Numerical MicroLocal Analysis (NMLA)

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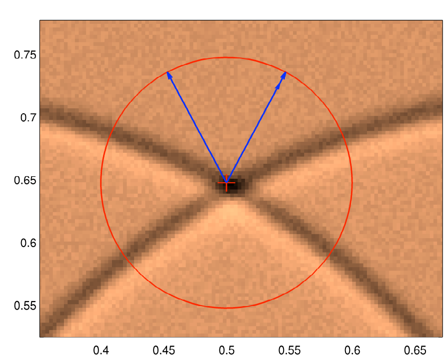
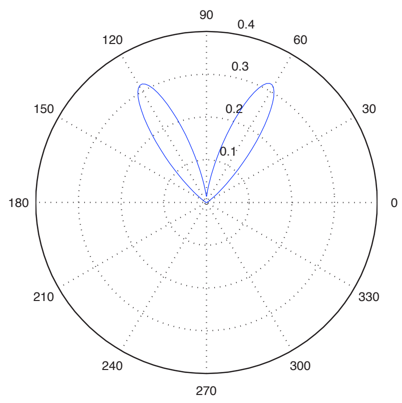
"Given frequency domain wave data, the proposed algorithm gives a **pointwise** estimate of the the **number of rays** , their **slowness vectors** and corresponding **wavefront curvature**. With time domain wave data and assuming the source wavelet is given, the method also estimates the traveltime."



$\longrightarrow$



Geometric optics equations / Far field models : Find local asymptotic (large ω) solutions of

$$\frac{\omega^2}{c^2(x)}\hat{u}(x;\omega) + \Delta\hat{u}(x;\omega) = 0$$

$\hat{u}$ replaced by *"ansatz"*

$$\hat{u} \simeq \hat{u}^{ray}(x;\omega) = A(x)e^{i\omega\varphi(x)}$$

yields

$$\begin{cases} \|\nabla\varphi(x)\| = \dfrac{1}{c(x)} \\ 2\nabla\varphi(x)\cdot\nabla A(x) + A(x)\Delta\varphi(x) = 0 \end{cases}$$

For constant medium and far field data, linear (plane wave) phase approximation is a popular choice (beamforming, DOA) :

$$\hat{u} \simeq \hat{u}^{ray}(x;\omega) = B(x)e^{i\vec{k}\cdot x}, \qquad \|\vec{k}\| = \frac{\omega}{c}$$

Rq. : $\Leftrightarrow$ Plane wave approximation around a point x_0

$$\varphi(x) \simeq \varphi(x_0) + (x - x_0)\cdot\nabla\varphi(x_0) + \frac{1}{2}(x-x_0)^T\, H\varphi(x_0)\,(x-x_0) + \ldots$$

yields

$$\hat{u}(x;\omega) \simeq B(x_0)e^{i\omega(x-x_0)\cdot\nabla\varphi(x_0)}$$

where

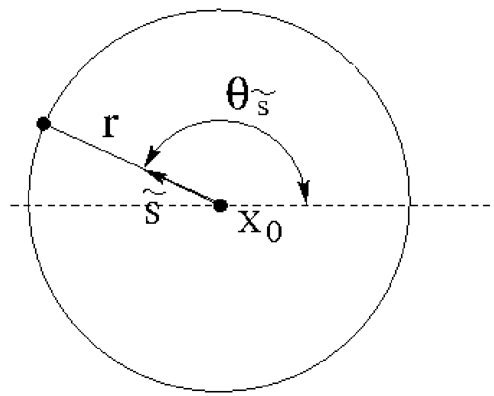
$$B(x_0) = A(x_0)e^{i\omega\varphi(x_0)}.$$

The first "general" N-rays ansatz we consider is :

$$\boxed{\hat{u}(x;\omega) \simeq \sum_{n=1}^{N} B_n(x_0)e^{i\omega(x-x_0)\cdot\nabla\varphi_n(x_0)} \quad x \text{ near } x_0}$$

NMLA observable in the frequency domain

The observable data ($\vec{s} \in \{\|\vec{s}\| = 1\}$)



$$U_{\alpha}(\vec{s}) = \frac{c(x_0)}{i\omega}\frac{\partial \hat{u}}{\partial r}(x_0 + r\vec{s}; \omega) + \hat{u}(x_0 + r\vec{s}; \omega), \qquad \boxed{r = \frac{\alpha c(x_0)}{\omega}}.$$

