Hybrid numerical-asymptotic methods for high frequency scattering

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Joint work with: Simon Chandler-Wilde, Steve Langdon, Ashley Twigger, Samuel Groth (University of Reading), Markus Melenk (TU Vienna)
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Oxford Mathematical Institute
25th March 2014

Solving scattering problems

Model problem - Helmholtz equation:

$$(\Delta + k^2)u = 0, \qquad k = \text{wavenumber} = \frac{\text{frequency}}{\text{wavespeed}} = \frac{2\pi}{\text{wavelength}}$$

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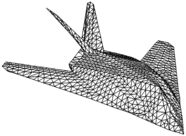
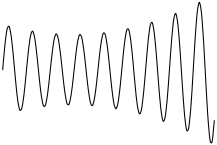
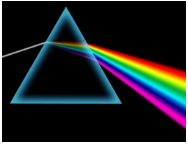
increasing frequency →

Numerical methods (FEM, BEM,...)

Asymptotic methods (Geometrical Optics, ray tracing, GTD,...)







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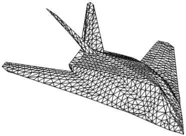

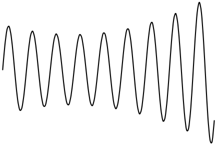

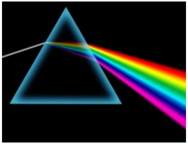

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computationally infeasible at high frequencies

computational cost independent of frequency
accurate only at high frequencies

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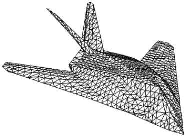

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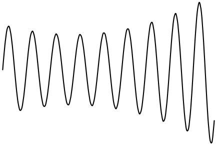
What to do here??



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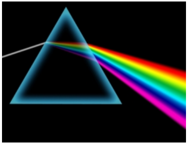

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Hybrid Numerical-Asymptotic (HNA) methods

Fuse conventional numerical methods with high frequency asymptotics to create algorithms that are controllably accurate and computationally feasible over the whole frequency range.

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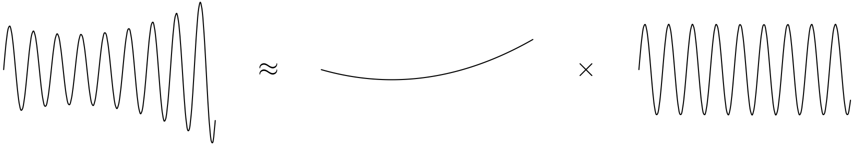
$$v(\mathbf{x}, k) \approx v_0(\mathbf{x}, k) + \sum_{m=1}^{M} v_m(\mathbf{x}, k)\, \mathrm{e}^{\mathrm{i}k\psi_m(\mathbf{x})},$$

- v_0 is some **known** leading order **asymptotic** behaviour
- ψ_m, $m = 1, \ldots, M$ are **specified** phase functions, from **asymptotics**
- v_m, $m = 1, \ldots, M$ are **unknown** amplitude functions, found **numerically**



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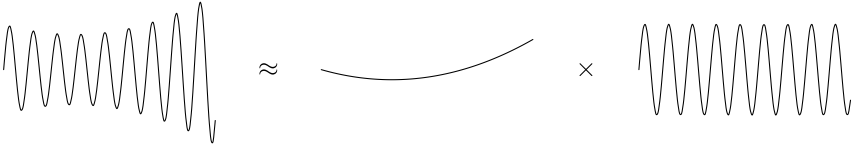
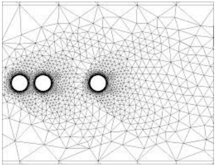


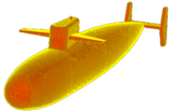
Expectation: If v_0 and ψ_m are chosen appropriately, v_m, $m = 1,\ldots,M$, will be **slowly varying**, and **less expensive** to approximate than v

Why do mathematicians like FEM/BEM?

FEM = Finite Element Method,



BEM = Boundary Element Method ("Method of Moments")



- General
- Systematic
- Flexible
- Controllably accurate
- Established frameworks for error analysis

...

