Shifted Laplace and related preconditioning for the Helmholtz equation

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Outline of talk:

- seismic imaging (Schlumberger Gould Research)
- high-frequency Helmholtz, variable wave speed
- Conventional FE discretization, efficient solvers?
- Iterative method: GMRES convergence rates?
- preconditioners based on absorption
- brief summary of mathematical results
- numerical experiments

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Chandler-Wilde, IGG, Langdon, Spence Acta Numerica 2012: Numerical-asymptotic boundary integral methods in high-frequency scattering

See also Dave Hewett's talk

Survey configuration

Marine seismic

Seismic inversion

Inverse problem: reconstruct material properties of subsurface (characterised by wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \;=\; f \quad \text{or its elastic variant}$$

Frequency domain:

$$-\Delta u - \left(\frac{\omega}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.

Seismic inversion

Inverse problem: reconstruct material properties of subsurface (wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{\partial^2 u}{\partial t^2} = f \quad \text{or its elastic variant}$$

Frequency domain:

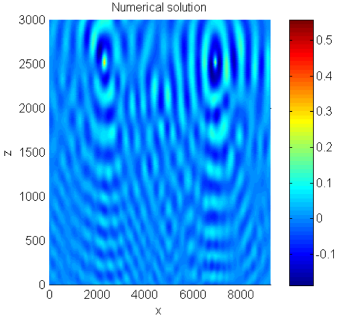
$$-\Delta u - \left(\frac{\omega L}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

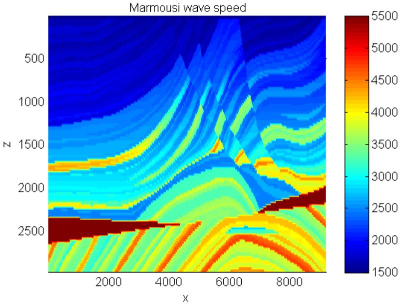
solve for u with approximate c.

Large domain of characteristic length L.

effectively high frequency

Marmousi Model Problem



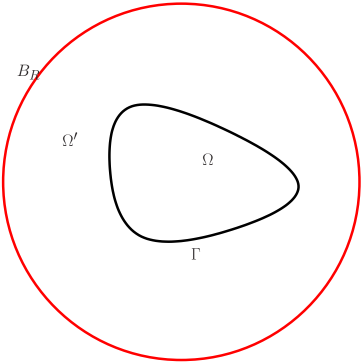


- Time domain: explicit finite difference methods (slow)
- Frequency domain: large linear systems for each ω
- Solver of choice (2007) based on principle of limited absorption (Erlangga, Osterlee, Vuik, 2004)...
- This work: develop better solvers, (parallel algorithms?)

Model interior impedance problem

$$
\begin{aligned}
-\Delta u - k^2 u &= f \quad \text{in} \quad \text{bounded domain} \quad \Omega \\
\frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \Gamma := \partial\Omega
\end{aligned}
$$

Results all hold for truncated sound-soft scattering problems in Ω' (large R)



Linear algebra problem

- weak form ,

$$a\ (u,v) := \int_\Omega (\nabla u.\nabla \overline{v} - k^2\ u\overline{v}) - \mathrm{i}k \int_\Gamma u\overline{v}$$
$$= \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}$$

- finite element discretization

$$\mathbf{A}\ \mathbf{u} := (\mathbf{S} - k^2\ \mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

Often: $h \sim k^{-1}$ **but pollution effect:**
need $h \sim k^{-2}$?? , $h \sim k^{-3/2}$??

Less dispersion: higher order FD or FE methods

Linear algebra problem

- weak form **with absorption** $k^2 \to k^2 + \mathrm{i}\varepsilon$,

$$a_\varepsilon(u,v) := \int_\Omega \left(\nabla u . \nabla \overline{v} - (k^2 + i\varepsilon) u \overline{v}\right) - \mathrm{i}k \int_\Gamma u \overline{v}$$
$$= \int_\Omega f \overline{v} + \int_\Gamma g \overline{v} \quad \text{“Shifted Laplacian''}$$

[Equivalently $k^2 + \mathrm{i}\varepsilon \longleftrightarrow (k + \mathrm{i}\rho)^2$]

- Finite element discretization

$$\mathbf{A}_\varepsilon \mathbf{u} := (\mathbf{S} - (k^2 + i\varepsilon)\mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

A_ε somehow **“better behaved”** than A.