"should" fit the ansatz form

$$\boxed{U_{\alpha}(\vec{s}) \simeq \textstyle\sum_{n=1}^{N} (\,\vec{s} \cdot \vec{d}_n + 1)\, B_n\, e^{i\alpha \vec{s}\cdot\vec{d}_n}}, \qquad \vec{d}_n = c(x_0)\,\nabla\varphi_n(x_0)$$

Relaxation towards a linear system

Set

$$\vec{d} = (\cos\theta_{\vec{d}}, \sin\theta_{\vec{d}}).$$

$$U_\alpha(\vec{s}) \simeq \int_0^{2\pi} (\vec{s}\cdot\vec{d} + 1)\,\beta(\vec{d})\,e^{i\alpha\vec{s}\cdot\vec{d}}\,d\theta_{\vec{d}}$$

$$\to \boxed{U = K_\alpha\,\beta}.$$

(Discretization)

$$K_{\alpha,m,n} = (\vec{d}_m\cdot\vec{d}_n + 1)\,e^{i\alpha\vec{d}_m\cdot\vec{d}_n}, \quad U_m = U_\alpha(\vec{d}_m), \quad B_m = \beta(\theta_m).$$

K_α is compact but a regularized inverse is easy to compute **(Filter + truncate Fourier modes of β)** and its norm is bounded independantly of α (Stability).

NMLA filter

$$\beta := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\{\widehat{\beta}_\ell\}), \qquad \widehat{\beta}_\ell = H_\ell \mathcal{F}(\{U\})_\ell$$

with

$$\begin{aligned} H_\ell \; &= D_\ell^{-1} \text{ if } |\ell| < L(\alpha) \\ &= 0 \text{ else.} \end{aligned}$$

where

$$D_\ell(\alpha) = 2\pi i^\ell (J_\ell(\alpha) - i\; J'_\ell(\alpha))$$

and

$$L(\alpha) = \min\{\alpha, \alpha + \alpha^{1/3} - 2.5\}$$

Chosen such that $\|{K'}_\alpha^{-1}\| < 3 \rightarrow$ (Stability Theorem)

For an exact Plane wave :

If wavefield is a perfect plane wave of direction $\vec{d}$ with amplitude A then

$$U^{plwa}(\vec{s}) = (\,\vec{d}\cdot\vec{s}+1)\;A\,e^{i\omega\varphi(x_0)}\,e^{i\alpha\vec{d}\cdot\vec{s}}$$

and

$$\beta^{plwa}(\vec{s}) = A\,e^{i\omega\varphi(x_0)}\,S_\alpha(\theta_{\vec{d}}-\theta_{\vec{s}})$$

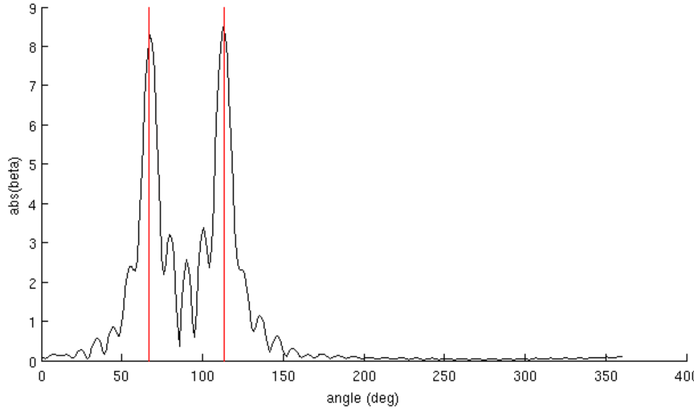
$$S_\alpha(\theta) = \frac{\sin((L(\alpha)+\frac{1}{2})\theta)}{(2L(\alpha)+1)\sin(\frac{\theta}{2})}$$