Basics of FEM

Starting point: Partial Differential Equation (PDE) written in "weak form":

Given $l \in V^*$, find $u \in V$ such that $\quad a(u, v) = l(v), \quad \forall v \in V$

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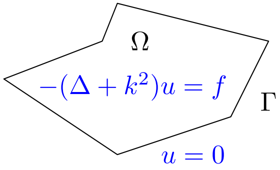
FEM example:



$$-(\Delta + k^2)u = f \text{ in } \Omega \quad \text{with } u = 0 \text{ on } \Gamma$$

$$a(u,v) := \int_D (\nabla u \cdot \overline{\nabla v} - k^2 u \overline{v})\, \mathrm{d}\mathbf{x}, \qquad V = H_0^1(D)$$

$$l(v) = \int_D f \overline{v}\, \mathrm{d}\mathbf{x}, \qquad V^* = H^{-1}(D)$$

Basics of BEM

Starting point: Boundary Integral Equation (BIE) written in "weak form":

Given $l \in V^*$, find $\phi \in V$ such that $\quad a(\phi, \psi) = l(\psi), \quad \forall \psi \in V$

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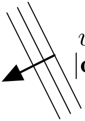
BEM example:

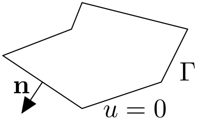
$$(\Delta + k^2)u = 0$$

D



$u^i = \mathrm{e}^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}$

$|\mathbf{d}| = 1$



$u^s := u - u^i$ outgoing at infinity

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad \mathbf{x} \in D$$

$$S\frac{\partial u}{\partial \mathbf{n}} = u^i \text{ on } \Gamma, \qquad S\phi(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),\ \mathbf{x} \in \Gamma$$

$$a(\phi,\psi) := \langle S\phi, \psi\rangle = \int_\Gamma (S\phi)(\mathbf{y})\overline{\psi(\mathbf{y})}\,\mathrm{d}s(\mathbf{y}), \quad V = H^{-1/2}(\Gamma)$$

$$l(\psi) = \int_\Gamma u^i(\mathbf{y})\overline{\psi(\mathbf{y})}\,\mathrm{d}s(\mathbf{y}), \quad V^* = H^{1/2}(\Gamma)$$

Basics of FEM/BEM

"Continuous" problem:

$$\boxed{\text{Find } u \in V \text{ such that} \quad a(u,v) = l(v), \qquad \forall v \in V}$$

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"Continuous" problem:

Find $u \in V$ such that $\quad a(u,v) = l(v), \qquad \forall v \in V$

To approximate this numerically, choose a finite dimensional subspace $V_N \subset V$ and consider the "discrete" problem:

Find $u^N \in V_N$ such that $\quad a(u^N, v^N) = l(v^N), \qquad \forall v^N \in V_N$

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Let $\{\phi_j\}_{j=1}^N$ be a basis for V_N. Write $u^N = \sum_{j=1}^N u_j \phi_j$, then

$$A\mathbf{u} = \mathbf{l}, \qquad A_{ij} = a(\phi_j, \phi_i),\ \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},\ \mathbf{l} = \begin{pmatrix} l(\phi_1) \\ \vdots \\ l(\phi_N) \end{pmatrix}$$

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Well-posedness and quasi-optimality

If $a(\cdot, \cdot)$ is "nice" (continuous and coercive) then the continuous and discrete problems both have unique solutions satisfying

$$\left\| u - u^N \right\|_V \leq C \min_{v^N \in V_N} \left\| u - v^N \right\|_V \quad \leftarrow \text{Best approx. error in } V_N$$

How to choose V_N for wave problems?

$$\left\|u - u^N\right\|_V \leq C \min_{v^N \in V_N} \left\|u - v^N\right\|_V$$

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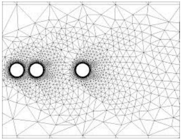
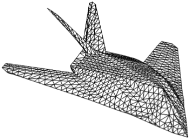
Conventional choice:

$V_N = \{$piecewise polynomials on a triangulation of Ω (or Γ)$\}$





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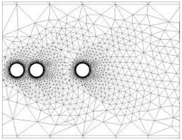