How bad things can be $\varepsilon = 0$

Solving $\mathbf{A}\mathbf{x} = \mathbf{f} = \mathbf{1}$ on unit square $h \sim k^{-3/2}$

Using GMRES

(minimises residual in Krylov space: $\mathrm{span}\{\mathbf{f}, A\mathbf{f}, \ldots A^{k-1}\mathbf{f}\}$)

k	n	# GMRES
25	15876	467
30	22801	633
35	44521	966
40	58081	$>$ 1000

Preconditioning with $\mathrm{A}_\varepsilon^{-1}$

Solve instead:

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

Theorem (with Martin Gander and Euan Spence)
For Lipschitz star-shaped domains:

If ϵ/k is sufficiently small then have GMRES converges independent of k.

Proof: uses (high frequency) analysis of continuous problem.

Warning: not a practical method (yet!)

Shifted Laplacian preconditioner $\varepsilon = k$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**6**
$h \sim k^{-3/2}$	20	**6**
	40	**6**
	80	**6**

Shifted Laplacian preconditioner $\varepsilon = k^{3/2}$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

$h \sim k^{-3/2}$

k	# GMRES
10	**8**
20	**11**
40	**14**
80	**16**

Shifted Laplacian preconditioner $\varepsilon = k^2$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

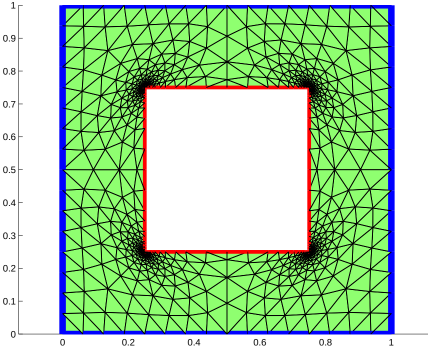
	k	# GMRES
	10	**13**
$h \sim k^{-3/2}$	20	**24**
	40	**48**
	80	**86**

Exterior scattering problem with refinement

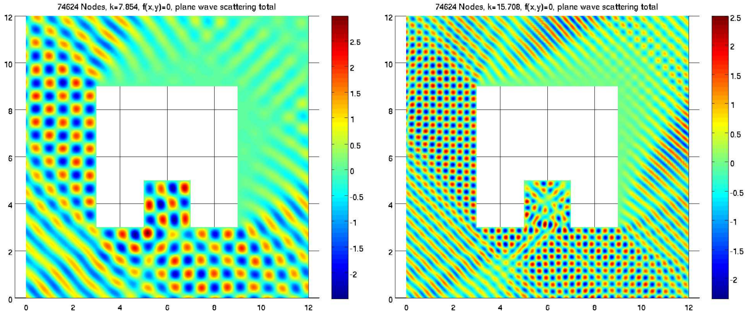
$h \sim k^{-1}$, $\quad$ # **GMRES**

with diagonal scaling

k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
20	**5**	**8**
40	**5**	**11**
80	**5**	**13**
160	**5**	**16**



A trapping domain



k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
$10\pi/8$	**18**	**29**
$20\pi/8$	**19**	**41**
$40\pi/8$	**21**	**60**
$80\pi/8$	**22**	**89**

Betcke, Chandler-Wilde, IGG, Langdon, Lindner, 2010

Part II: Domain decomposition for $\mathbf{A}_\varepsilon^{-1}$

Approximate by solves with A_ε in subspaces:

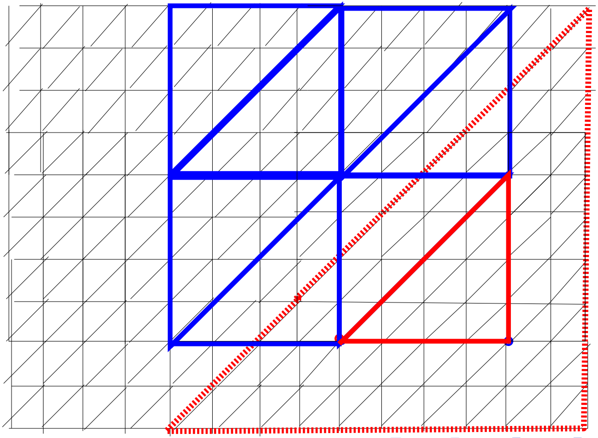
subdomains
coarse grid

Questions: Convergence, Scalability?