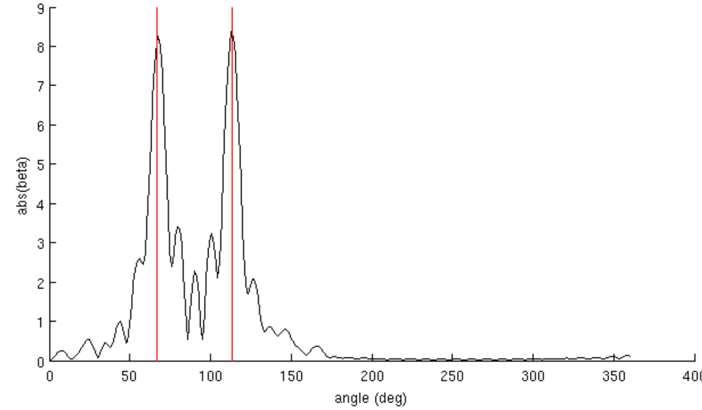
where $L(\alpha)$ is the (explicitely given) Number of Fourier Modes used to filter β. Fourier modes are also given analytically

$$\boxed{\widehat{\beta}_\ell = A\,e^{i\omega\varphi(x_0)}\,e^{i\ell\theta_{\vec{d}}}}$$

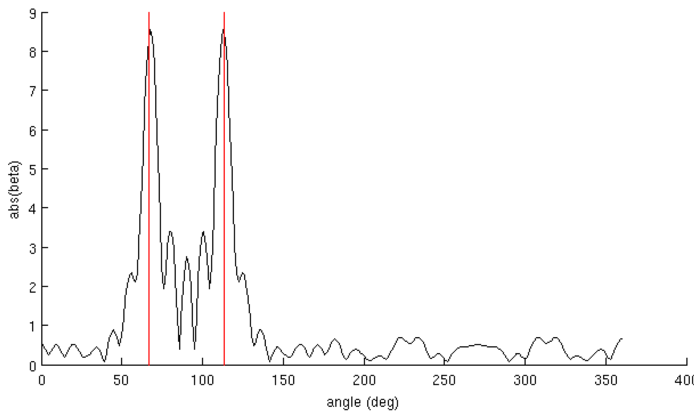
Test 2 sources , homogeneous medium NMLA stability.

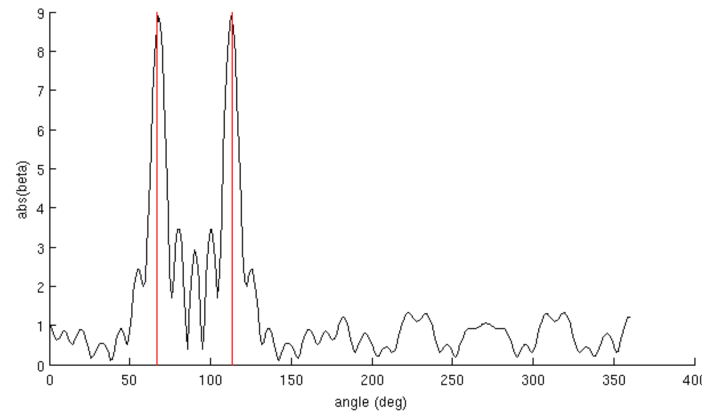
$|\beta(\theta_{\widehat{s}})|$ White noise (20%-40%)





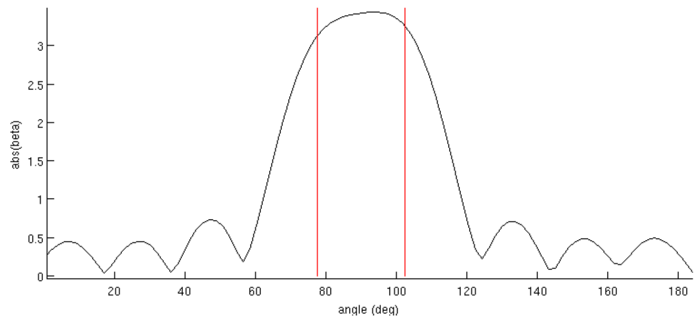
Correlated noise (20%-40%)