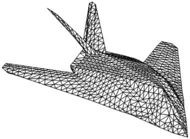

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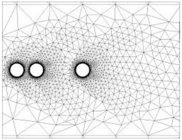

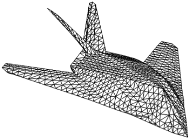

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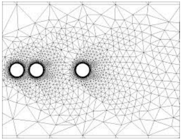

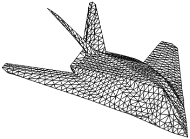

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Attraction: if chosen correctly, **oscillatory functions should approximate the solution more efficiently** (i.e. with smaller N) than piecewise polynomials alone

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Choose oscillations based on high frequency asymptotics of solution

- FEM e.g. Giladi and Keller (2001).
- BEM e.g. Chandler-Wilde, Langdon, Hewett, Groth, Gibbs, Melenk, Graham, Dominguez, Smyshlyaev, Bruno, Huybrechs, Vandewalle, Ganesh, Hawkins...

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Many mathematical challenges:

- high frequency behaviour of solution
- estimation of $\min_{v^N \in V_N} \left\| u - v^N \right\|_V$
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- evaluation of $A_{ij} = a(\phi_j, \phi_i)$ (highly oscillatory integrals)

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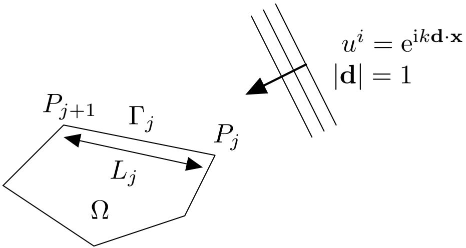
Current work: generalise to **3D, penetrable and nonconvex scatterers**.

High frequency asymptotics - convex polygons

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\, \mathrm{d}s(\mathbf{y}), \qquad \mathbf{x} \in D$$



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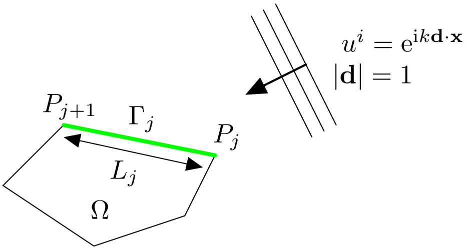

According to Geometrical Optics/Geometrical Theory of Diffraction, on a "lit" side

$$\frac{\partial u}{\partial \mathbf{n}} \sim \underbrace{2\frac{\partial u^i}{\partial \mathbf{n}}}_{\text{incident + reflected}} + \underbrace{A^+\mathrm{e}^{\mathrm{i}ks} + A^-\mathrm{e}^{-\mathrm{i}ks}}_{\text{diffracted}}, \qquad k \to \infty$$

where s is arc length along the side, measured from P_j

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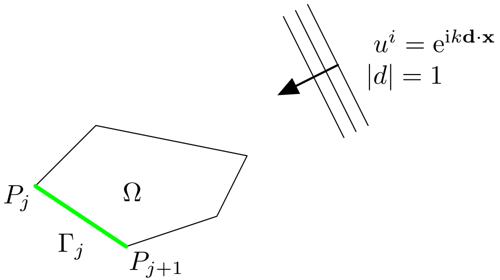

On an "unlit" (or "shadow") side

$$\frac{\partial u}{\partial \mathbf{n}} \sim \underbrace{A^+ \mathrm{e}^{\mathrm{i}ks} + A^- \mathrm{e}^{-\mathrm{i}ks}}_{\text{diffracted}}, \qquad k \to \infty$$

Regularity results - convex polygons

Theorem (Hewett, Langdon, Melenk (2013))

Let Ω *be a convex polygon. Then on any side* Γ_j

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \Psi(\mathbf{x}(s)) + v_j^+(s)\mathrm{e}^{\mathrm{i}ks} + v_j^-(L_j - s)\mathrm{e}^{-\mathrm{i}ks}, \qquad 0 < s < L_j,$$

where

(i) $\Psi := 2\frac{\partial u^i}{\partial \mathbf{n}}$ *if* Γ_j *is lit and* $\Psi := 0$ *otherwise,*

(ii) $v_j^{\pm}(s)$ *are* **analytic** *in* $\mathrm{Re}\,[s] > 0$, *with*

$$|v_j^+(s)| \leq Ck^2 \begin{cases} |ks|^{\pi/\Omega_j - 1}, & 0 < |s| \leq 1/k, \\ |ks|^{-1/2}, & |s| > 1/k, \end{cases}$$

where Ω_j *is the exterior angle at the vertex* P_j.

