Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions

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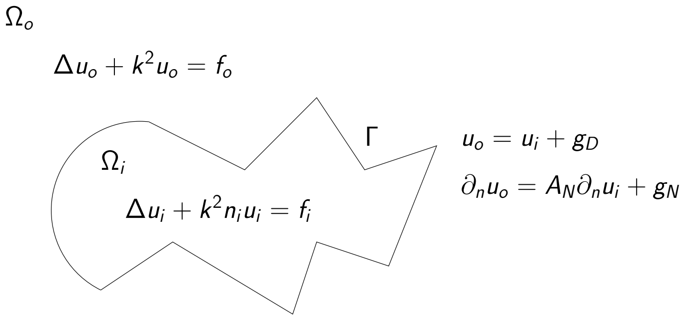
Helmholtz transmission problem (one single penetrable obstacle)

SRC



Data: $f_i \in L^2(\Omega_i)$, $f_o \in L^2_{\text{comp}}(\Omega_o)$, $g_D \in H^1(\Gamma)$, $g_N \in L^2(\Gamma)$,
$A_N > 0$, $n_i > 0$.

Solution exists and is unique for Ω_i Lipschitz and $k \in \mathbb{C} \setminus \{0\}$ with $\Im k \geq 0$ (Torres, Welland (1999)).

Goal and motivation

From Fredholm theory we have

$$\left\| \begin{pmatrix} u_i \\ u_o \end{pmatrix} \right\|_{\Omega_{i/o}} \leq \mathcal{C}_1 \left\| \begin{pmatrix} f_i \\ f_o \end{pmatrix} \right\|_{\Omega_{i/o}} + \mathcal{C}_2 \left\| \begin{pmatrix} g_D \\ g_N \end{pmatrix} \right\|_{\Gamma}$$

Goal: find out how $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on k, n_i, and A_N and deduce results about resonances.

Motivation: increasing interest in NA of Helmholtz problems with variable wavenumber:

- Brown, Gallistl, Peterseim (2015)
- Barucq, Chaumont-Frelet, Gout (2015)
- Ohlberger, Verfürth (2016)
- Graham, Sauter (in preparation)

and with random wavenumber (from "UQ" perspective):

- Feng, Lin, Lorton (2015).

Plan of talk

- Part 1: $n_i < 1$ ☺
- Part 2: $n_i > 1$ ☹
- Part 3: $n_i > 1$ 😐

(For simplicity, take $g_D = g_N = 0$.)

“Cut-off resolvent”: $R_\chi(k)$

Solution operator:

$$R(k, \cancel{n_i, A_N}) : \begin{pmatrix} f_i \\ f_o \end{pmatrix} \mapsto \begin{pmatrix} u_i \\ u_o \end{pmatrix}.$$

Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$ s.t. $\chi_j \equiv 1$ in a neighbourhood of Ω_i.
Let

$$R_\chi(k) := \chi_1 R(k) \chi_2,$$

then

$$R_\chi(k) : L^2(\Omega_i) \oplus L^2(\Omega_o) \to H^1(\Omega_i) \oplus H^1(\Omega_o).$$

Can show $R_\chi(k)$ is holomorphic on $\Im k > 0$.
Resonances: poles of meromorphic continuation of $R_\chi(k)$ to $\Im k < 0$.

Part 1: $n_i < 1$ ☺

- Cardoso, Popov, Vodev (1999):

Ω_i smooth, convex, with strictly positive curvature,
$n_i < 1$, $A_N > 0$,

$$\|R_\chi(k)\|_{L^2\to L^2} \leq \frac{C_0}{k}, \qquad \|R_\chi(k)\|_{L^2\to H^1} \leq C_1 \quad \text{for all } k \geq k_0 \quad (\star)$$

C_0, C_1 not explicit in n_i, A_N.

- Moiola, S. (2017):

Ω_i star-shaped Lipschitz obstacle,

$$n_i \leq \frac{1}{A_N} \leq 1$$

bound $(\star)$ with C_0, C_1 explicit in n_i, A_N (and geometry).

(One of) the Moiola-S. bounds in gory detail...

Ω_i is star-shaped, $g_N = g_D = 0$, $k > 0$, and

$$n_i \leq \frac{1}{A_N} \leq 1$$

Given $R > 0$ such that $\operatorname{supp} f_o \subset B_R$, let $D_R := \Omega_o \cap B_R$.

$$\|\nabla u_i\|^2_{L^2(\Omega_i)} + k^2 n_i \|u_i\|^2_{L^2(\Omega_i)} + \frac{1}{A_N}\left(\|\nabla u_o\|^2_{L^2(D_R)} + k^2 \|u_o\|^2_{L^2(D_R)}\right)$$
$$\leq \left[4\,\mathrm{diam}(\Omega_i)^2 + \frac{1}{n_i}\left(2R + \frac{d-1}{k}\right)^2\right]\|f_i\|^2_{L^2(\Omega_i)}$$
$$+ \frac{1}{A_N}\left[4R^2 + \left(2R + \frac{d-1}{k}\right)^2\right]\|f_o\|^2_{L^2(D_R)}.$$

Link with resonances

- Vodev (1999):

If $\exists C_0,\ k_0 > 0$ s.t.

$$\|R_\chi(k)\|_{L^2\to L^2} \le \frac{C_0}{k} \quad \text{for all } k \ge k_0 \quad (\star)$$

then $\exists \widetilde{C_0},\ \widetilde{k_0},\ \delta > 0$ s.t. $R_\chi(k)$ is holomorphic in

$$\Re k \ge \widetilde{k_0}, \quad \Im k \ge -\delta$$

and satisfies

$$\|R_\chi(k)\|_{L^2\to L^2} \le \frac{\widetilde{C_0}}{k} \quad \text{in this region,}$$

i.e. $\exists$ a strip (width δ) underneath $\mathbb{R}$ free of resonances.

