Integral equations for composite structures

MATHmONDES 2012

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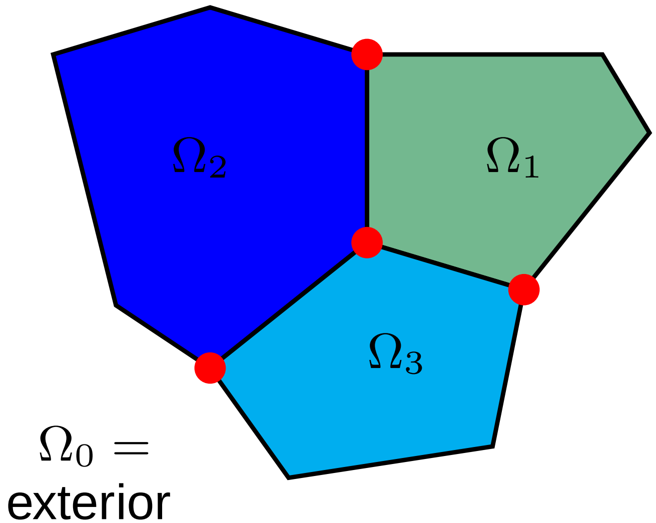
joint work with R.Hiptmair, SAM, ETH Zürich

Multi-subdomain scattering problem



Geometry

$\Omega_j =$ Lipschitz open set

with $\Omega_j \cap \Omega_k = \emptyset$ si $j \neq k$

and Ω_j borné si $j \neq 0$

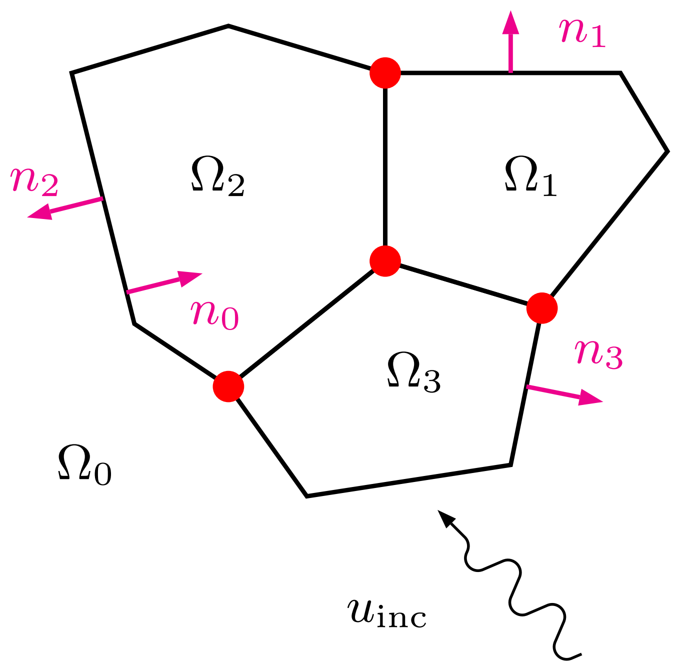
$\mathbb{R}^d = \cup_{j=0}^{n} \overline{\Omega}_j$

$\Gamma = \cup_{j=0}^{n} \partial\Omega_j$ (skeleton)

Important: different from the case of an homogeneous scatterer

- 3 subdomains (or more) may be adjacent to each other.
- the skeleton Γ is not an orientable surface.

Multi-subdomain scattering problem



Notations

$\kappa_j, \mu_j \in \mathbb{R}_+^*$ material carac. in Ω_j

n_j = normal to $\partial\Omega_j$

toward the exterior of Ω_j

Transmission problem (well posed):

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_{\rm loc}(\Delta, \overline{\Omega}_j) \text{ such that} \\ \Delta u + \kappa_j^2 u = 0 \quad \text{in} \quad \Omega_j,\ j = 0, \dots n \\ u - u_{\rm inc} \quad \text{outgoing in } \Omega_0 \end{cases}$$

$$\begin{cases} u|_{\partial\Omega_j} - u|_{\partial\Omega_k} = 0 \\ \mu_j^{-1}\partial_{n_j} u|_{\partial\Omega_j} + \mu_k^{-1}\partial_{n_k} u|_{\partial\Omega_k} = 0 \\ \qquad\qquad \text{on} \quad \partial\Omega_j \cap \partial\Omega_k, \forall j, k \end{cases}$$

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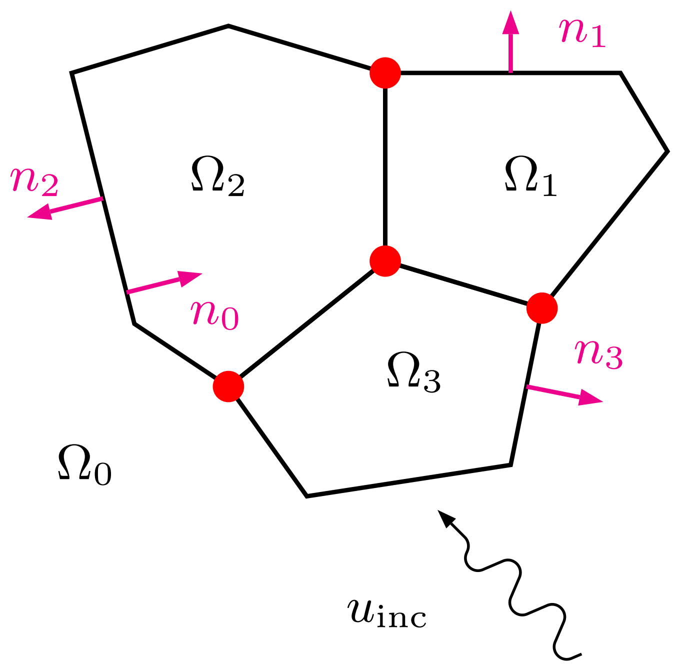

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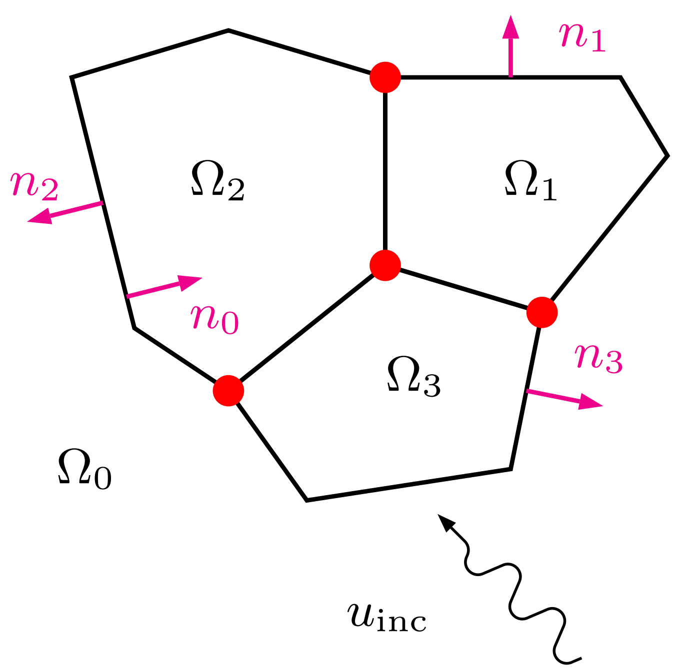
In this talk we assume $\mu_0 = \cdots = \mu_n = 1$

Multi-subdomain scattering problem



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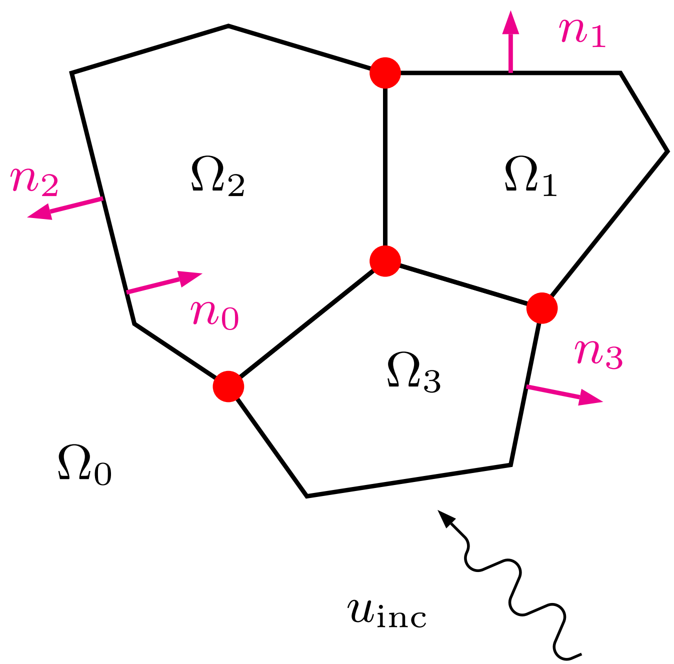

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We want to solve this problem by means of a boundary element method (BEM) constructed upon a suitable integral equation.