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$$|v_j^-(s)| \leq Ck^2 \begin{cases} |ks|^{\pi/\Omega_{j+1}-1}, & 0 < |s| \leq 1/k, \\ |ks|^{-1/2}, & |s| > 1/k, \end{cases}$$

where Ω_{j+1} *is the exterior angle at the vertex* P_{j+1}.

Regularity results - convex polygons

Theorem (Hewett, Langdon, Melenk (2013))

Let Ω *be a convex polygon. Then on any side* Γ_j

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \Psi(\mathbf{x}(s)) + v_j^+(s)\mathrm{e}^{\mathrm{i}ks} + v_j^-(L_j - s)\mathrm{e}^{-\mathrm{i}ks}, \qquad 0 < s < L_j,$$

where

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where Ω_{j+1} *is the exterior angle at the vertex* P_{j+1}*.*

To form **HNA approximation space** V_N, replace $v_j^{\pm}$ by piecewise polynomials

Best approximation error - convex polygons

Theorem (Hewett, Langdon, Melenk (2013))

Under appropriate assumptions on the piecewise polynomial approximation, there exist constants $C, \tau > 0$, independent of k, such that

$$\min_{v^N \in V_N} \left\| \frac{\partial u}{\partial \mathbf{n}} - v^N \right\|_{L^2(\Gamma)} \leq C k^2 \, \mathrm{e}^{-\tau \sqrt{N}}.$$

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$$\min_{v^N \in V_N} \left\| \frac{\partial u}{\partial \mathbf{n}} - v^N \right\|_{L^2(\Gamma)} \leq Ck^2 \, \mathrm{e}^{-\tau\sqrt{N}}.$$

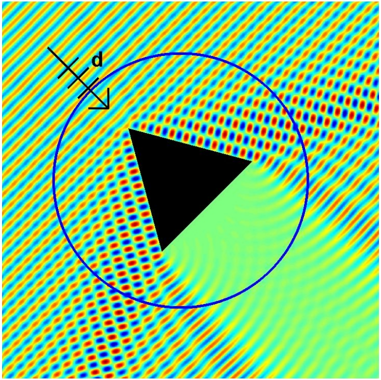
Result

We can provably achieve **any required approximation accuracy** with N growing only like $\log^2 k$ as $k \to \infty$, rather than like k, as for a conventional BEM.

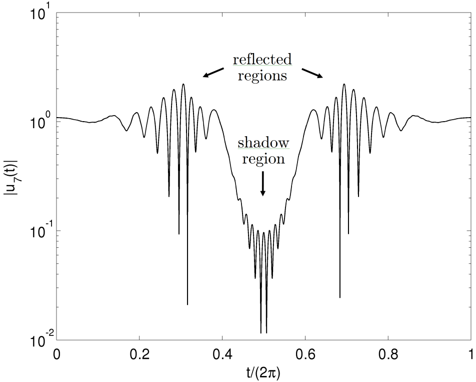
Numerical results - convex polygon

Plot the field arising from the numerical boundary solution:

$$u^N(\mathbf{x}) := u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)^N (\mathbf{y})\, \mathrm{d}s(\mathbf{y}), \qquad \mathbf{x} \in D$$



$k = 10$



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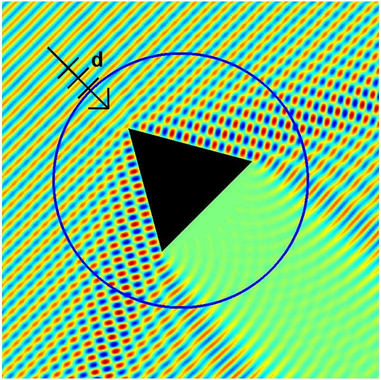