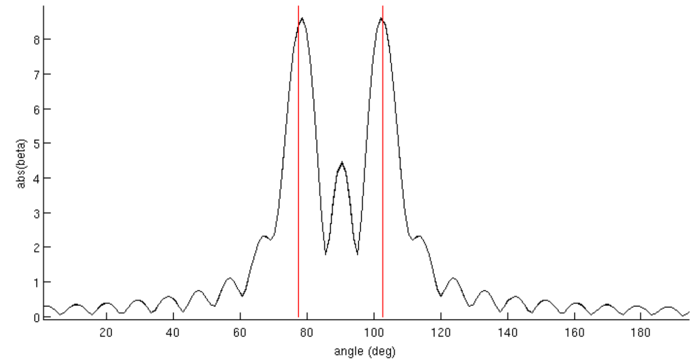


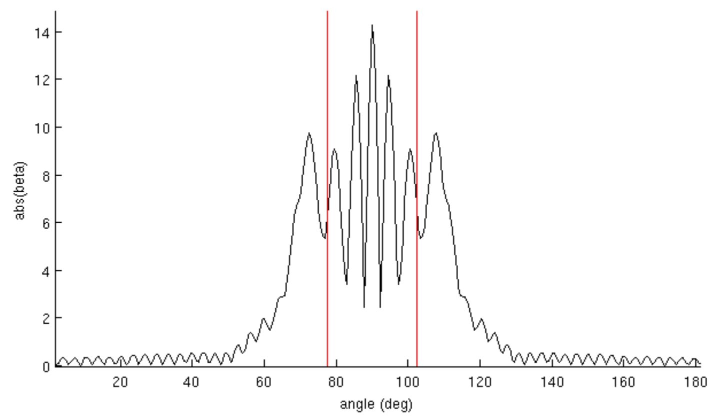


Red lines : exact ray angles.

Varying $\alpha = \frac{\omega r}{c(x_0)} \simeq L(\alpha)$: 10, 20, 50







NMLA 2^{nd} order / near field application ?

- $\alpha = \frac{\omega r}{c(x_0)} \simeq L(\alpha)$ bounds the number of Fourier modes, while we hope to recover dirac masses ...

- Cannot increase α because of the plane wave approximation.

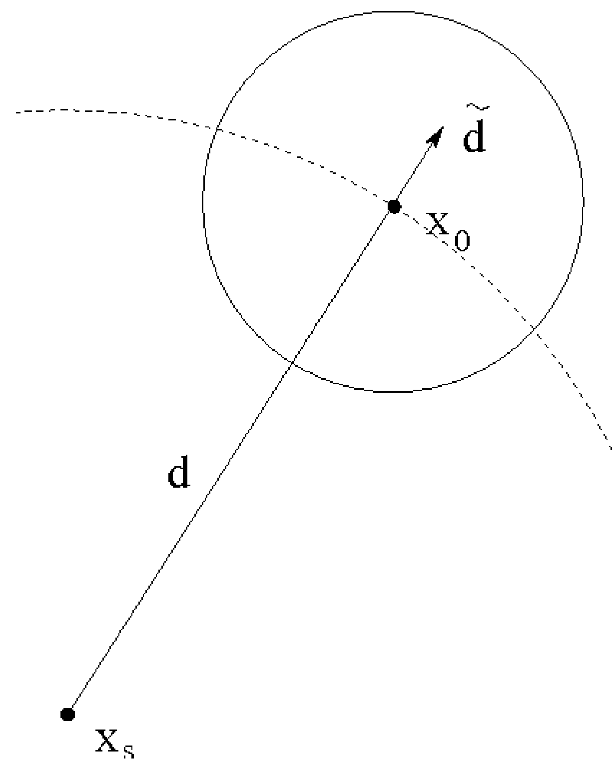
  $$\varphi(x) \simeq \varphi(x_0) + (x - x_0) \cdot \nabla\varphi(x_0) + \frac{1}{2}(x - x_0)^T H\varphi(x_0)\,(x - x_0) + ...$$

  Recall $x - x_0 = \alpha\frac{c(x_0)}{\omega}$, $r = \|x - x_0\|$.

$\rightarrow$ Need to estimate 2^{nd} order terms.

The simplest 2nd order approximation

Assume only one ray in the solution.
Constant curvature HF asymptotics

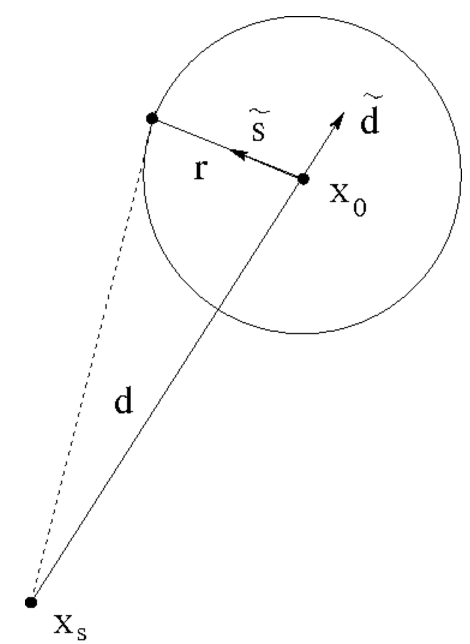


corresponds to the point source solution $H_0^1(\frac{\omega}{c}|x - x_s|)$.