Preconditionning issue with integral equations

The preconditionning issue
Boundary element methods imply the inversion of fully populated linear systems. For industrial applications, iterative solvers become mandatory. Unfortunately, high resolution BEM induce poorly conditionned matrices (except with 2nd kind integral equations. . .), so that iterative solvers turn inefficient unless some preconditionner is used.

The Calderón preconditionner
In the case of homogeneous scatterers treated with 1st kind boundary integral equations, the Calderón preconditionner has emerged as a popular and efficient preconditionning strategy for a reasonable frequency range.

[Steinbach & Wendland, 1998], [Christiansen & Nédélec, 2000], [Antoine & Boubendir, 2008], [Cools, Andriulli & Olyslager, 2009],. . .

Main idea: the 1st kind integral operator A of the scattering pb satisfies

$$\mathrm{A}\cdot\mathrm{A} \quad \simeq \quad \mathrm{Id} + \text{compact} \qquad \text{(due to Calderón formula).}$$

As $\mathrm{Id} + \text{compact} =$"easy to solve with an iterative solver" (in many cases. . .), this suggests A as preconditionner for itslef.

Main difficulty and literature

Main objective: Devise an accurate boundary element method posed on Γ that lends itself to the Calderón preconditionning strategy.

Main difficulty: In the case of homogeneous scatterer, the derivation of classical integral equations relies on some orientation of interfaces. In our case, the skeleton Γ is not orientable.

Already available:

- **Rumsey principle/PMCHWT = "single trace formulation"**

- **Boundary element tearing and interconnecting method (BETI)** [Steinbach & Windisch, 2010], [Langer & Steinbach, 2003], [Hsiao, Steinbach & Wendland, 2000], . . .

- **Local multi-trace formulation [Jerez & Hiptmair, 2011].**

We present the global multi-trace formulation, another formulation where Calderón preconditionning is applicable.

OUTLINE

I. A single subdomain:
integral representation results

II. Two subdomains:
the gap idea

III. General case: Rumsey principle

IV. Global multi-trace formulation

V. Numerical results

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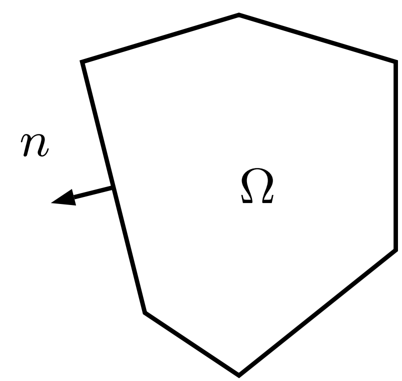
Notations for traces

$\Omega =$ open Lipschitz set

$\mathbb{H}(\partial\Omega) := \mathrm{H}^{\frac{1}{2}}(\partial\Omega) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)$.



Trace operator

$$\gamma(v) := \begin{bmatrix} v|^{\text{int}}_{\partial\Omega} \\ \partial_n v|^{\text{int}}_{\partial\Omega} \end{bmatrix}, \quad \gamma_{\boldsymbol{c}}(v) := \begin{bmatrix} v|^{\text{ext}}_{\partial\Omega} \\ \partial_n v|^{\text{ext}}_{\partial\Omega} \end{bmatrix},$$

$\{\gamma\} := \frac{1}{2}(\gamma + \gamma_c)$ and $[\gamma] := \gamma - \gamma_c$.

Representation theorem

Potential operator

$\mathscr{G}_\kappa(\mathbf{x}) := \exp(i\kappa|\mathbf{x}|)/(4\pi|\mathbf{x}|) =$ is the outgoing Green kernel for Helmholtz eq.

$$\mathrm{G}_\kappa\left(\begin{bmatrix} v \\ q \end{bmatrix}\right)(\mathbf{x}) := \int_{\partial\Omega} q(\mathbf{y})\,\mathscr{G}_\kappa(\mathbf{x}-\mathbf{y}) - v(\mathbf{y})\,n(\mathbf{y})\cdot\nabla_{\mathbf{y}}\big(\mathscr{G}_\kappa(\mathbf{x}-\mathbf{y})\big)\,d\sigma(\mathbf{y})$$

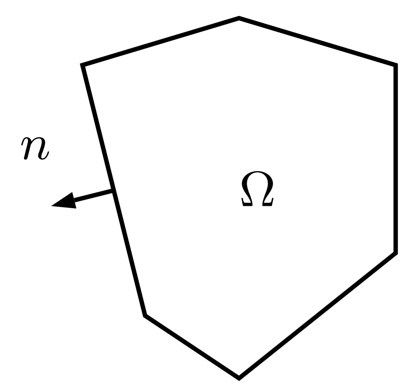
Theorem

If $u \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega})$ such that $\Delta u + \kappa^2 u = 0$ in Ω (+ u outgoing if Ω unbounded) then

$$\mathrm{G}_\kappa\big(\,\gamma(u)\,\big)(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega \\ 0 & \text{if } \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega} \end{cases}$$

Remark

$\mathrm{G}_\kappa(V)(\mathbf{x})$ is solution to Helmholtz equation in $\mathbb{R}^d \setminus \partial\Omega$, for any $V = (v,q)$.



Calderón projector

Cauchy data local to Ω:

$$\mathscr{C}_\kappa(\partial\Omega) := \{\ \gamma(u) \mid \Delta u + \kappa^2 u = 0 \ \text{ in } \Omega \ \ (+u \text{ outgoing if } \Omega \text{ unbounded})\ \}$$

If v is a solution to the Helmholtz equation in Ω then

$$\gamma(v) \in \mathscr{C}_\kappa(\partial\Omega) \quad \Longrightarrow \quad v \ = \ \mathrm{G}_\kappa(\gamma(v)) \quad \text{in } \Omega.$$

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If v is a solution to the Helmholtz equation in Ω then

$$\gamma(v) \in \mathscr{C}_\kappa(\partial\Omega) \quad \Longrightarrow \quad \gamma(v) = \gamma \cdot \mathrm{G}_\kappa(\gamma(v))$$

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Caracterization of Cauchy data
The operator $\gamma \cdot \mathrm{G}_\kappa : \mathbb{H}(\partial\Omega) \to \mathscr{C}_\kappa(\partial\Omega) \subset \mathbb{H}(\partial\Omega)$ is a continuous projector called Calderón projector interior to Ω. We have

$$V \in \mathscr{C}_\kappa(\partial\Omega) \quad \Longleftrightarrow \quad V = \gamma \cdot \mathrm{G}_\kappa(V)$$

Jump formula: $[\gamma] \cdot \mathrm{G}_\kappa = \mathrm{Id}$

Calderón identity: $(2\,\mathrm{A}_\kappa)^2 = \mathrm{Id}$ with $\mathrm{A}_\kappa = \{\gamma\} \cdot \mathrm{G}_\kappa$

