High frequency scattering by nonconvex polygons

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- Marko Lindner (Hamburg)
- Ivan Graham, Euan Spence, Tatiana Kim (Bath)

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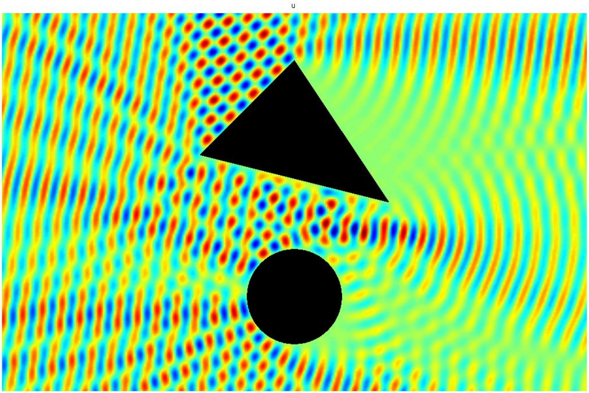
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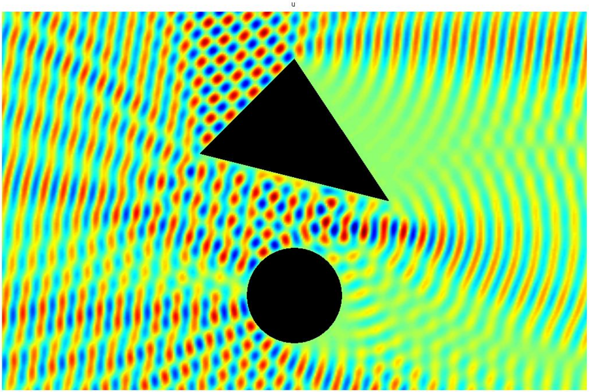
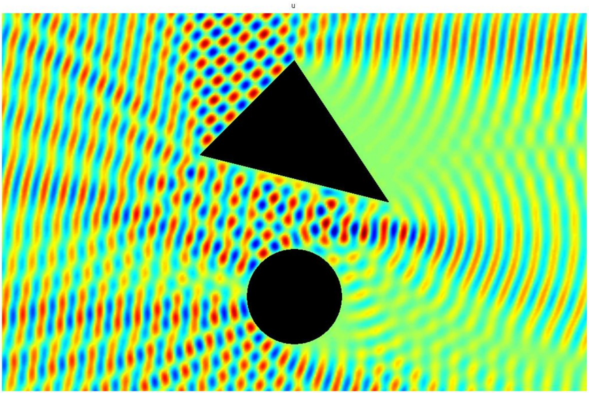
High frequency scattering



$$\Delta u + k^2 u = 0, \quad \text{in exterior domain.}$$



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Difficult when k is large!

Difficulties at high frequencies

- Solutions oscillate in space with wavelength $\lambda = 2\pi/k$.
- Conventional (piecewise polynomial) boundary elements lead to full matrices of dimension at least $N = \mathcal{O}(k^{d-1})$, as $k \to \infty$.
- Domain finite elements lead to sparse matrices but require even larger N.

Improved schemes for high frequencies

- Main idea is to incorporate knowledge of the high frequency asymptotic behaviour into the approximation space.
- High frequency asymptotics have a long history, e.g. Keller et al., Fock, Buslaev, Babich, Ludwig, Grimshaw, Ursell, etc. (1960s); Melrose and Taylor (1980s).
- First combined with numerical scheme by Uncles (1976), in the acoustics literature.
- Similar ideas utilised by: Chandler-Wilde (1988), James (1990), Wang (1991) and Aberegg and Peterson (1995).
- First numerical analysis by Abboud, Nédélec and Zhou (1994), demonstrating $O(k^{2/3})$ degrees of freedom for smooth convex 3D scatterers.
- Since Bruno, Sei and Caponi (2000), many (close to) $O(1)$ schemes developed, for simple geometries; main challenges are **proving** $O(1)$ cost, and extending to more complicated scatterers.

Hybrid methods

Idea: Can we reduce required number of degrees of freedom by including appropriate oscillatory functions in the approximation space?

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Question: How to choose ϕ_m?