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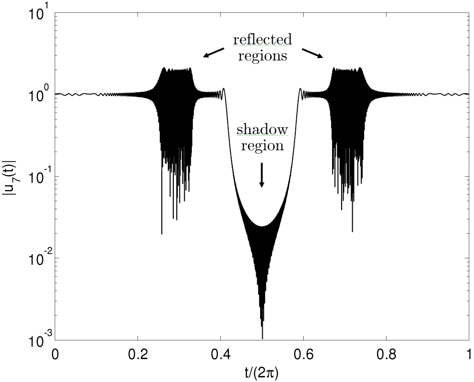
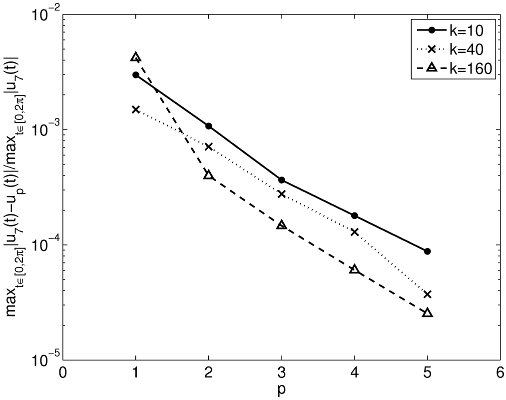


$k = 160$

Numerical results - convergence of u^N

Theorem

$$(\textit{Relative maximum error in } D) \leq Ck^2\, e^{-\tau\sqrt{N}}$$

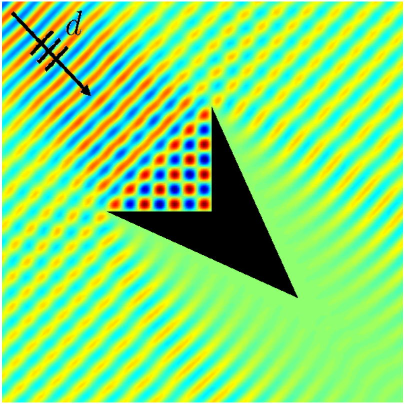


(Here $p \propto \sqrt{N}$ is the maximum polynomial degree used)

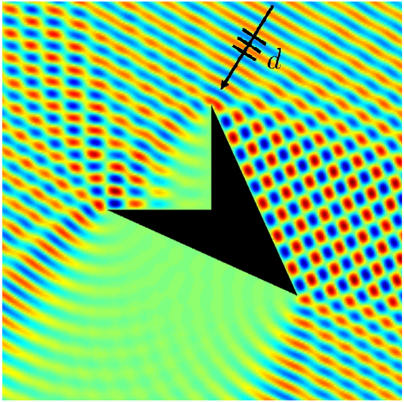
Accuracy actually improves as k gets larger!

Nonconvex polygons

High frequency asymptotic behaviour on Γ is more complicated:



Multiple reflections



Partial illumination

Nonconvex polygons

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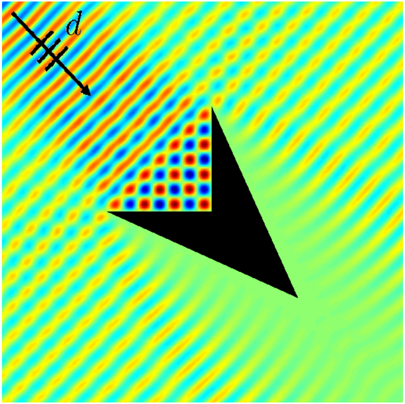

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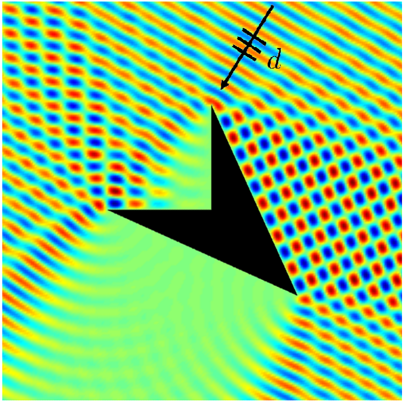


Partial illumination

Theorem (Chandler-Wilde, Hewett, Langdon, Twigger (2012))

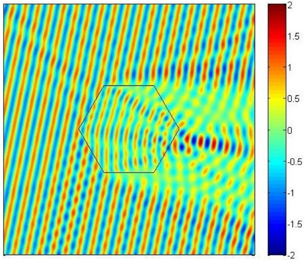
For a class of nonconvex polygons we can achieve any required accuracy of approximation with N growing only like $\log^2 k$ as $k \to \infty$.