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$(\mathrm{A}_\kappa + \mathrm{Id}/2)V$

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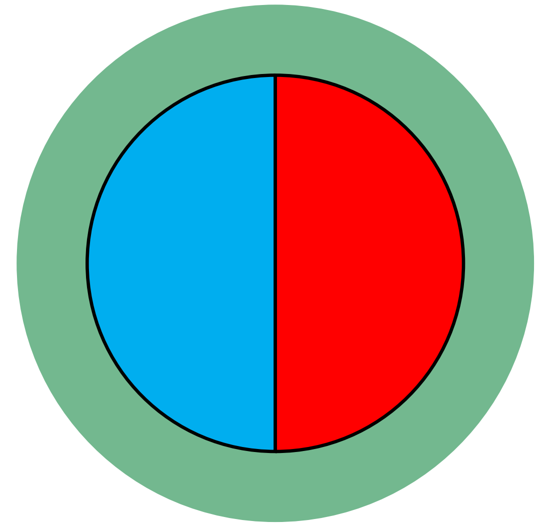
The gap idea



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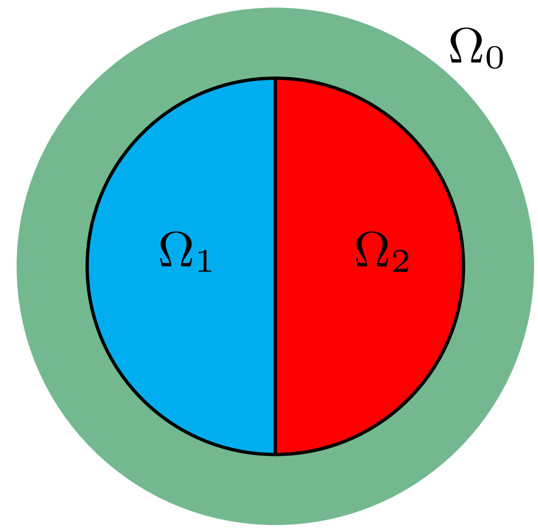

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In the perturbed geometry, all interfaces become orientable anew, and usual techniques become applicable.

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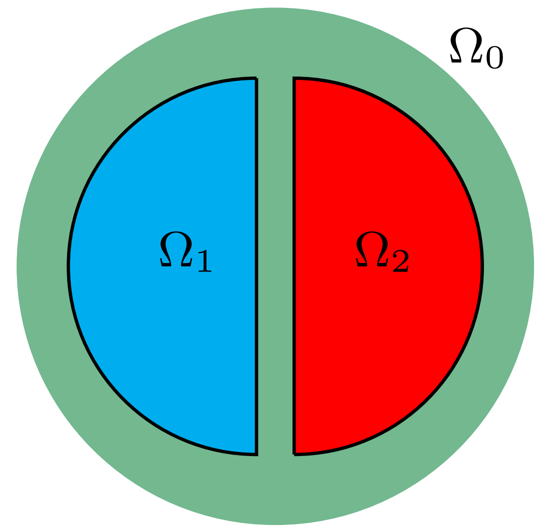

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Find $U_j = \gamma^j(u) \in \mathbb{H}(\partial\Omega_j), j = 0, 1, 2$ such that

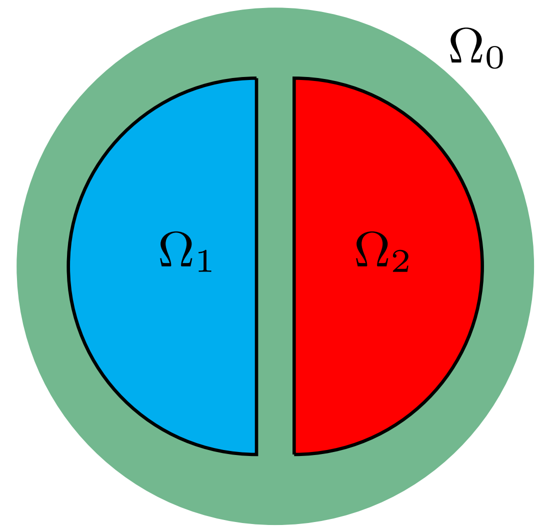
$$U_1 = Q \cdot U_0|_{\partial\Omega_1}$$

$$U_2 = Q \cdot U_0|_{\partial\Omega_2} \quad \text{where } Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(\mathrm{A}^2_{\kappa_2} - \mathrm{Id}/2)U_2 = 0$$

$$(\mathrm{A}^1_{\kappa_1} - \mathrm{Id}/2)U_1 = 0$$

$$(\mathrm{A}^0_{\kappa_0} - \mathrm{Id}/2)U_0 = \gamma^0(u_{\mathrm{inc}})$$



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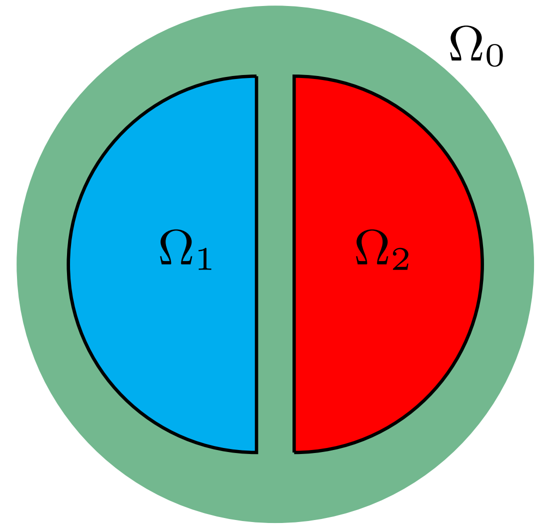

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caracterization of
Cauchy data

$\Delta u + \kappa_j^2 u = 0$ in $\Omega_j, \forall j$

$u - u_{\mathrm{inc}}$ outgoing

transmission
conditions

$u|^1_{\partial\Omega_1} = u|^0_{\partial\Omega_1}$

$\partial_{n_1} u|^1_{\partial\Omega_1} = -\partial_{n_0} u|^0_{\partial\Omega_1}$

$u|^2_{\partial\Omega_2} = u|^0_{\partial\Omega_2}$

$\partial_{n_2} u|^2_{\partial\Omega_2} = -\partial_{n_0} u|_{\partial\Omega^0_2}$

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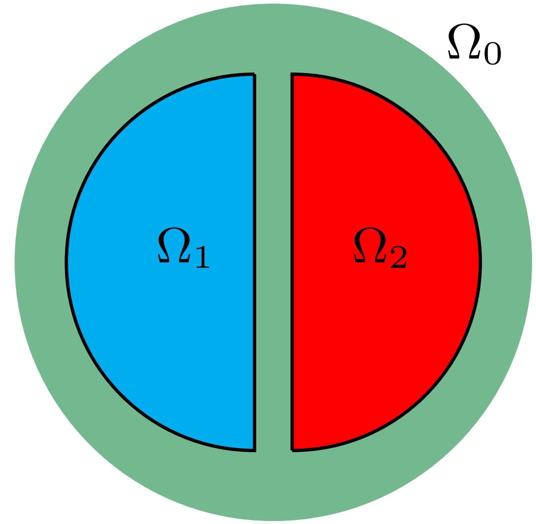

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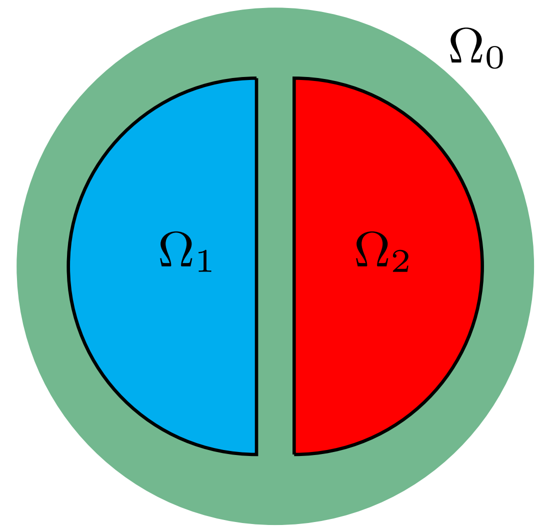