Approximate NMLA data with H_0^1

$$U_\alpha(\vec{s}) \simeq \frac{A_0(x_0)}{|\frac{i}{4}H_0^{(1)}\left(\frac{\omega}{c}|d\,d + r\,\vec{s}|\right)|}\, e^{i\omega(\varphi(x_0)-d)}\,\frac{i}{4}H_0^{(1)}\left(\frac{\omega}{c}|d\,\vec{d} + r\,\vec{s}|\right)$$

$\vec{d}$ and d yet to be found



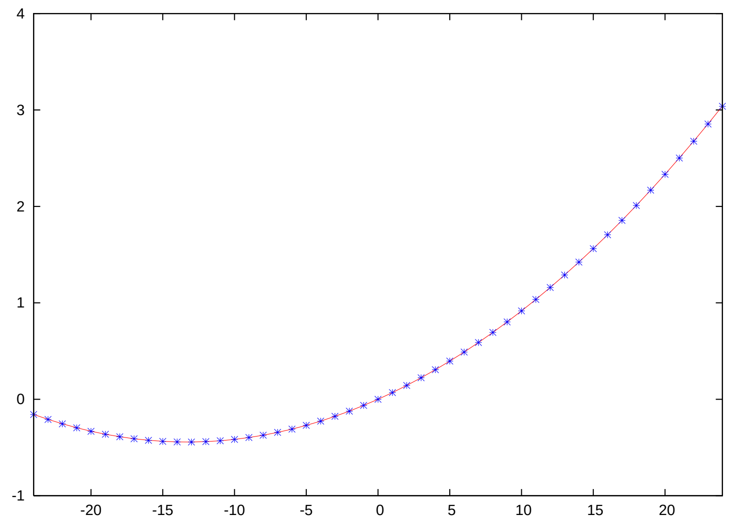
$\theta_{\vec{d}}$ and $\frac{1}{d}$ are the local

ray direction and mean curvature.

Use FMM type asymptotic expansions $\gamma = \frac{\omega\, d}{c(x_0)}$ is the large parameter. This "new ansatz" yields a curvature correction to the NMLA Fourier modes

$$\widehat{\beta}_\ell \simeq A e^{i\omega\varphi(x_0)} e^{i\left(\ell\theta_{\vec{d}} + \boxed{\frac{(\ell^2 - \frac{1}{4})}{2\gamma}}\right)}$$

