Vlasov Equation, Violent Relaxation and some comments on the nature of Entropy

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Boltzmann’s Kinetic theory

- Boltzmann equation:
  \( \frac{df}{dt} = \Gamma[f] \)

  \( f(x,v,t) \): phase-space density for a typical/test particle

- \( \frac{df}{dt} \): time reversible

- \( \Gamma[f] \): collisional/relaxation term – irreversible
  introduces the arrow of time

- self - gravitational N-body systems → ‘collisionless’ (\( t_{\text{coll}} > t_{\text{cr}} \)):
  \( \Gamma[f] = 0 \)

- can be described by the Vlasov-Poisson equation:
  \( \frac{df}{dt} = 0 \)  → time – reversible!

- violent relaxation: clearly non-reversible & assumed described by Vlasov-Poisson - “Fundamental paradox of stellar dynamics” (Ogorodnikov 1965)
testing Vlasov

• transport equation:
  \[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \phi}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \Gamma[f] \]

• Shannon entropy:
  \[ S = -\int f \ln f d\mathbf{r}d\mathbf{v} \]

• entropy evolution:
  \[ \frac{dS}{dt} = -\int (1 + \ln f) \Gamma[f] d\mathbf{r}d\mathbf{v} \]

• collisionless system:
  \[ \Gamma[f] = 0 \implies \frac{dS}{dt} = 0 \]

• let’s test Vlasov by looking the entropy!

• N-body simulations with NBODY6, NBODY2, GADGET-2
  \( N = 10^4, 10^5, 10^6 \)

• ICs: uniform sphere, \( Q_0 = T/|W| = 0.5 \)

• entropy estimation:
  \[ \hat{S} = -\frac{1}{N} \sum_i \ln \hat{f}_i \]

\( f \) is estimated in the phase-space by 3 methods: adaptive kernel, k-NN and EnBiD
(Sharma & Steinmetz 2006)
testing Vlasov

- overall increase of entropy
- entropy oscillations during violent relaxation:
  H theorem: monotonic increase of S if the system is described by the Boltzmann equation → violation!
- oscillations: result of conversion of kinetic to potential energy and vice-versa
- Prigogine & Severne (1966), Jaynes (1971): behaviour expected in systems with appreciable potential energy – they present correlations, memory

Vlasov doesn’t work for VR!
long-term evolution

- (Orbit-averaged) Fokker-Planck: $\frac{df}{dt} = \Gamma_{FP}[f]$

- Weak encounters, static potential, $f = f(E)$

$\Gamma_{FP}[f(E)] \propto \ln \Lambda$... Theory: $\ln \Lambda \approx \ln(0.4 \cdot N)$ (free parameter)

Integration with AGAMA (Vasiliev 2019)
two phenomena, two time-scales:

- violent relaxation
  fast - $t_{cr}$
inconsistent with Vlasov

- collisional relaxation
  slow – $t_{coll}$
consistent with Fokker-Planck

What causes irreversibility?

- chaotic mixing?
  Merritt & Valluri 1996, Prigogine 1999
- time-varying potential?
  Lynden-Bell 1967
- large $N$ and phase mixing?
  Lebowitz 1993; 1999

tool: orbit integration (with AGAMA) in fixed external potential
harmonic potential

- potential:
  \[ \Phi(r) = \frac{1}{2} \Omega^2 r^2 \]

- all particles have the same frequency
- macroscopic reversibility
- IC: sample of a Plummer model
- entropy changes due to “mechanical” phase-mixing
Plummer potential

- potential:
  \[ \phi(r) = -\frac{GM/a}{\sqrt{1 + (r/a)^2}} \]

- IC: uniform sphere
- IC: self-consistent Plummer sample
  \[ f(E) \propto (-E)^{7/2} \]

- No spurious entropy production
- Agreement with 2nd law; similar to N-body sims.

- Entropy increase due to “mechanical” phase-mixing (no chaos)
chaotic mixing

- ellipsoidal model
Chaos and N-dependence

- Chaos accelerates entropy production
- N-dependence:

\[
\frac{T_{\Delta S/2}}{\tau_{cr}} \approx 0.1 \times N^{\alpha/6}
\]

\(\alpha \approx 1\) for integrable systems
\(\alpha \lesssim 1\) for non-integrable systems
entropy and violent relaxation

- entropy grows by phase-mixing:
  - mechanical phase-mixing
  - chaotic phase-mixing

- VR: both processes
  - oscillations - mechanical phase mixing
  - overall entropy increase: mechanical+chaotic phase-mixing

- oscillations - possibly associated to the long-range nature of gravitational interactions
  => correlations among the particles
entropy and violent relaxation

• violent relaxation:
• usual interpretation – driven by the time-changing collective gravitational potential
  (King 1962, Hénon 1964, Lynden-Bell 1967)

• entropy production: follows the N-dependence
  => time-changing potential is not the main driver!
  => the relaxation is discreteness driven!

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• entropy increases whenever the phase-space distribution of a system is not in dynamical equilibrium with the potential
entropy and violent relaxation

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summary:
simulations in integrable and non-integrable potentials:

• VR is not explained by the Vlasov-Poisson equation
• long-term N-body evolution well described by the Fokker-Planck model
• entropy measures phase-mixing, either mechanical or chaotic
• there is a clear N-dependency for entropy production
• VR is better explained by this N-dependence than as an effect of a time-changing potential
IoA, beginning of the 90’s
N-body codes

$\bar{d} = R/N^{1/3} \approx 0.02$

entropy estimators
why collisionless?

- two main time-scales:
  \[ t_{cr} \sim \frac{R}{v} \]
  \[ t_{col} \sim \left(\frac{N}{\ln N}\right) t_{cr} \]

- violent relaxation:
  \[ t_{vr} \sim t_{cr} \]

- globular clusters:
  \[ N \sim 10^6 \text{ stars}, \quad t_{col} \sim 10^9 \text{ yr} \]
  \[ \rightarrow \text{collisional} \]
  (wrt Hubble time)

- elliptical galaxies:
  \[ N \sim 10^{11} \text{ stars}, \quad t_{col} \sim 10^{17} \text{ yr} \]
  \[ \rightarrow \text{collisionless} \]