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Plugging transmission conditions in the 3rd Calderón identity yields:

$$\begin{bmatrix} \mathrm{A}^1_{\kappa_0} + \mathrm{Id}/2 & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & \mathrm{A}^2_{\kappa_0} + \mathrm{Id}/2 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \end{bmatrix}$$

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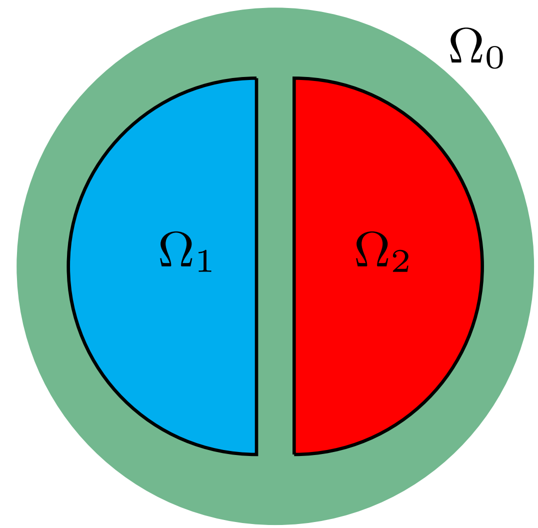

$$(\mathrm{A}^2_{\kappa_2} - \mathrm{Id}/2)U_2 = 0$$

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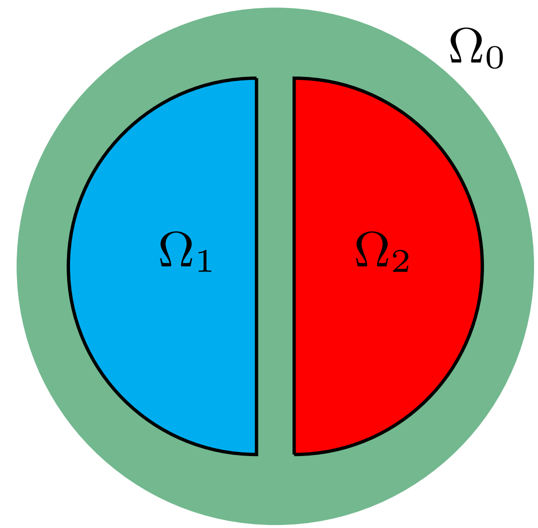


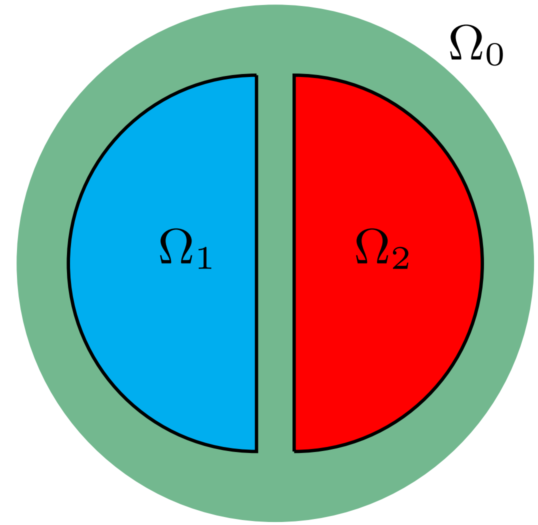
$(\mathrm{A}^2_{\kappa_2} - \mathrm{Id}/2)U_2 = 0$ (1)

$(\mathrm{A}^1_{\kappa_1} - \mathrm{Id}/2)U_1 = 0$ (2)

$$\begin{bmatrix} \mathrm{A}^1_{\kappa_0} + \mathrm{Id}/2 & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & \mathrm{A}^2_{\kappa_0} + \mathrm{Id}/2 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \end{bmatrix} \quad \begin{matrix} (3) \\ (4) \end{matrix}$$

The gap idea

Find $U_j = \gamma^j(u) \in \mathbb{H}(\partial\Omega_j), j = 1, 2$ such that



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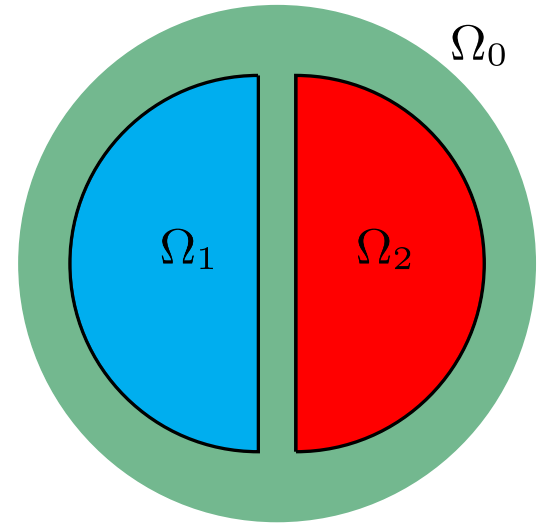

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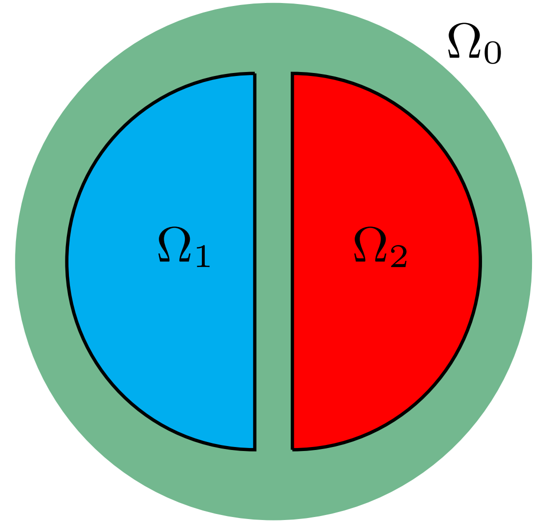


Observation 1

The pair of traces U_0 does not come into play anymore, and U_1, U_2 are independent to each other (transmission conditions do not appear explicitely).

Observation 2

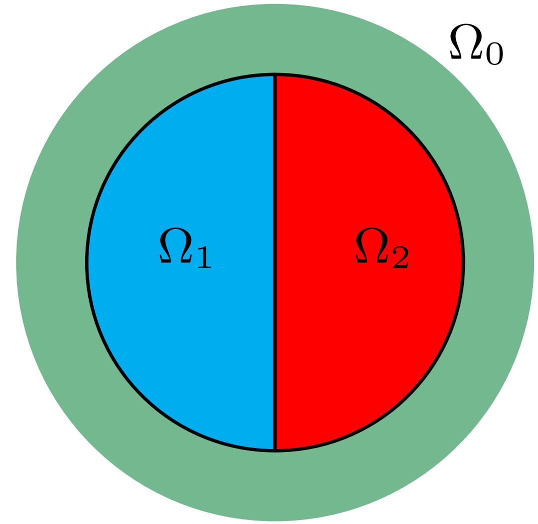
This formulation satisfies a Garding inequality, and a generalized Calderón identity when $\kappa_0 = \kappa_1 = \kappa_2$

$$\begin{bmatrix} 2\,\mathrm{A}^1_{\kappa_0} & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & 2\,\mathrm{A}^2_{\kappa_0} \end{bmatrix} \cdot \begin{bmatrix} 2\,\mathrm{A}^1_{\kappa_0} & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & 2\,\mathrm{A}^2_{\kappa_0} \end{bmatrix} = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$$

Observation 3

Without any gap, this formulation remains meaningful (every operators are well defined). However the derivation we have presented does not hold anymore.