Transmission problems - penetrable scatterers

Joint work with S. Groth and S. Langdon
(EPSRC CASE award with Met Office, Industrial supervisor A. Baran)

Motivating application: scattering by ice crystals in cirrus clouds

- First steps: 2D acoustic case, convex polygon
- High frequency asymptotic solution involves infinitely many refractions/reflections/diffractions
- **Infinitely many phases to consider, even for a convex scatterer**



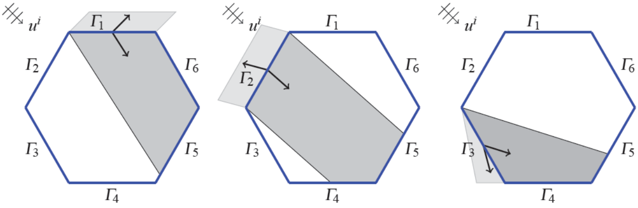
$$u_1(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \left(u_1(\mathbf{y}) \frac{\partial \Phi_1(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} - \Phi_1(\mathbf{x},\mathbf{y}) \frac{\partial u_1(\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Omega_1,$$

$$u_2(\mathbf{x}) = \int_\Gamma \left(\Phi_2(\mathbf{x},\mathbf{y}) \frac{\partial u_2(\mathbf{x})}{\partial \mathbf{n}(\mathbf{y})} - u_2(\mathbf{y}) \frac{\partial \Phi_2(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Omega_2,$$

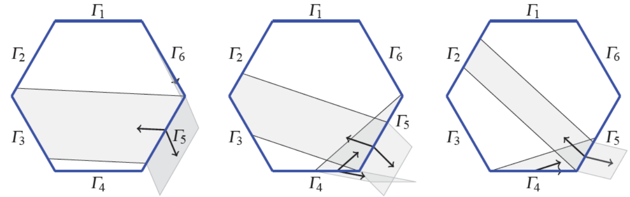
Here Ω_1 is exterior (k_1), Ω_2 is interior (k_2)

HNA approximation space - GO terms

Compute Geometrical Optics (GO) approximation using a beam tracing algorithm:



(a) Primary beams from Γ_1 (b) Primary beams from Γ_2 (c) Primary beams from Γ_3



(d) Secondary beams arising from transmitted beam in (a) (e) Secondary beams arising from transmitted beam in (b) (f) Secondary beams arising from transmitted beam in (c)

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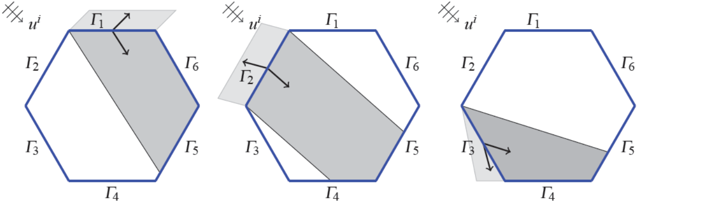

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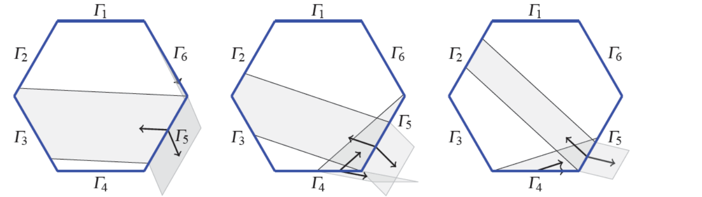
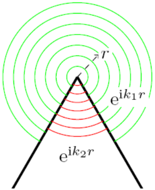