$\frac{1}{i}\log(\frac{\widehat{\beta}_\ell}{\widehat{\beta}_0})$ versus ℓ

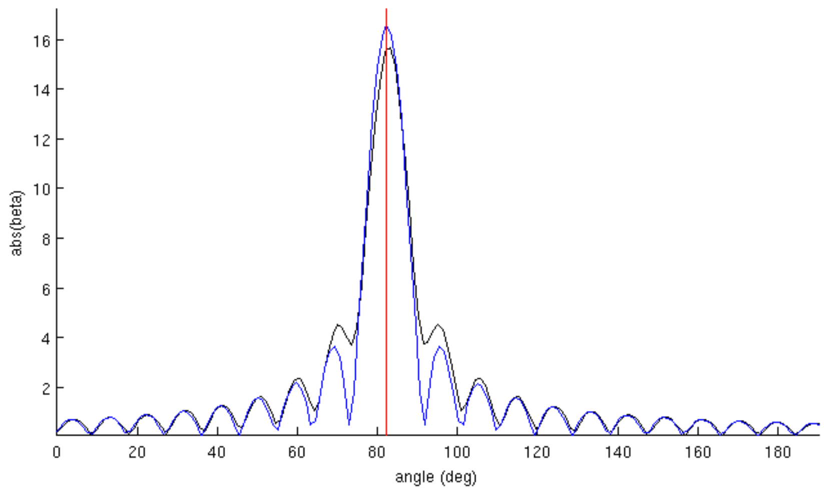


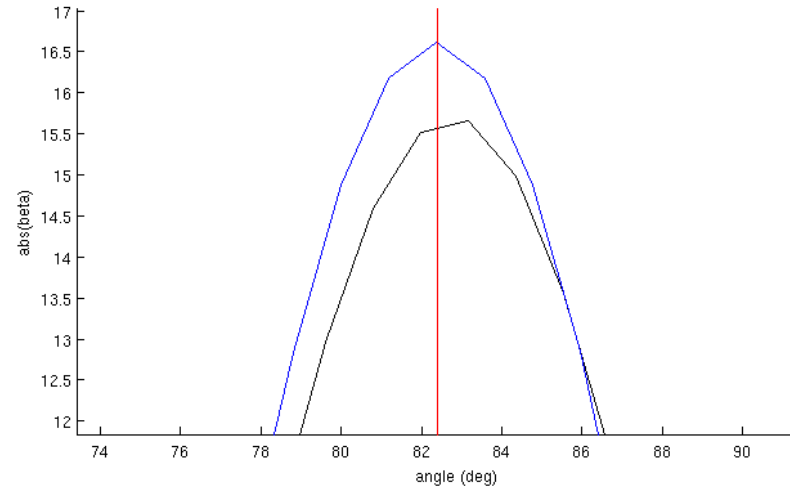
Recall : Plane wave approximation Fourier modes were

$$\widehat{\beta}_\ell \simeq A e^{i\omega\varphi(x_0)} e^{i\ell\theta_{\vec{d}}}$$

Test 1 source , homogeneous medium

Black : NMLA - Blue : NMLA 2nd order - Red : exact direction.





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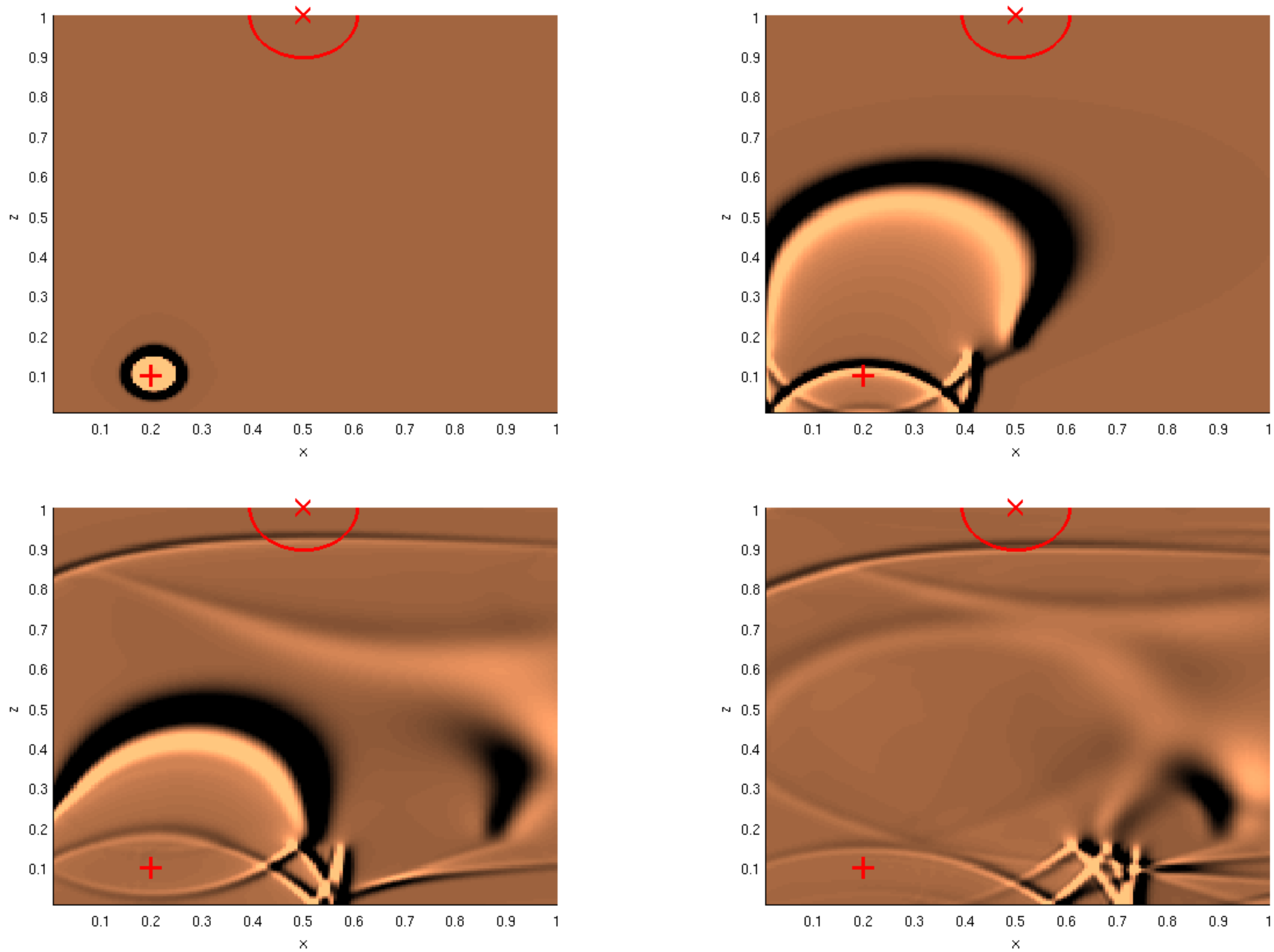
zoom