General strategy - use asymptotics

Use high frequency asymptotics (GO, GTD) to inform choice of phase functions $\phi_m(x)$

- BEM for rough surface scattering, Bruno et al. (2000, 2002).
- FEM e.g. Giladi and Keller (2001).
- BEM for half-plane with impedance boundary conditions, Chandler-Wilde et al. (2004), Langdon and Chandler-Wilde (2006).
- BEM for smooth obstacles e.g. Bruno et al. (2004), Dominguez et al. (2007), Huybrechs and Vandewalle (2007), Ganesh and Hawkins (2011).
- BEM for non-smooth obstacles e.g. Chandler-Wilde and Langdon (2007), Langdon et al. (2010), Chandler-Wilde et al. (2012), Hewett et al. (2012), Chandler-Wilde et al. (2012).

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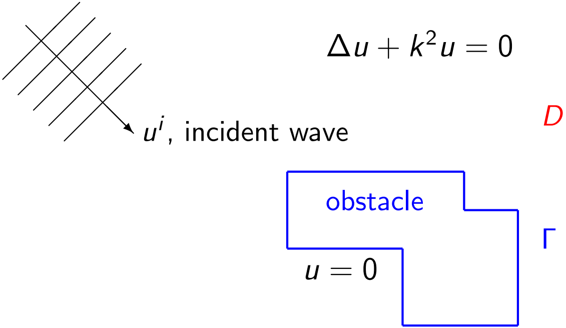
Advantage of BEM

Only need asymptotic behaviour on the **boundary**.

Sound soft scattering - BIE formulation



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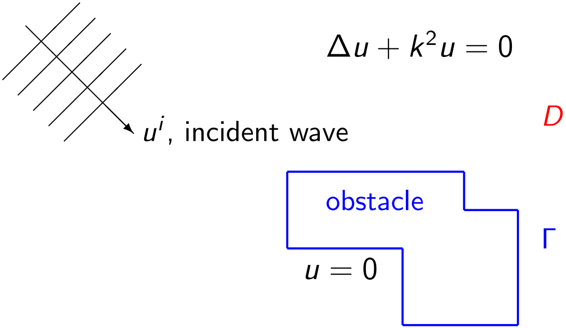
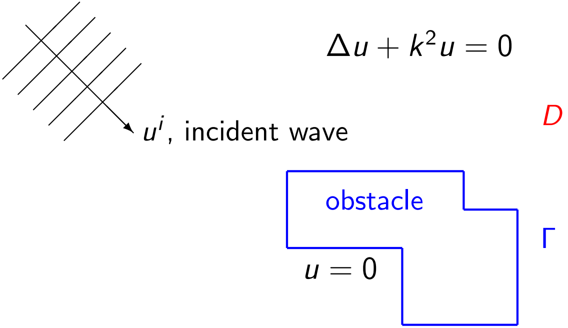


To make use of high frequency asymptotics, we require a **direct** boundary integral equation formulation.



To make use of high frequency asymptotics, we require a **direct** boundary integral equation formulation.
Using Green's representation theorem we reformulate the Helmholtz scattering problem as a BIE:

$$Av = f, \quad \text{where } v := \frac{\partial u}{\partial n}, \quad A : V \mapsto V',$$

and V is some Hilbert space.

Galerkin method

To solve $Av = f$ numerically:

- choose a finite-dimensional approximation space $V_N \subset V$;
- select an approximation to v from V_N using the Galerkin method: find $v_N \in V_N$ such that

$$\langle Av_N, w_N \rangle = \langle f, w_N \rangle, \quad \forall w_N \in V_N.$$

Algorithmic and mathematical challenges

This leads to two significant questions:

Q1

Can we design k-dependent approximation spaces V_N, of dimension N, which keep

$$\inf_{w_N \in V_N} \|v - w_N\| \leq \epsilon_{TOL},$$

with N growing slowly or not at all as $k \to \infty$?