The gap idea



Find $U_j = \gamma^j(u) \in \mathbb{H}(\partial\Omega_j), j = 1, 2$ such that

$$\begin{bmatrix} \mathrm{A}^1_{\kappa_1} + \mathrm{A}^1_{\kappa_0} & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & \mathrm{A}^2_{\kappa_2} + \mathrm{A}^2_{\kappa_0} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \end{bmatrix}$$

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OUTLINE

I. A single subdomain:
integral representation results

II. Two subdomains:
the gap idea

III. General case: Rumsey principle

IV. Global multi-trace formulation

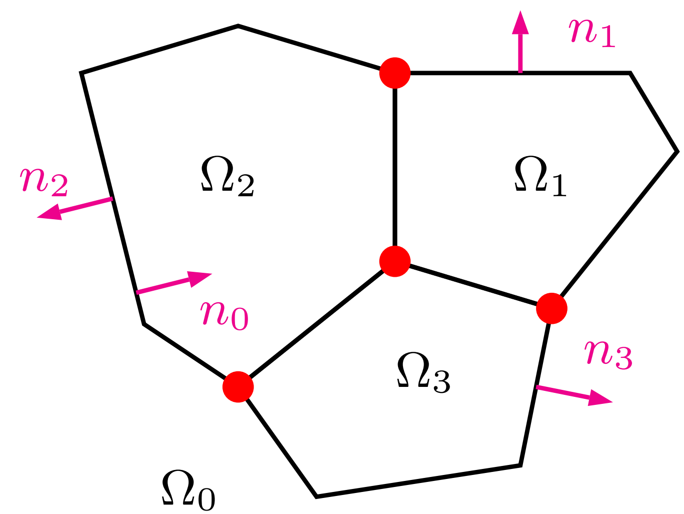
V. Numerical results

Back to the general problem…

General transmission problem

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \text{ such that} \\ \Delta u + \kappa_j^2 u = 0 \quad \text{in} \quad \Omega_j,\ j = 0, \ldots n \\ u - u_{\mathrm{inc}} \quad \text{outgoing in } \Omega_0 \end{cases}$$

$$\begin{cases} u|_{\partial\Omega_j} - u|_{\partial\Omega_k} = 0 \\ \partial_{n_j} u|_{\partial\Omega_j} + \partial_{n_k} u|_{\partial\Omega_k} = 0 \\ \qquad\qquad \text{on} \quad \partial\Omega_j \cap \partial\Omega_k, \forall j, k \end{cases}$$



Notations

$\gamma^j = \begin{bmatrix} \gamma^j_{\mathrm{D}} \\ \gamma^j_{\mathrm{N}} \end{bmatrix}$ = Dirichlet and Neumann traces on $\partial\Omega_j$

n_j = normal to $\partial\Omega_j$.

Multi/Single trace spaces

Multi-trace space:

$$\mathbb{H}(\Gamma) := \mathbb{H}(\partial\Omega_0) \times \cdots \times \mathbb{H}(\partial\Omega_n) \quad \text{with} \quad \mathbb{H}(\partial\Omega_j) = \mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)$$

Duality pairing on $\mathbb{H}(\boldsymbol{\Gamma})$:

$$\mathrm{B}(U, V) = \sum_{j=0}^{n} \mathrm{B}_j\Big(\begin{bmatrix} u_j \\ p_j \end{bmatrix}, \begin{bmatrix} v_j \\ q_j \end{bmatrix} \Big) = \sum_{j=0}^{n} \int_{\partial\Omega_j} u_j q_j - p_j v_j d\sigma,$$

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Single trace space:

$\mathbb{X}(\Gamma) =$ closure of $\{ \; (\gamma^j(v))_{j=0\dots n} \; \mid \; v \in \mathrm{H}^1(\mathbb{R}^d), \; \Delta v \in \mathrm{L}^2(\mathbb{R}^d) \; \}$ for $\| \;\; \|_{\mathbb{H}(\Gamma)}$

$\mathbb{X}(\Gamma) =$ elements of $\mathbb{H}(\Gamma)$ that satisfy transmission conditions.

Multi/Single trace spaces

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$\mathbb{X}(\Gamma) =$ elements of $\mathbb{H}(\Gamma)$ that satisfy transmission conditions.

Lemma:
For $U \in \mathbb{H}(\Gamma)$, we have: $U \in \mathbb{X}(\Gamma) \iff \mathrm{B}(U,V) = 0 \; \forall V \in \mathbb{X}(\Gamma)$.

Reformulation the scattering problem

General transmission problem

$$\begin{cases} u|_{\partial\Omega_j} - u|_{\partial\Omega_k} = 0 \\ \partial_{n_j} u|_{\partial\Omega_j} + \partial_{n_k} u|_{\partial\Omega_k} = 0 \\ \qquad\qquad \text{sur} \quad \partial\Omega_j \cap \partial\Omega_k, \forall j, k \end{cases}$$

$$\begin{cases} \text{Trouver } u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \text{ tel que} \\ \Delta u + \kappa_j^2 u = 0 \quad \text{in} \quad \Omega_j, \ \forall j \\ u - u_{\mathrm{inc}} \quad \text{sortant dans } \Omega_0 \end{cases}$$

Reformulation the scattering problem

General transmission problem

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$$\iff \quad U = (U_j) := \big(\, \gamma^j(u) \,\big)_{0 \leq j \leq n} \in \mathbb{X}(\Gamma)$$

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Reformulation the scattering problem

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$$\Longleftrightarrow \quad U = (U_j) := \big(\gamma^j(u)\big)_{0\leq j\leq n} \in \mathbb{X}(\Gamma)$$

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$$\Longleftrightarrow \quad \big(-\mathrm{Id}/2 + \mathrm{A}^j_{\kappa_j}\big)\big(U_j - U_j^{\mathrm{inc}}\big) = 0, \ \forall j$$

with $U_j^{\mathrm{inc}} = \gamma^j(u_{\mathrm{inc}})$

Reformulation the scattering problem

$$\begin{cases} \text{Find } U \in \mathbb{X}(\Gamma) \\ \quad (-\mathrm{Id}/2 + \mathbf{A}_\kappa)U \quad = \quad F \end{cases}$$

with

$$\mathbf{A}_\kappa = \begin{bmatrix} \mathrm{A}^0_{\kappa_0} & 0 & \cdots & 0 \\ 0 & \mathrm{A}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{A}^n_{\kappa_n} \end{bmatrix}$$

Reformulation the scattering problem

$$\begin{cases} \text{Find } U \in \mathbb{X}(\Gamma) \\ \mathrm{B}\big((-\mathrm{Id}/2 + \mathbf{A}_\kappa)U, V\big) = \mathrm{B}(F, V) \quad \forall V \in \mathbb{H}(\Gamma) \end{cases}$$

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[VonPetersdorff, 1989]

Reformulation the scattering problem

$$\begin{cases} \text{Find } U \in \mathbb{X}(\Gamma) \\ -\frac{1}{2}\mathrm{B}(U,V) + \mathrm{B}(\mathbf{A}_\kappa U, V) = \mathrm{B}(F,V) \quad \forall V \in \mathbb{X}(\Gamma) \end{cases}$$

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Rumsey principle/PMCHWT

$$\begin{cases} \text{Find } U \in \mathbb{X}(\Gamma) \\ \mathrm{B}(\mathbf{A}_\kappa U, V) = \mathrm{B}(F, V) \quad \forall V \in \mathbb{X}(\Gamma) \end{cases}$$

$$\mathbf{A}_\kappa = \begin{bmatrix} \mathrm{A}^0_{\kappa_0} & 0 & \cdots & 0 \\ 0 & \mathrm{A}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{A}^n_{\kappa_n} \end{bmatrix}$$

We obtain our formulation as a modified Rumsey principle by eliminating the contributions associated to Ω_0.