(d) Secondary beams arising from transmitted beam in (a) (e) Secondary beams arising from transmitted beam in (b) (f) Secondary beams arising from transmitted beam in (c)

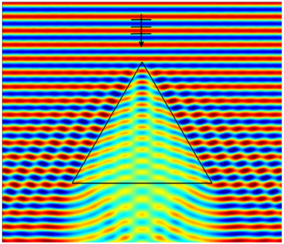
Using this alone in integral representation corresponds to Physical-Geometrical Optics Hybrid (PGOH) method of Bi et al ('11), see also Yang and Liou ('95,'96,'97), Muinonen ('89). **We want to include diffracted field.**

HNA approximation space - diffraction terms

Problem! No closed form solution yet known for canonical diffraction problem (transmission wedge), cf. Rawlins '99



- Use "heuristic" choice of phases for diffracted field
- Need to include oscillations at both interior and exterior wavenumbers
- Compare GO alone with (1) adding diffraction from adjacent corners and (2) adding diffraction from opposite corners too



Compute "numerical best approximation errors" by comparison with a reference solution computed using a standard BEM

(Full HNA BEM currently being implemented)

In our experiments we use **fix** $N = 168$ and vary $k = 5, 10, 20, 40, 80, 160$

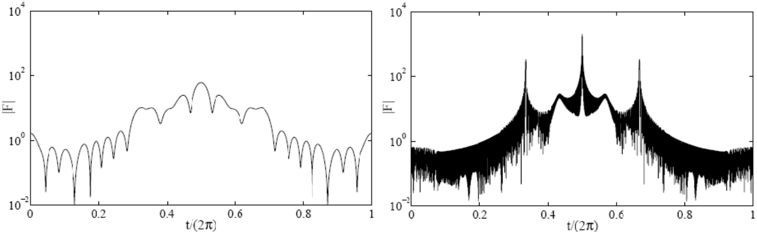
Best approx. errors on the boundary

k_1	ξ	$\frac{\|\|u-u_{go}\|\|}{\|\|u\|\|}$	$\frac{\|\|u-U_1\|\|}{\|\|u\|\|}$	$\frac{\|\|u-U_2\|\|}{\|\|u\|\|}$
5	0.05	1.88×10^{-1}	1.66×10^{-2}	2.57×10^{-3}
10	0.05	1.37×10^{-1}	1.03×10^{-2}	1.35×10^{-3}
20	0.05	1.00×10^{-1}	8.41×10^{-4}	3.72×10^{-4}
40	0.05	7.25×10^{-2}	2.23×10^{-4}	2.20×10^{-4}
80	0.05	5.19×10^{-2}	2.58×10^{-4}	2.58×10^{-4}
160	0.05	3.69×10^{-2}	2.31×10^{-4}	2.31×10^{-4}
5	0.0125	2.48×10^{-1}	4.05×10^{-2}	8.02×10^{-3}
10	0.0125	1.84×10^{-1}	7.88×10^{-2}	9.46×10^{-3}
20	0.0125	1.28×10^{-1}	4.53×10^{-2}	9.42×10^{-3}
40	0.0125	9.13×10^{-2}	1.05×10^{-2}	2.66×10^{-3}
80	0.0125	6.69×10^{-2}	1.87×10^{-3}	1.79×10^{-3}
160	0.0125	4.84×10^{-2}	7.52×10^{-4}	7.52×10^{-4}
5	0	2.57×10^{-1}	5.30×10^{-2}	1.16×10^{-2}
10	0	2.15×10^{-1}	1.43×10^{-1}	1.95×10^{-2}
20	0	1.79×10^{-1}	1.48×10^{-1}	2.82×10^{-2}
40	0	1.50×10^{-1}	1.34×10^{-1}	3.07×10^{-2}
80	0	1.25×10^{-1}	1.17×10^{-1}	3.17×10^{-2}
160	0	1.04×10^{-1}	1.00×10^{-1}	2.81×10^{-2}

Refractive index is $k_2/k_1 = 1.31 + \xi\mathrm{i}$

Smaller ξ (**less absorption**) $\Rightarrow$ need to include **more diffracted terms**
Smaller k (**lower frequency**) $\Rightarrow$ need to include **more diffracted terms**