Classical additive Schwarz

To solve a problem on a fine grid FE space $\mathcal{S}_h$

- **Coarse space** $\mathcal{S}_H$ (here linear FE) **on a coarse grid**
- **Subdomain spaces** $\mathcal{S}_i$ **on subdomains** Ω_i, overlap δ



Classical additive Schwarz p/c for matrix $\mathbf{C}$

Approximation of C^{-1}:

$$\sum_i \mathbf{R}_i^T \mathbf{C}_i^{-1} \mathbf{R}_i \; + \; \mathbf{R}_H^T \mathbf{C}_H^{-1} \mathbf{R}_H$$

$\mathbf{R}_i$ = restriction to $\mathcal{S}_i$, $\quad$ $\mathbf{R}_H$ = restriction to $\mathcal{S}_H$
$\mathbf{C}_i = \mathbf{R}_i \mathbf{C} \mathbf{R}_i^T$ $\quad$ $\mathbf{C}_H = \mathbf{R}_H \mathbf{C} \mathbf{R}_H^T$

Apply to A_ε to get $\mathrm{B}_\varepsilon^{-1}$

Convergence results (work in progress)

Theorem IGG, E. Spence, E. Vainikko, 2014

Consider solving

$$B_\varepsilon^{-1} A_\varepsilon \mathbf{x} = B_\varepsilon^{-1} \mathbf{f}$$

$n =$ number of GMRES iterates to achieve fixed accuracy

$$n \sim \left(\frac{k^2}{\varepsilon}\right)^4$$

provided $\quad kH \leq c \left(\frac{\varepsilon}{k^2}\right)^{3/2}$

So $\varepsilon \sim k^2 \implies$ robust method no pollution in coarse grid.

Actually we want to solve

$$B_\varepsilon^{-1} A \mathbf{x} = B_\varepsilon^{-1} \mathbf{f} \qquad \varepsilon \sim k?$$