Algorithmic and mathematical challenges

This leads to two significant questions:

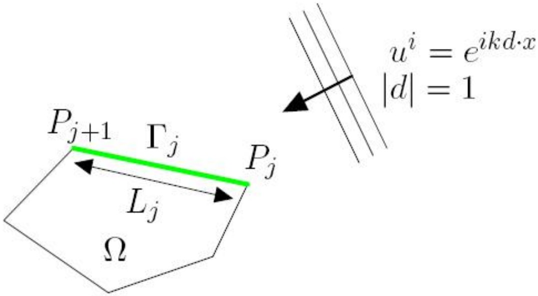
Q2

Does the Galerkin method achieve anything close to the best approximation? Can we show **quasi-optimality**, that

$$\|v - v_N\| \leq C \inf_{w_N \in V_N} \|v - w_N\|,$$

and understand how C depends on k?

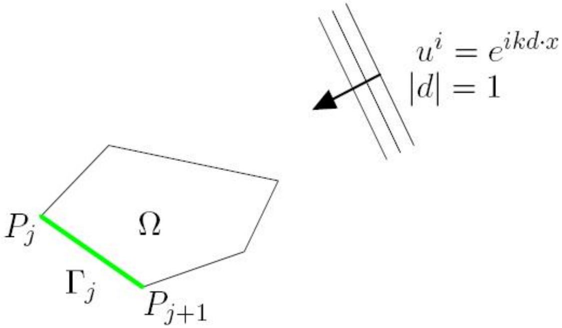
High frequency asymptotics - convex polygons



According to GTD, for a **convex** polygon, the leading-order asymptotic behaviour on a “lit” side is

$$\frac{\partial u}{\partial n} \sim 2\frac{\partial u^i}{\partial n} + A\mathrm{e}^{iks} + B\mathrm{e}^{-iks}, \qquad k \to \infty$$

where s is arc length along the side.



On an “unlit” side it is just

$$\frac{\partial u}{\partial n} \sim A\mathrm{e}^{iks} + B\mathrm{e}^{-iks}, \qquad k \to \infty.$$

Regularity results - convex polygons

Theorem (Hewett, Langdon and Melenk (2012))

Let Ω be a convex polygon. Then on any side Γ_j

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + \mathrm{e}^{iks} v_j^+(s) + \mathrm{e}^{-iks} v_j^-(L_j - s), \qquad x \in \Gamma_j,$$

where

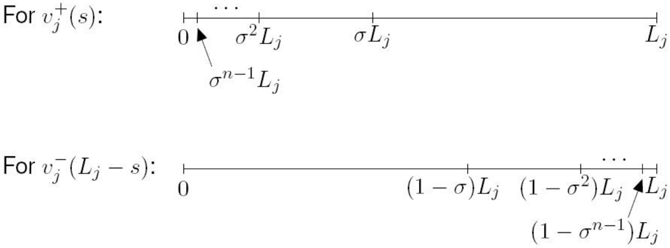
- $\Psi := 2\frac{\partial u^i}{\partial n}$ *if* Γ_j *is lit and* $\Psi := 0$ *otherwise;*
- *The functions* $v_j^{\pm}(s)$ *are analytic in* $\mathrm{Re}[s] > 0$*, with:*

$$|v_j^+(s)| \leq C \begin{cases} k^{3/2} \log^{1/2}(2+k) |ks|^{\pi/\Omega_j - 1}, & 0 < |s| \leq 1/k, \\ k^{3/2} \log^{1/2}(2+k) |ks|^{-1/2}, & |s| > 1/k, \end{cases}$$

where Ω_j *is the exterior angle at the vertex* P_j*.*

hp approximation space V_N

Approximate $v_j^{\pm}$ by piecewise polynomials on overlapping geometric meshes, graded towards the corner singularities



Here σ is a grading parameter - typically $\sigma \approx 0.15$.

For simplicity, we assume the same number of layers n on each mesh, and the same degree p of polynomial approximation on each element.

Best approximation error - convex polygons

Theorem (Hewett, Langdon and Melenk (2012))

If $c, k_0 > 0$ *and* $n \geq cp$, $k \geq k_0$, *then, for some* $C, \tau > 0$,

$$\inf_{w_N \in V_N} \left\| \frac{\partial u}{\partial n} - w_N \right\|_{L^2(\Gamma)} \leq C k^{1/2+\alpha} \log^{1/2}(2+k) \mathrm{e}^{-p\tau},$$

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Total number of degrees of freedom $N = O\left(n(p+1)\right)$

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Total number of degrees of freedom $N = O\left(n(p+1)\right)$

We can achieve any required accuracy with N growing like $\log^2 k$ as $k \to \infty$, rather than like k, as for a standard BEM.