How the Moiola-S. bound was obtained

Multiply the PDE by the "test function"

$$
\begin{aligned}
A_N \left(\mathbf{x} \cdot \nabla u - \mathrm{i}kRu + \frac{d-1}{2} u \right) \qquad & \text{in } \Omega_i, \\
\mathbf{x} \cdot \nabla u - \mathrm{i}kRu + \frac{d-1}{2} u \qquad & \text{in } D_R, \\
\mathbf{x} \cdot \nabla u - \mathrm{i}k|\mathbf{x}|u + \frac{d-1}{2} u \qquad & \text{in } \mathbb{R}^d \setminus D_R,
\end{aligned}
$$

and integrate by parts.

These type of test functions for Helmholtz introduced by Morawetz in 1960s/1970s.

Part 2: $n_i > 1$

- Popov, Vodev (1999):

Ω_i smooth, convex, with strictly positive curvature,
$n_i > 1$, $A_N > 0$,

$\exists$ complex sequence $(k_j)_{j=1}^{\infty}$, with $|k_j| \to \infty, \Re k_j \geq 1$, and
$0 > \Im k_j = \mathcal{O}(|k_j|^{-\infty})$ s.t.

$$\|R_\chi(k_j)\|_{L^2 \to L^2} \text{ blows up super-algebraically}$$

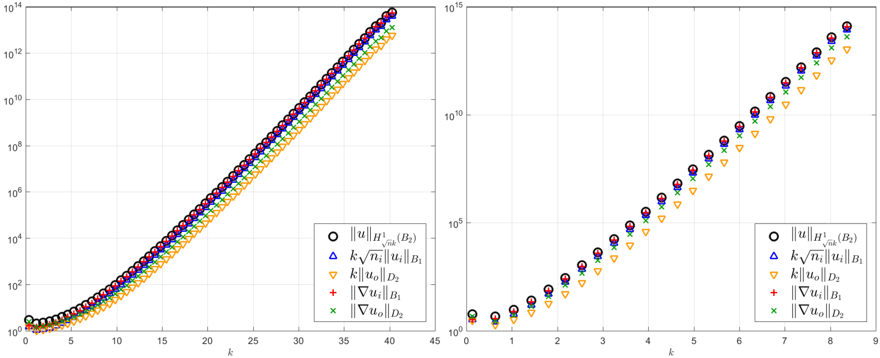
- Bellassoued (2003)

Ω_i smooth, $n_i > 0$, $A_N > 0$, $\exists C_1, C_2, k_0 > 0$, s.t.

$$\|R_\chi(k)\|_{L^2 \to L^2} \leq C_1 \exp(C_2 k) \quad \text{for all } k \geq k_0$$

Part 2: $n_i > 1$ ☹

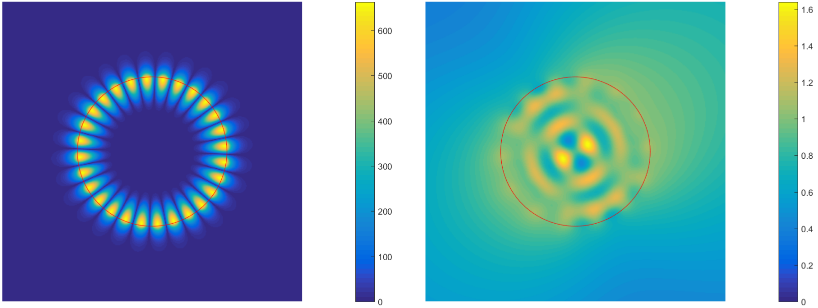
Ω_i = unit ball in 2-d



Left: $n_i = 3$ Right: $n_i = 10$.

Part 3: $n_i > 1$

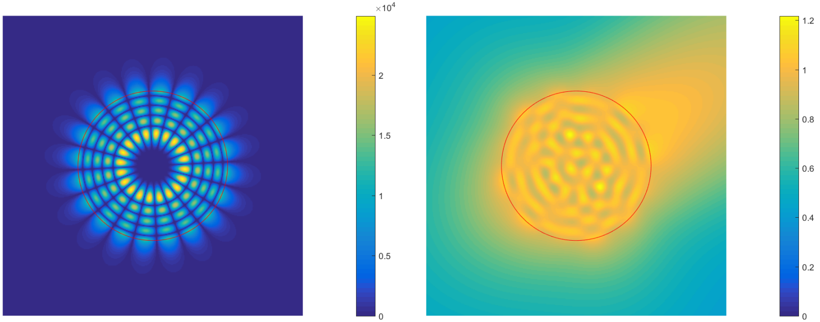
Ω_i = unit ball in 2-d, $\quad n_i = 100$



Left: $k = 1.631889489833541$ Right: $k_3 = 1.631889489833$

Part 3: $n_i > 1$

Ω_i = unit ball in 2-d, $\quad n_i = 100$



Left: $k_2 = 2.722270996079$ $\qquad$ Right: $k = 2.72227$

Summary of talk

- Part 1: $n_i < 1$ ☺ - resolvent bounded uniformly in k

- Part 2: $n_i > 1$ ☹ - exponential growth through $(k_j)_{j=1}^{\infty}$

- Part 3: $n_i > 1$ 😐 - growth very sensitive to $(k_j)_{j=1}^{\infty}$

Further information

Distribution of resonances

- Cardoso, Popov, Vodev (2001)
- Galkowski (2015)

Detailed bounds in the case that Ω_i is a ball

- Capdeboscq (2012)
- Capdeboscq, Leadbetter, Parker (2012)
- (summarised in Alberti, Capdeboscq (2016))