Numerical experiments: unit square, impedance BC

B_ε^{-1} as preconditioner for A_ε $\quad \varepsilon = k^2$

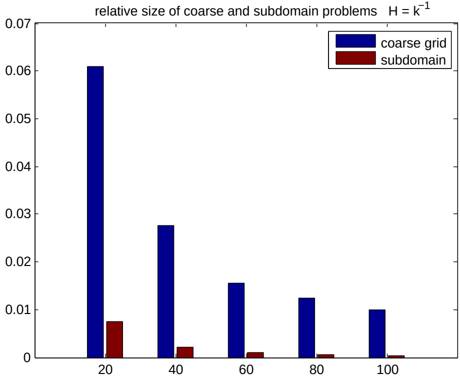
$h \sim k^{-3/2}$,

$kH \sim 1$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.07$

k	#GMRES
20	8
40	8
60	8
80	8
100	8



B_ε^{-1} as preconditioner for A_ε $\quad \varepsilon = k^2$

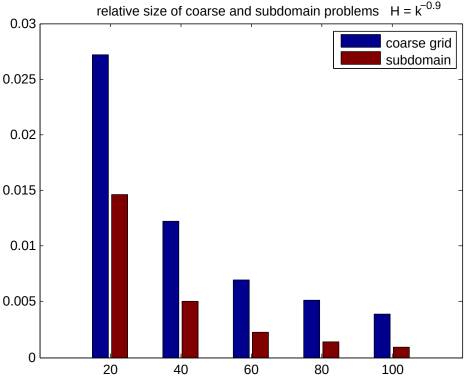
$h \sim k^{-3/2}$,

$kH \sim k^{0.1}$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.03$

k	#GMRES
20	9
40	10
60	10
80	10
100	10



B_ε^{-1} as preconditioner for A_ε $\quad \varepsilon = k^2$

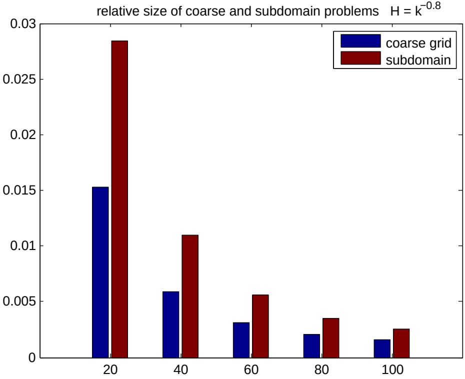
$h \sim k^{-3/2}$,

$kH \sim k^{0.2}$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.03$

k	**#GMRES**
20	**10**
40	**10**
60	**11**
80	**11**
100	**11**



"Aggressive coarseing"

$\mathbf{B}_\varepsilon^{-1}$ as preconditioner for $\mathbf{A}_\varepsilon$ and $\mathbf{A}$ $\quad \varepsilon = k$

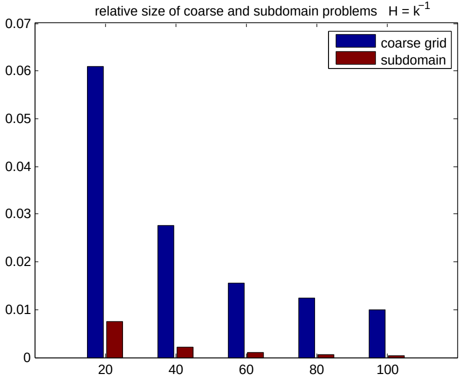
$h \sim k^{-3/2}$,

$kH \sim 1$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.07$

k	for $\mathbf{A}_\varepsilon$	for $\mathbf{A}$
20	**11**	**12**
40	**14**	**15**
60	**19**	**20**
80	**24**	**26**
100	**31**	**33**



When $\varepsilon \sim k$

Introduce more "wavelike" components:

Impedance boundary condition on subdomains

Hybrid restricted additive Schwarz

B_ε^{-1} as preconditioner for A $\quad \varepsilon = k$

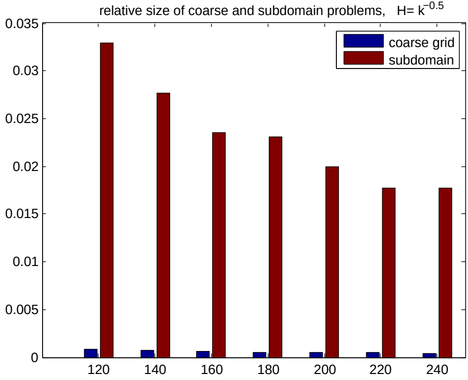
20 grid points per wavelength, Hybrid RAS,
Impedance subdomain problems

$kH \sim k^{0.5}$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.035$

k	**#GMRES**
120	**51**
140	**56**
160	**59**
180	**57**
200	**61**
220	**64**
240	**65**



Complexity (serial): $\sim n \log n, \quad h \gtrsim 10^{-10}$ in 2D

$\mathrm{B}_\varepsilon^{-1}$ as preconditioner for A $\quad \varepsilon = k$

20 grid points per wavelength, Hybrid RAS, $kH \sim k^{0.5}$

k	**GMRES**, impedance	**GMRES**, Dirichlet
120	**51**	**487**
140	**56**	**595**
160	**59**	$>$ **600**
180	**57**	$>$ **600**

Summary

- k and ϵ explicit analysis allows **rigorous explanation** of some empirical observations and formulation of new methods.

- When $\epsilon \sim k$, $\mathbf{A}_\epsilon^{-1}$ is optimal preconditioner for $\mathbf{A}$

- When $\epsilon \sim k^2$, $\mathbf{B}_\varepsilon^{-1}$ is optimal preconditioner for $\mathbf{A}_\varepsilon$

- When preconditioning $\mathbf{A}$ with $\mathbf{B}_\varepsilon^{-1}$, **best choice is $\varepsilon \sim k$**

- Then BC of subdomain problems very important (impedance, PML,...).

- **New analysis** of Domain Decomposition methods.