Accuracy of the Galerkin method - convex polygons

Using the "star-combined formulation" (Spence, Chandler-Wilde, Graham and Smyshlyaev (2011)), i.e.

$$A := (x \cdot n)\left(\frac{1}{2}\mathcal{I} + \mathcal{D}'\right) + x \cdot \nabla_\Gamma \mathcal{S} - \mathrm{i}(k|x| + \mathrm{i}/2)\mathcal{S},$$

we can show that the Galerkin solution v_N satisfies, for all $k \geq k_0$,

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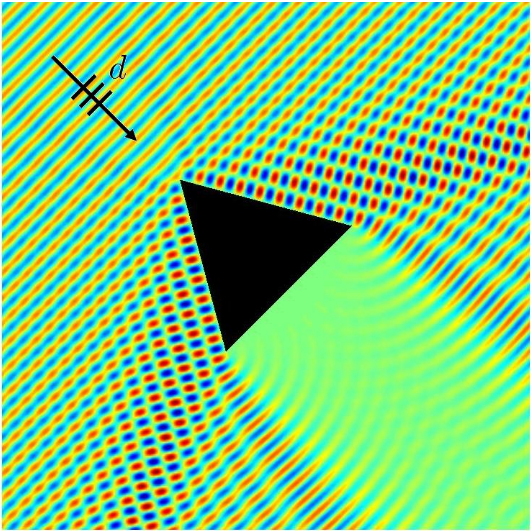
First formulation and algorithm that provably achieves any required accuracy, uniformly in the wavenumber k, with sub-algebraic growth in N ($N \sim \log^2 k$).

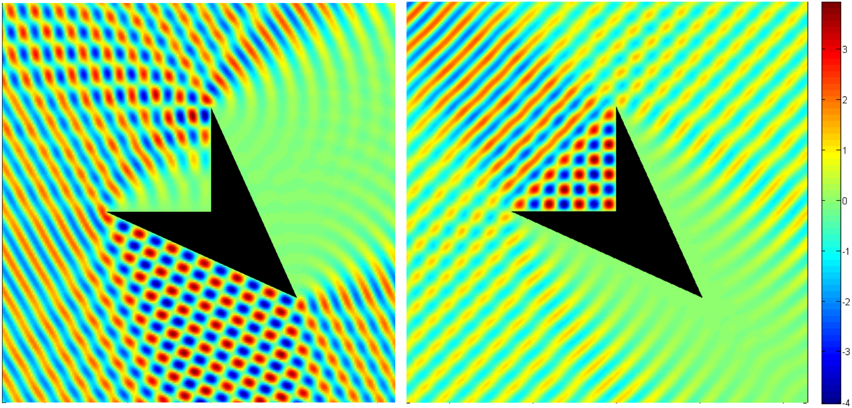
Numerical results - equilateral triangle



Numerical results, fixed $N = 300$, triangle

k	$\frac{N}{L/\lambda}$	$(1/k)\|\partial u/\partial n - v_{300}\|_{L^2(\Gamma)}$	COND	cpt(s)
5	20.00	1.96×10^{-1}	3.50×10^{2}	621
10	10.00	1.48×10^{-1}	2.77×10^{1}	612
20	5.00	1.12×10^{-1}	3.51×10^{1}	600
40	2.50	8.50×10^{-2}	4.60×10^{1}	691
80	1.25	6.44×10^{-2}	6.12×10^{1}	665
160	0.63	4.88×10^{-2}	8.27×10^{1}	648
320	0.31	3.70×10^{-2}	1.12×10^{2}	746
640	0.16	2.80×10^{-2}	1.53×10^{2}	746
1280	0.08	2.16×10^{-2}	2.08×10^{2}	764
2560	0.04	1.65×10^{-2}	2.83×10^{2}	826
5120	0.02	1.26×10^{-2}	3.85×10^{2}	823

The leading-order asymptotic behaviour on Γ is more complicated:



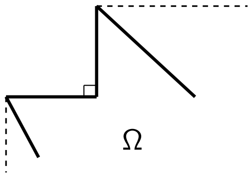
Partial illumination

Re-reflections

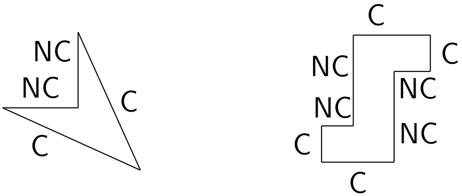
Restrict attention to a particular class of nonconvex polygons

Assume that:

1. Each exterior angle is either a right angle or greater than π.
2. At each right angle, the obstacle lies within the dashed lines:



Examples:



On a “convex” (C) side, $\partial u / \partial n$ behaves as in convex case

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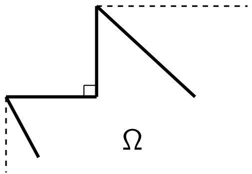

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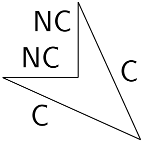
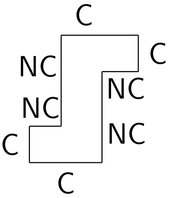
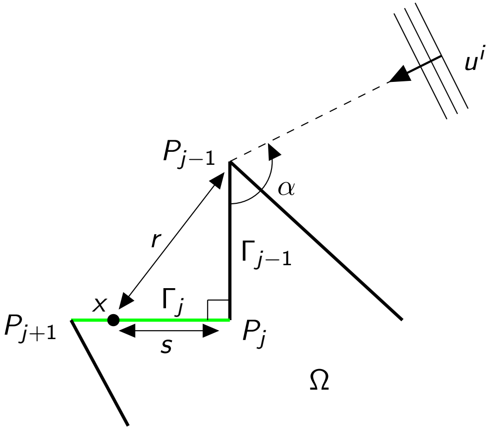




On a "convex" (C) side, $\partial u/\partial n$ behaves as in convex case

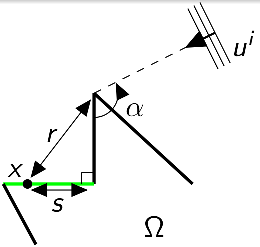
Question: What happens on a "nonconvex" (NC) side?

Geometry near a typical nonconvex side Γ_j



Expect diffraction from P_{j-1} and P_{j+1}, and reflection from Γ_{j-1}

Regularity results on a nonconvex side



For $x \in \Gamma_j$ the following representation holds

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + v_j^+(L_j + s)e^{iks} + v_j^-(L_j - s)e^{-iks} + \tilde{v}_j(s)e^{ikr}$$

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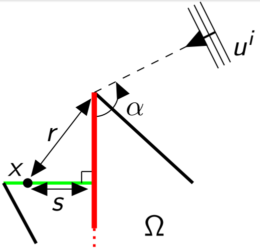

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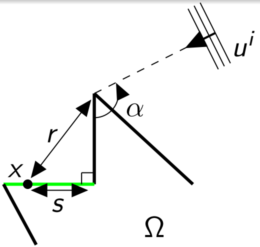
Leading order behaviour

$$\Psi(x) := \begin{cases} 2\frac{\partial u^d}{\partial n}(x), & \frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

where u^d is the known solution of a canonical diffraction problem.

Regularity results on a nonconvex side



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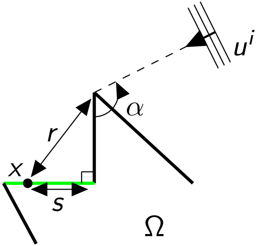
Theorem

The functions $v_j^{\pm}$ have the same properties as those for the convex sides, in particular are analytic in the right hand complex plane.

