx 0.1 \times N^{\alpha/6}$$

$\alpha \approx 1$ for integrable systems

$\alpha \lesssim 1$ for non-integrable systems

entropy and violent relaxation

- entropy grows by phase-mixing:
  - mechanical phase-mixing
  - chaotic phase-mixing



- VR: both processes

  -oscillations - mechanical phase mixing

  -overall entropy increase: mechanical+chaotic phase-mixing

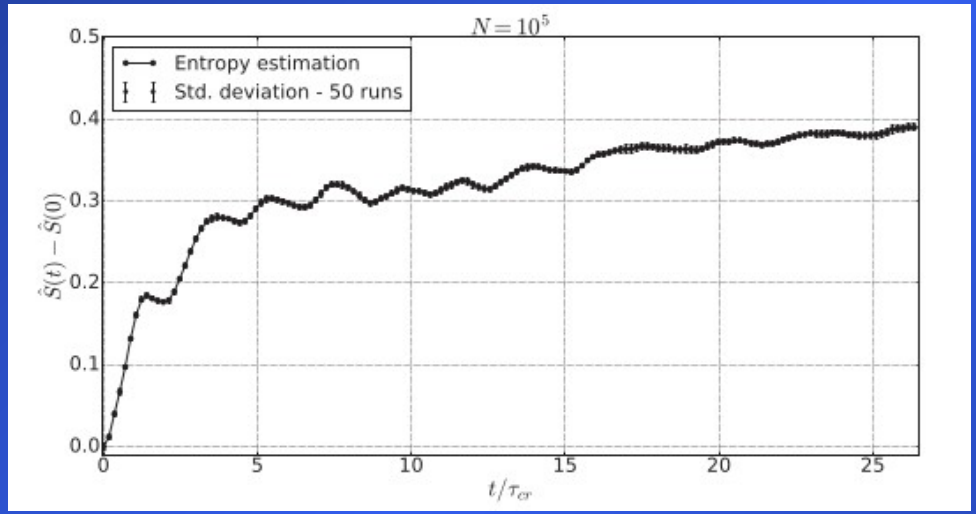
- oscillations - possibly associated to the long-range nature of gravitational interactions

  => correlations among the particles

entropy and violent relaxation

- violent relaxation:
- usual interpretation – driven by the time-changing collective gravitational potential

  (King 1962, Hénon 1964, Lynden-Bell 1967)



- entropy production: follows the N-dependence

  => time-changing potential is not the main driver!

  => the relaxation is discreteness driven!

$$\frac{T_{\Delta S/2}}{\tau_{cr}} \approx 0.1 \times N^{\alpha/6}$$

- entropy increases whenever the phase-space distribution of a system is not in dynamical equilibrium with the potential

entropy and violent relaxation

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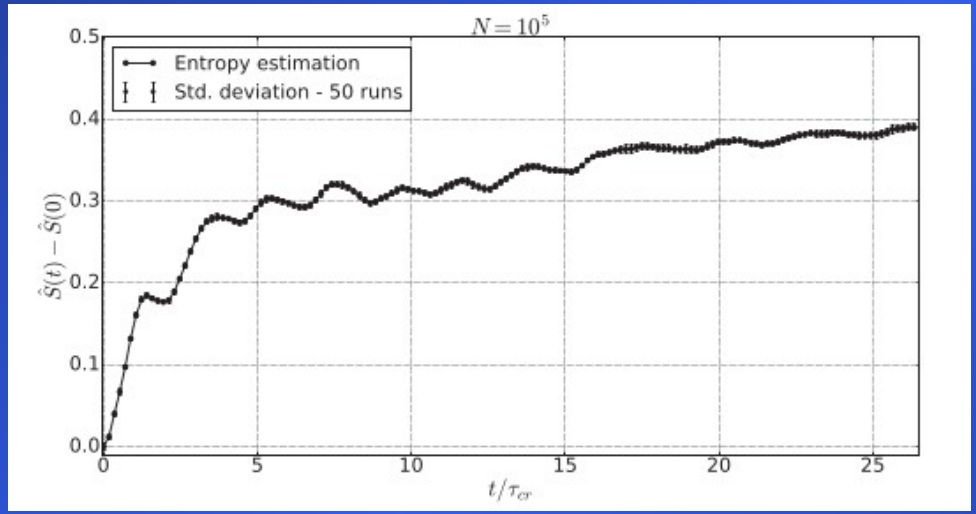