Best approx. errors: far-field pattern



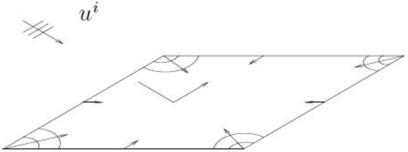
(a) $k_1 = 5$ (b) $k_1 = 160$

k_1	$\frac{\|\|F-F_{go}\|\|}{\|\|F\|\|}$	$\frac{\|\|F-F_1\|\|}{\|\|F\|\|}$	$\frac{\|\|F-F_2\|\|}{\|\|F\|\|}$
5	5.93×10^{-2}	2.72×10^{-3}	4.52×10^{-5}
10	3.67×10^{-2}	8.98×10^{-3}	9.08×10^{-4}
20	2.54×10^{-2}	7.16×10^{-4}	2.74×10^{-4}
40	1.85×10^{-2}	1.17×10^{-4}	1.14×10^{-4}
80	1.31×10^{-2}	1.04×10^{-4}	1.04×10^{-4}
160	9.35×10^{-3}	1.04×10^{-4}	1.04×10^{-4}

Refractive index is $k_2/k_1 = 1.31 + 0.05\mathrm{i}$

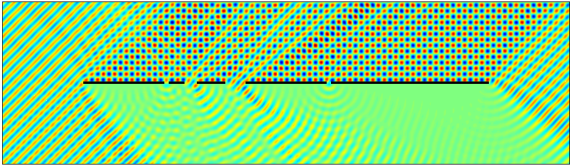
3D problems

Scattering by a planar screen in 3D



Complexity of high frequency asymptotics similar to that of the 2D transmission problem

- Numerical best approximation results are promising
- Currently implementing a BEM (with J. Hargreaves, Salford)
- Analysis would have to be in $\tilde{H}^{-1/2}(\Gamma)$. Already have:
  - full NA for 2D problem of multiple collinear screens (with S. Langdon and S. Chandler-Wilde)
  - k-explicit continuity and coercivity results for 2D and 3D case (with S. Chandler-Wilde)



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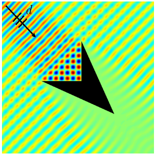

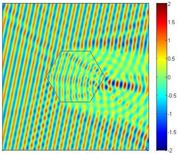

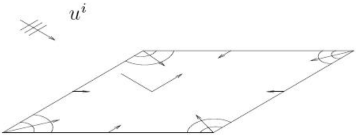

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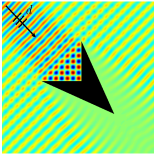

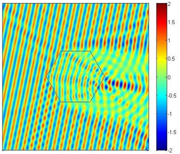

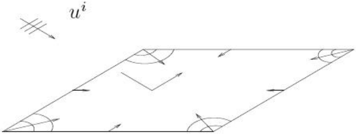

- Possible approach for attacking "real-world" problems: try a combination of conventional and HNA methods (Gibbs, Langdon, Chandler-Wilde)

References

- D. P. Hewett, S. Langdon, J. M. Melenk, *A high frequency hp boundary element method for scattering by convex polygons*, SIAM J. Num. Anal., 51(1), 2013
- S. N. Chandler-Wilde, D. P. Hewett, S. Langdon, A. Twigger, *A high frequency boundary element method for scattering by a class of nonconvex obstacles*, to appear in Numer. Math. 2014
- S. P. Groth, D. P. Hewett, S. Langdon, *Hybrid numerical-asymptotic approximation for high frequency scattering by penetrable convex polygons*, to appear in IMA J. Appl. Math., 2014
- D. P. Hewett, S. Langdon, S. N. Chandler-Wilde, *A frequency-independent boundary element method for scattering by two-dimensional screens and apertures*, under review

Preprints available at `www.maths.ox.ac.uk/~hewett`

For a more general review:

Chandler-Wilde, Graham, Langdon and Spence, *Numerical-asymptotic boundary integral methods in high frequency acoustic scattering*, Acta Numerica (2012).