Regularity results on a nonconvex side



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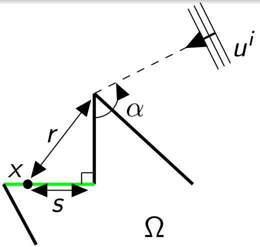
Theorem

The function $\tilde{v}_j$ is analytic in a complex k-independent neighbourhood D_ϵ of the side Γ_j with

$$|\tilde{v}_j(s)| \leq Ck \log^{1/2}(2 + k), \quad s \in D_\epsilon, \quad k \geq k_1.$$

Regularity results on a nonconvex side



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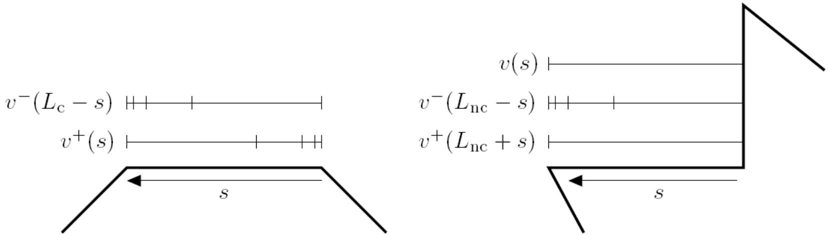
Approximation space:

- Replace v_j^- by a piecewise polynomial supported on a geometric mesh.
- Replace v_j^+ and $\tilde{v}_j$ by polynomials supported on the whole side.

Overlapping meshes

Approximate by piecewise polynomials on overlapping geometric meshes, graded towards the corner singularities



(a) Convex side Γ_c (b) Nonconvex side Γ_{nc}

Best approximation error - nonconvex polygons

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Total number of degrees of freedom $N = O\left(n(p+1)\right)$.

Again, we can achieve any required accuracy with N growing like $\log^2 k$ as $k \to \infty$, rather than like k, as for a standard BEM.

Boundary/domain/FFP estimates - nonconvex polygons

For star-like polygons, using V_N in a Galerkin method with the star-combined formulation we have, for all $k \geq k_0$,

$$\begin{aligned}
\left\| \frac{\partial u}{\partial n} - v_N \right\|_{L^2(\Gamma)} &\leq Ck^{1+\alpha} \log^{1/2}(2+k) \mathrm{e}^{-p\tau}, \\
\frac{\|u - u_N\|_{L^\infty(D)}}{\|u\|_{L^\infty(D)}} &\leq Ck \log(2+k) \mathrm{e}^{-p\tau}, \\
\|F - F_N\|_{L^\infty(\mathbb{S}^1)} &\leq Ck^{1+\alpha} \log^{1/2}(2+k) \mathrm{e}^{-p\tau}.
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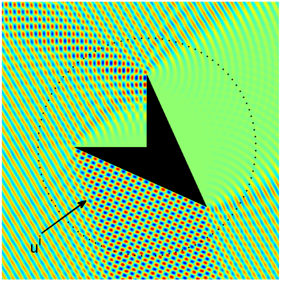
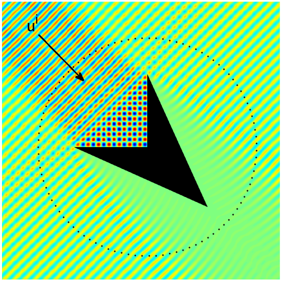
So $N \sim p^2$ growing like $\log^2 k$ provably maintains accuracy!