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Global multi-trace formulation

Theorem
$U = (U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$ is solution to Rumsey principle if and only if $\widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$ satisfies

$$\begin{cases} \widehat{U} \in \widehat{\mathbb{H}}(\Gamma) \quad \text{such that} \\ \widehat{\mathrm{B}}(\widehat{\mathrm{A}}_\kappa \widehat{U}, \widehat{V}) = \widehat{B}(\widehat{F}, \widehat{V}) \qquad \forall \widehat{V} \in \widehat{\mathbb{H}}(\Gamma). \end{cases}$$

with

$$\widehat{\mathbb{H}}(\Gamma) = \left[\mathrm{H}^{\frac{1}{2}}(\partial\Omega_{\mathbf{1}}) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_{\mathbf{1}})\right] \times \cdots \times \left[\mathrm{H}^{\frac{1}{2}}(\partial\Omega_n) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_n)\right]$$

$$\widehat{\mathrm{B}}(\widehat{U}, \widehat{V}) = \sum_{j=\mathbf{1}}^{n} \mathrm{B}_j(U_j, V_j)$$

and

$$\widehat{\mathrm{A}}_\kappa \widehat{U} = \begin{bmatrix} \mathrm{A}^1_{\kappa_1} + \mathrm{A}^1_{\kappa_0} & \gamma^1 \cdot \mathrm{G}^2_{\kappa_0} & \cdots & \gamma^1 \cdot \mathrm{G}^n_{\kappa_0} \\ \gamma^2 \cdot \mathrm{G}^1_{\kappa_0} & \mathrm{A}^2_{\kappa_2} + \mathrm{A}^2_{\kappa_0} & \cdots & \gamma^2 \cdot \mathrm{G}^n_{\kappa_0} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^n \cdot \mathrm{G}^1_{\kappa_0} & \gamma^n \cdot \mathrm{G}^2_{\kappa_0} & \cdots & \mathrm{A}^n_{\kappa_n} + \mathrm{A}^n_{\kappa_0} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

Three key results for the proof

Recall that:

$$\mathrm{G}^j_{\kappa_0}\left(\begin{bmatrix} v \\ q \end{bmatrix}\right)(\mathbf{x}) := \int_{\partial\Omega_j} q(\mathbf{y})\,\mathscr{G}_\kappa(\mathbf{x}-\mathbf{y}) - v(\mathbf{y})\,n(\mathbf{y})\cdot\nabla_{\mathbf{y}}\big(\mathscr{G}_\kappa(\mathbf{x}-\mathbf{y})\big)\;d\sigma(\mathbf{y})$$

Proposition

$\sum_{j=0}^{n} \mathrm{G}^j_{\kappa_0}(U_j)(\mathbf{x}) = 0 \qquad \forall \mathbf{x}\in\mathbb{R}^d,\;\; \forall U = (U_0,\dots,U_n)\in\mathbb{X}(\Gamma)$.

Theorem

$\mathbb{H}(\Gamma) = \mathbb{X}(\Gamma)\oplus\mathscr{C}_\kappa(\Gamma)$ where $\mathscr{C}_\kappa(\Gamma) = \mathscr{C}_{\kappa_0}(\partial\Omega_0)\times\cdots\times\mathscr{C}_{\kappa_n}(\partial\Omega_n)$

Proposition

For $U\in\mathbb{H}(\Gamma)$ we have: $U\in\mathscr{C}_\kappa(\Gamma) \iff \mathrm{B}(U,V) = 0 \;\;\forall V\in\mathscr{C}_\kappa(\Gamma)$

Remarkable properties

Notation:

$$\Theta\Big(\begin{pmatrix} u_j \\ p_j \end{pmatrix}_{1\leq j\leq n} \Big) = \begin{pmatrix} -\overline{u}_j \\ \overline{p}_j \end{pmatrix}_{1\leq j\leq n}$$

Generalized Gårding inequality

For any $\kappa_0, \dots, \kappa_n \in \mathbb{R}_+$, the operator $\widehat{\mathrm{A}}_\kappa : \widehat{\mathbb{H}}(\Gamma) \to \widehat{\mathbb{H}}(\Gamma)$ is an isomorphism and $\exists\, C > 0$ and $\exists\, \mathrm{K} : \widehat{\mathbb{H}}(\Gamma) \to \widehat{\mathbb{H}}(\Gamma)$ compact such that

$$\Re e\Big\{\ \widehat{\mathrm{B}}\Big(\,(\widehat{\mathrm{A}}_\kappa + \mathrm{K})\widehat{U}, \Theta(\widehat{U})\,\Big)\ \Big\} \ \geq\ C\,\|\widehat{U}\|^2_{\widehat{\mathbb{H}}} \qquad \forall \widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$$

Consequence:
Quasi-optimal convergence of conforming Galerkin discretizations.

Calderón identity

If $\kappa_0 = \cdots = \kappa_n$ we have: $\quad (\widehat{\mathrm{A}}_\kappa)^2 = \mathrm{Id}.$