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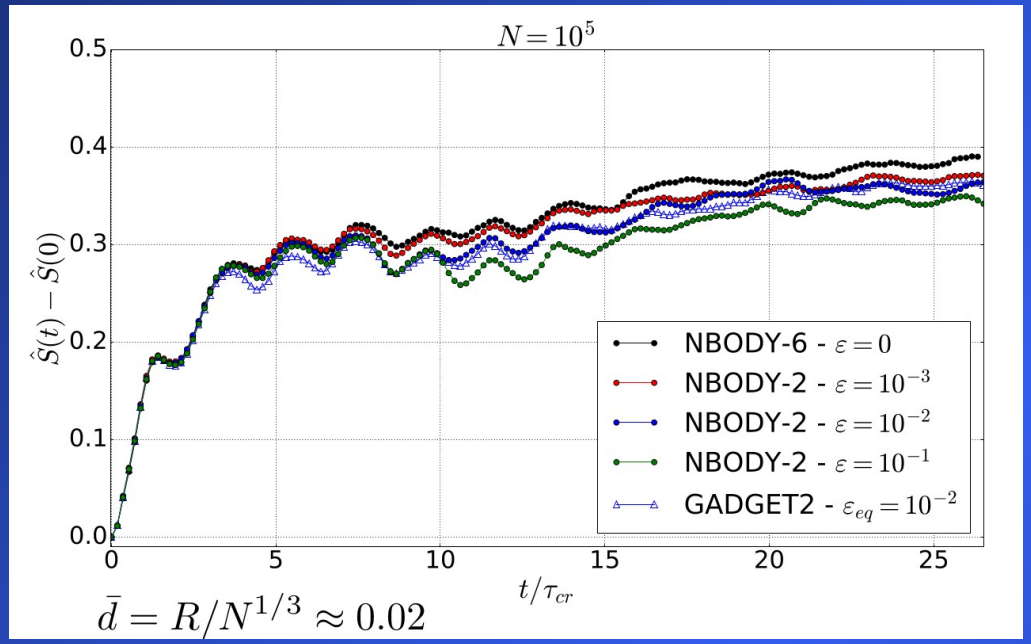
summary:

simulations in integrable and non-integrable potentials:

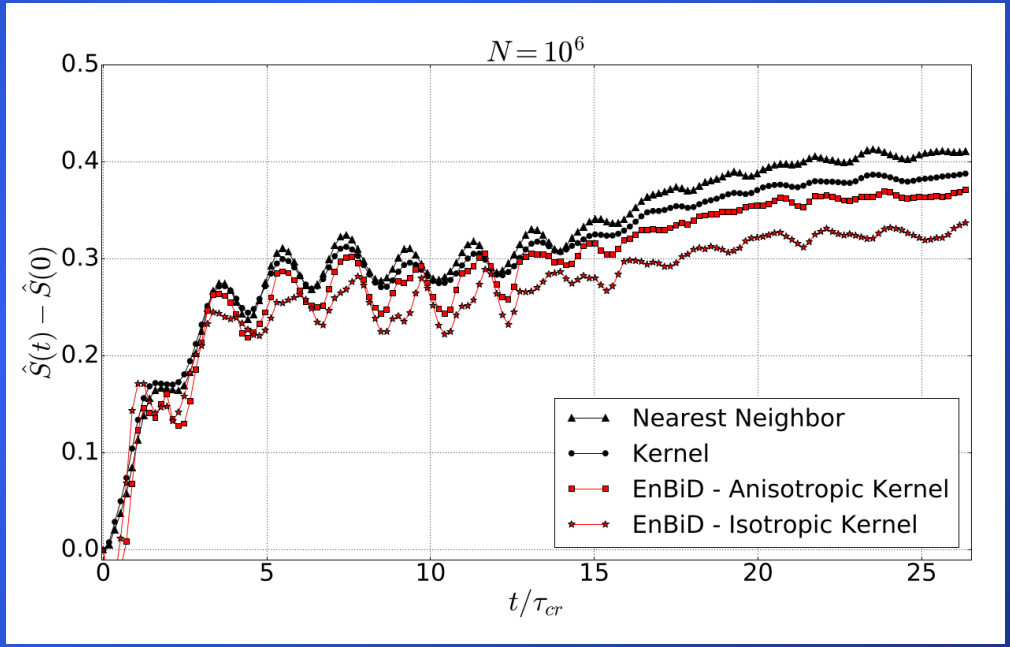
- VR is not explained by the Vlasov-Poisson equation
- long-term N-body evolution well described by the Fokker-Planck model
- entropy measures phase-mixing, either mechanical or chaotic
- there is a clear N-dependency for entropy production
- VR is better explained by this N-dependence than as an effect of a time-changing potential

IoA, beginning of the 90’s

N-body codes



entropy estimators



why collisionless?

- two main time-scales:

  $t_{cr} \sim R/v$

  $t_{col} \sim (N / \ln N)\, t_{cr}$

- violent relaxation:

  $t_{vr} \sim t_{cr}$

- globular clusters:

  $N \sim 10^6$ stars, $t_{col} \sim 10^9$ yr

  → collisional

  (wrt Hubble time)

- elliptical galaxies:

  $N \sim 10^{11}$ stars, $t_{col} \sim 10^{17}$ yr

  → collisionless