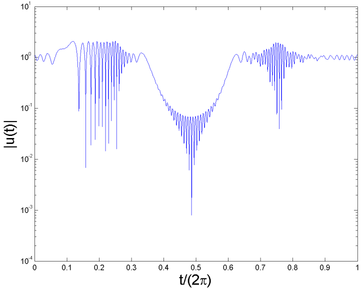


Partial illumination

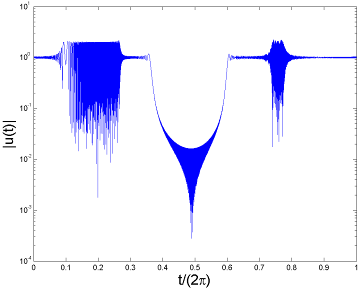


Re-reflections

Total field on circle in domain - partial illumination example

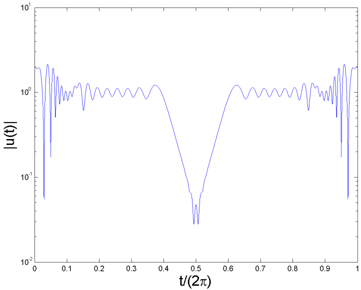


$k = 10$

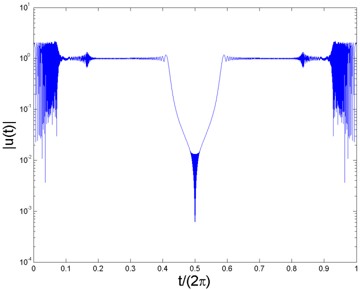


$k = 160$

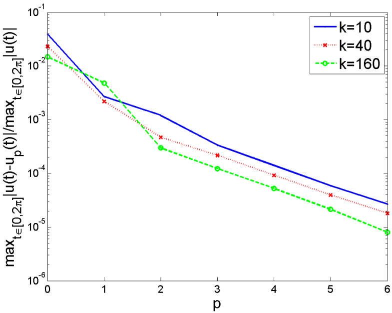
Total field on circle in domain - re-reflections example



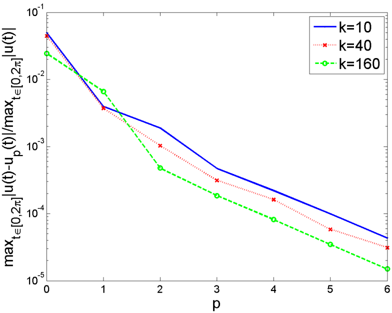
$k = 10$



$k = 160$

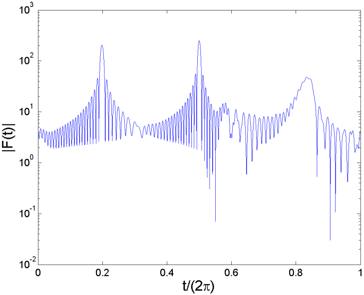


Partial illumination

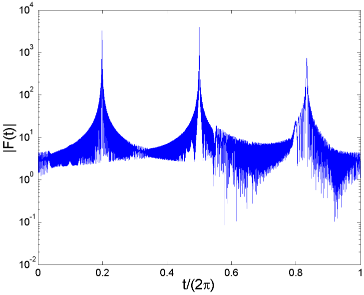


Re-reflections

FFP - partial illumination example

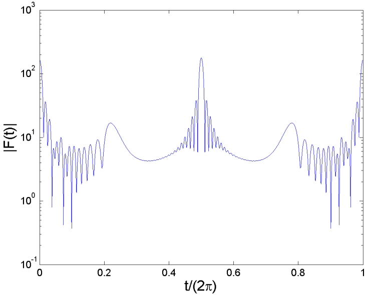


$k = 10$

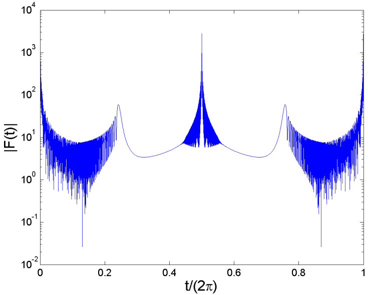


$k = 160$

FFP - re-reflections example

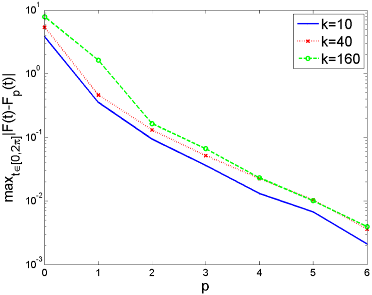


$k = 10$

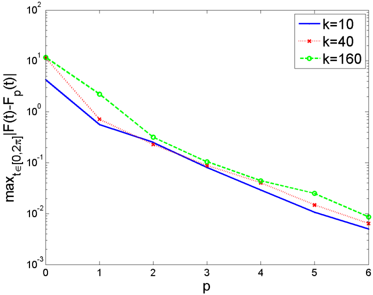


$k = 160$

Maximum absolute error in FFP



Partial illumination



Re-reflections

For a more detailed review

Chandler-Wilde, Graham, Langdon and Spence, *Numerical-Asymptotic Boundary Integral Methods in High-Frequency Acoustic Scattering*, Acta Numerica 21 (2012), pp. 89–305.