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Model problem

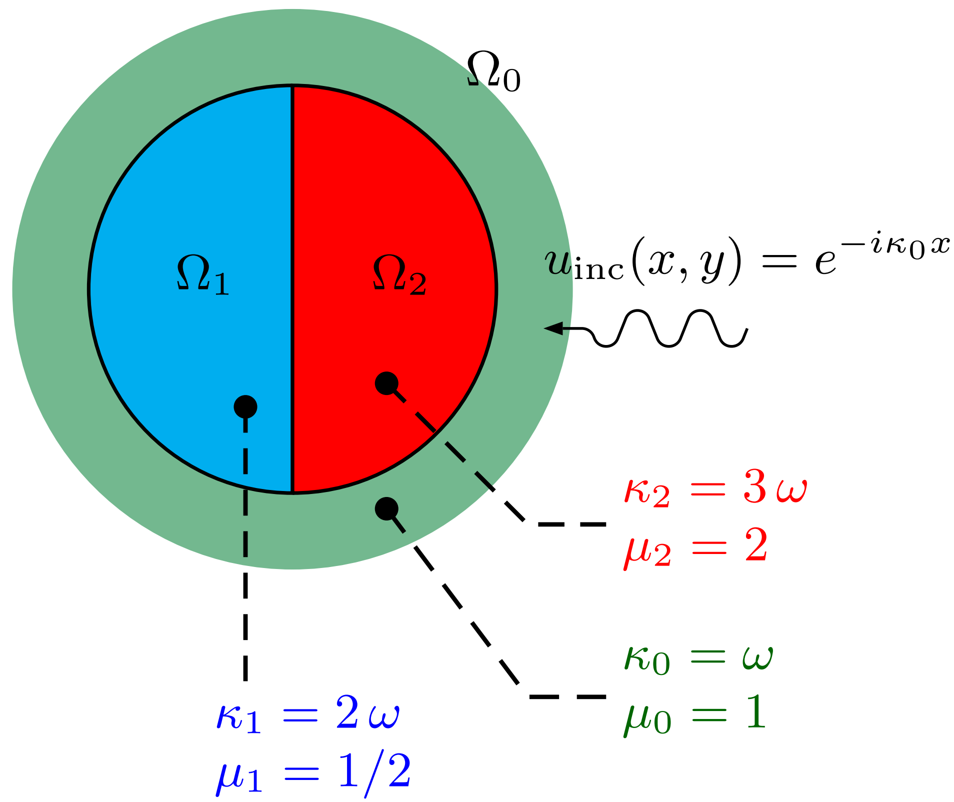
Propagation medium

$\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\mathrm{D}}(0,1)$

Transmission problem

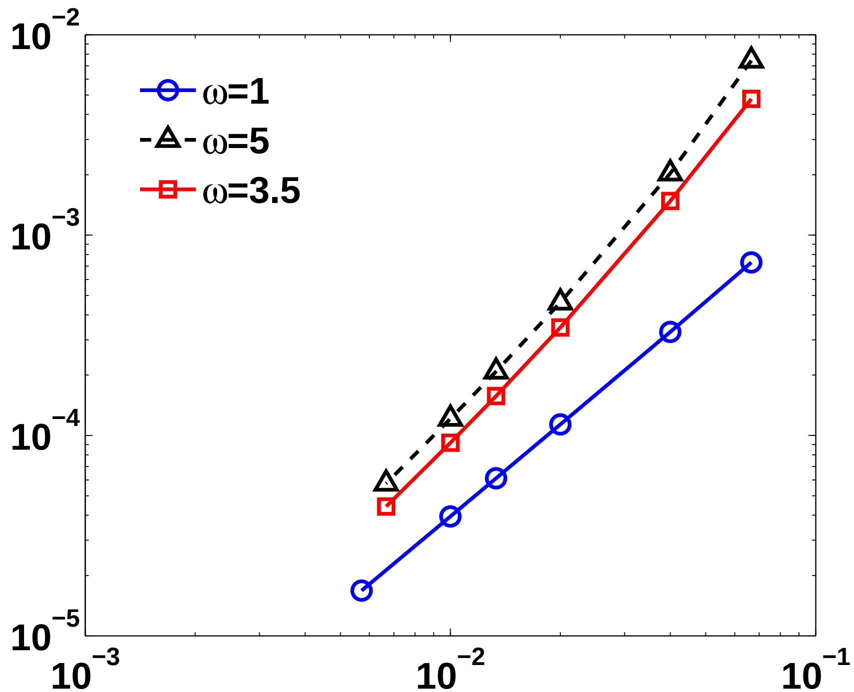
$$\begin{cases} \text{Find } u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \text{ such that} \\ \Delta u + \kappa_j^2 u = 0 \quad \text{in } \Omega_j \\ u - u_{\mathrm{inc}} \quad \text{outgoing in } \Omega_0 \end{cases}$$

$$\begin{cases} u|_{\partial\Omega_j} - u|_{\partial\Omega_k} = 0 \\ \mu_j^{-1} \partial_{n_j} u|_{\partial\Omega_j} + \mu_k^{-1} \partial_{n_k} u|_{\partial\Omega_k} = 0 \\ \qquad \text{on} \quad \partial\Omega_j \cap \partial\Omega_k, \forall j,k \end{cases}$$



Reference solution:
we solve numerically both our formulation and Rumsey principle.

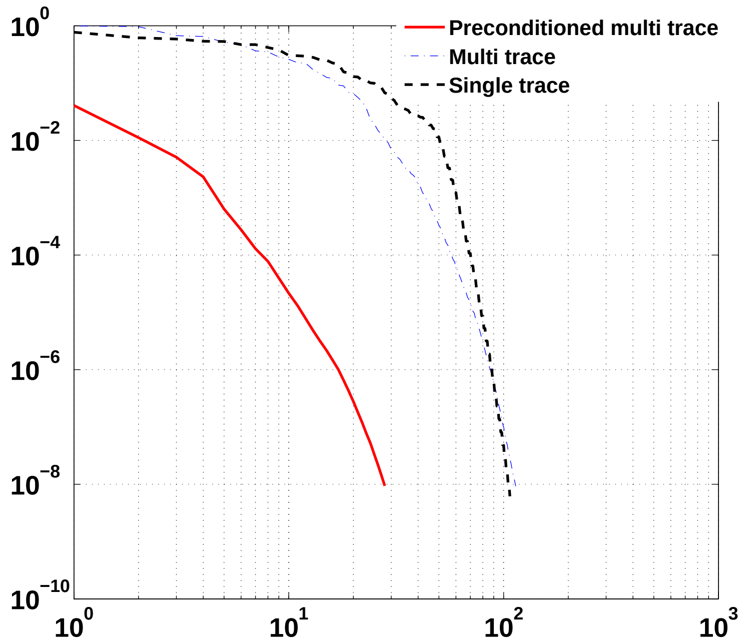
Consistency result



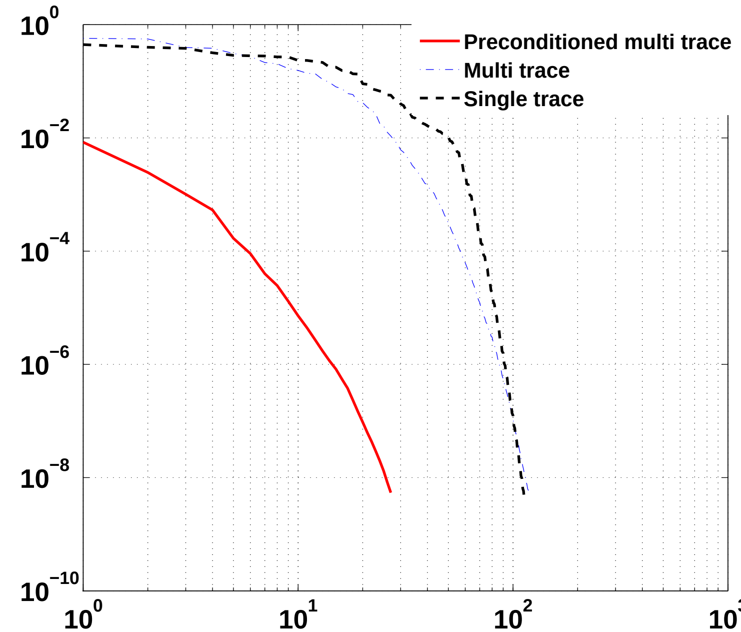
Relative error $\|U_h^{\mathrm{MTF}} - U_h^{\mathrm{Rumsey}}\| / \|U_h^{\mathrm{Rumsey}}\|$ versus step of the mesh h

Discretization: continuous piecewise linear for both Dirichlet and Neumann traces