Synthetic data numerical simulation (snapshots)

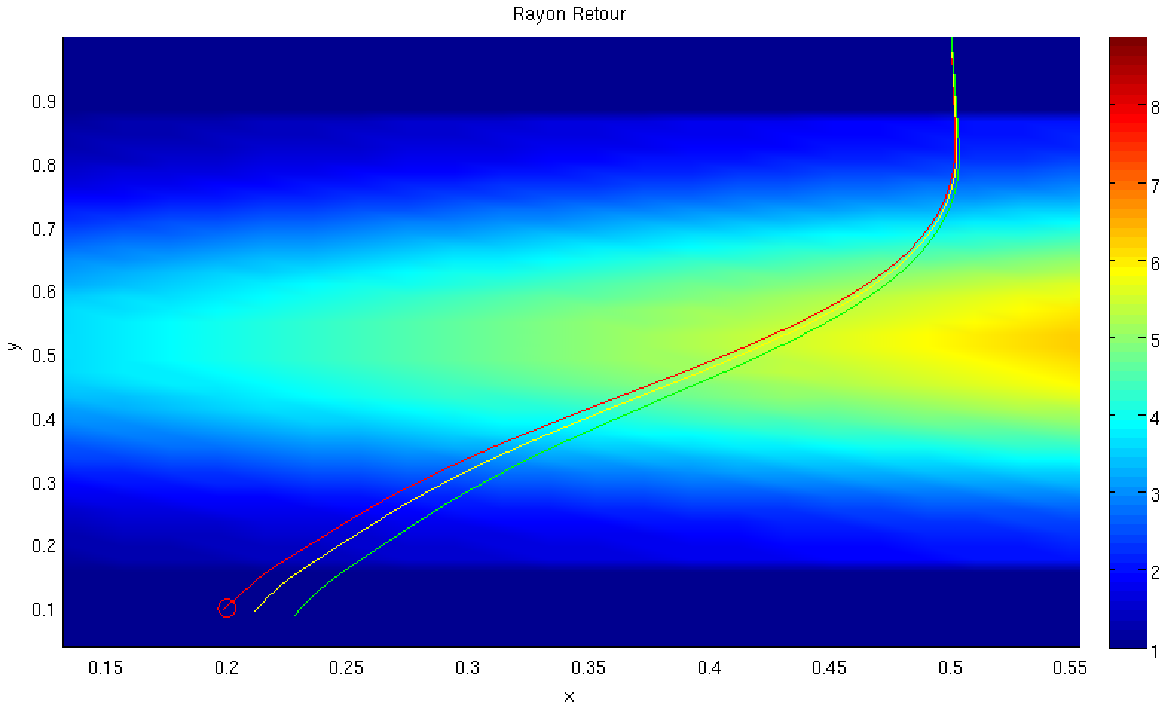


Generated using standard FDTD + ABCs

Source Point localization in Heteogeneous medium

Synthetic data - backward ray tracing using NMLA output (red) versus
Radon (green) and PWD (yellow).

Ray backward propagation and velocity model



Center of red circle is the exact source localization.