Convergence history of GMRES



$h = 0.02$



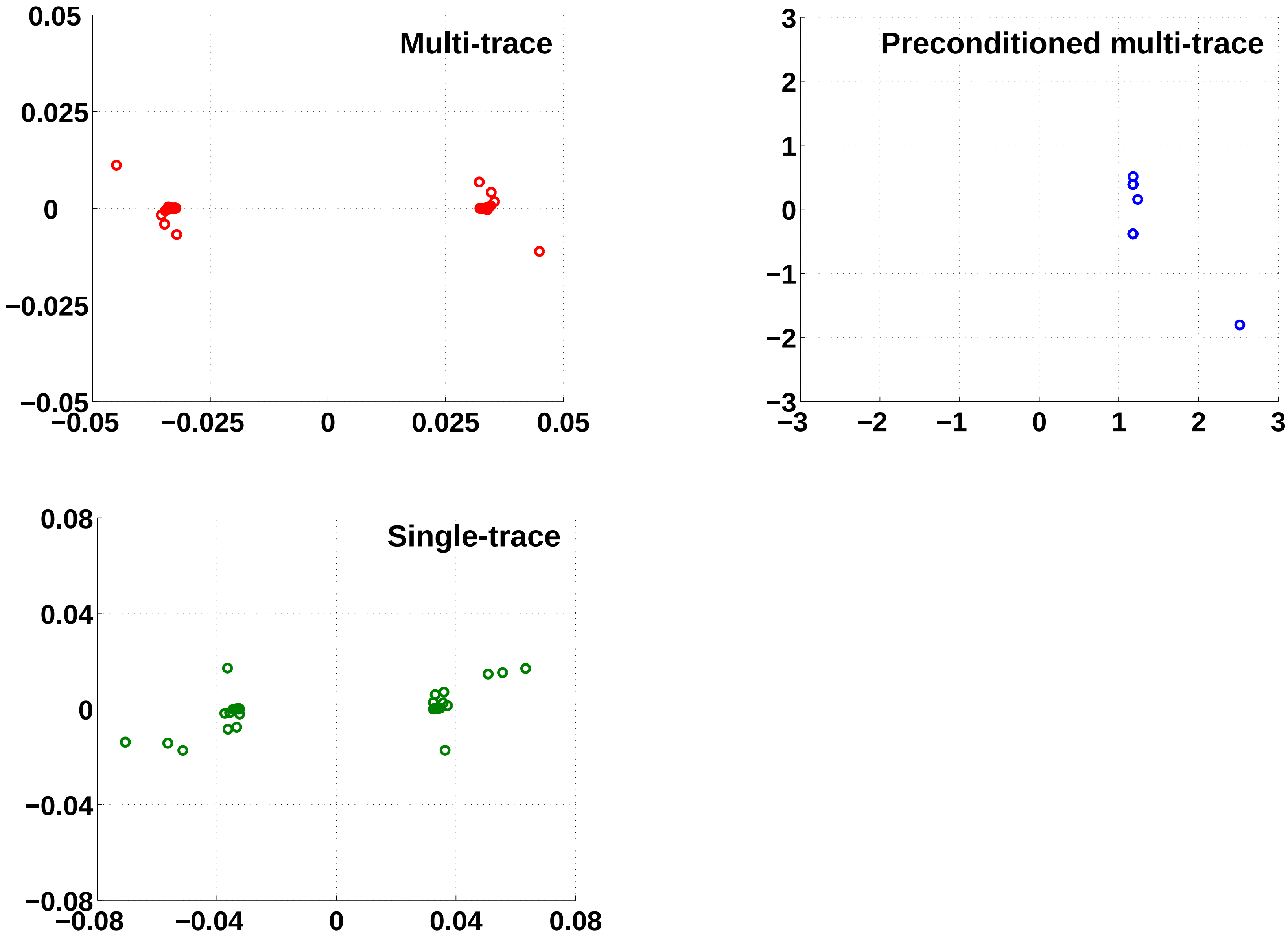
$h = 0.0066$

Quadratic norm of the residue of GMRES (without restart) versus number of iterations for $\omega = 2$.

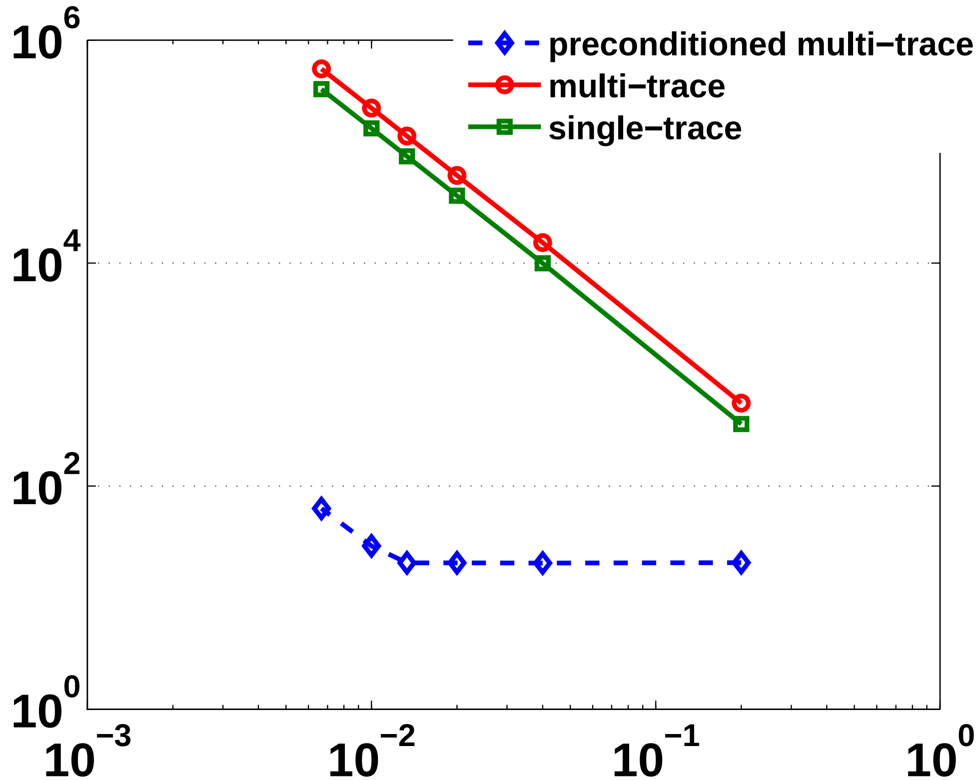
We took $\mathrm{M}_h^{-1}\mathrm{A}_h\mathrm{M}_h^{-1}$ as a preconditionner for A_h where

$\mathrm{A}_h =$ Galerkin matrix of $\widehat{\mathrm{A}}_\kappa$,
$\mathrm{M}_h =$ mass matrix associated to $\widehat{\mathrm{B}}(\ ,\ )$.

Location of eigenvalues ($h = 0.02, \omega = 1$)



Condition number



Condition number versus step of the mesh

with $\mu_0 = \mu_1 = \mu_2$ and $\kappa_0 = 1$, $\kappa_1 = 2$, $\kappa_2 = 3$

Conclusion

Possible extensions

• $\boldsymbol{\Im m\{\kappa_j\} \neq 0}$
as long as the transmission problem remains well posed (true if $\Im m\{\kappa_j\} \geq 0$, $\Re e\{\kappa_j\} \geq 0$ and $\kappa_j \neq 0$, $\forall j = 0 \dots n$).
Consequence: no spurious mode!

• **Any values of $\boldsymbol{\mu_0, \dots, \mu_n} \in \mathbb{R}_+$**
For Calderón identity we still need $\mu_0 = \cdots = \mu_n$.

• **Maxwell equations**
We have proved a counterpart of every results for Maxwell. The proof of quasi-optimal convergence makes use of the framework developped in [Buffa,2005].

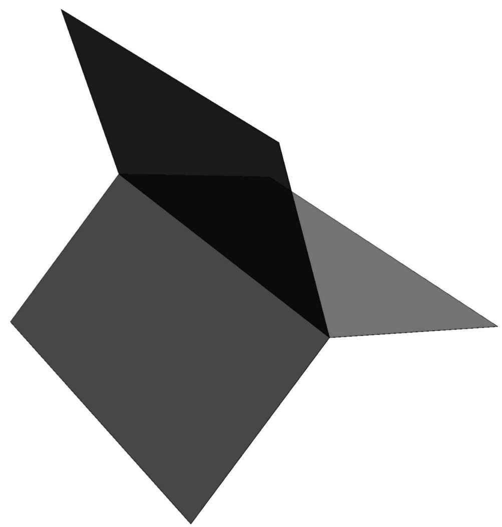
References

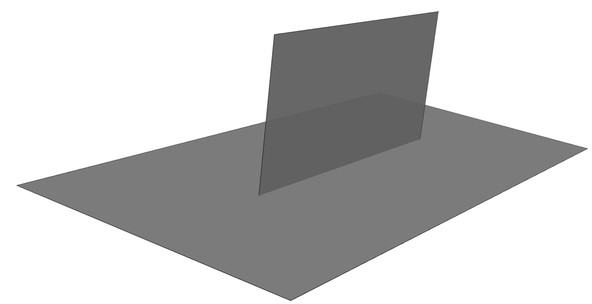
X.Claeys, "*A single trace integral formulation of the second kind for acoustic scattering*", SAM Report No.2011-14, ETH Zürich, 2011.

X.Claeys & R.Hiptmair, "*Boundary integral formulation of the first kind for acoustic scattering by composite structures*", accepted in Comm. Pure Appl. Math.

X.Claeys & R.Hiptmair, "*Electromagnetic Scattering at Composite Objects: A Novel Multi-trace Boundary Integral Formulation*", M2AN, 46 (2012) 1421-1445.

Future work: multi-screens





Thank you

for your attention