Wave propagation in complex domains — UCL — 30.3.2017

Sobolev spaces on non-Lipschitz domains

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with S.N. Chandler-Wilde (Reading) and D.P. Hewett (UCL)

From Dave's talk

Questions related to scattering by fractal screens (well-posedness, unique formulation, "audibility", BEM and prefractal convergence. . .) crucially depend on properties of Sobolev spaces on rough sets.

E.g.: for which compact K, open Γ and $s \in \mathbb{R}$ are
$\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset K\} = \{0\}$ and
$\mathcal{D}(\Gamma)$ dense in $\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Gamma}\}$?

Many results available (Maz'ya, Triebel, Polking, Adams, Hedberg,. . .) but not entirely clear/satisfactory/useful for us.

We need to learn more about Sobolev spaces on non-Lipschitz sets!

Part I

Definitions and duality

Basic definitions I: Sobolev spaces on $\mathbb{R}^n$

For $k \in \mathbb{N}_0$, $$W^k := \{u \in L^2(\mathbb{R}^n) : \partial^{\boldsymbol{\alpha}} u \in L^2(\mathbb{R}^n),\ \forall |\boldsymbol{\alpha}| \le k\},$$

$$\|u\|_{W^k}^2 := \sum_{|\boldsymbol{\alpha}| \le k} \int_{\mathbb{R}^n} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}.$$

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For $s \in \mathbb{R}$, $$H^s := \{u \in \mathcal{S}^*(\mathbb{R}^n) : \hat{u} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \|u\|_{H^s} < \infty\},$$
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- $(H^s)^* = H^{-s}$, with duality pairing

$$\langle u, v \rangle_{H^{-s}\times H^s} := \int_{\mathbb{R}^n} \hat{u}(\boldsymbol{\xi}) \overline{\hat{v}(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}.$$

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- $H^s \subset C(\mathbb{R}^n)$ for $s > n/2$ (Sobolev embedding theorem).
  $\delta_{\mathbf{x}_0} \in H^s \iff s < -n/2$ $\quad (\langle \delta_{\mathbf{x}_0}, \phi \rangle = \overline{\phi(\mathbf{x}_0)})$.

Basic definitions II: Sobolev sp. on subsets of $\mathbb{R}^n$

Notation: $\Gamma \subset \mathbb{R}^n$ open, $F \subset \mathbb{R}^n$ closed, $K \subset \mathbb{R}^n$ compact.

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$$H^s_F := \{u \in H^s : \operatorname{supp} u \subset F\} = \{u \in H^s : u(\varphi) = 0 \ \forall \varphi \in \mathcal{D}(F^c)\}$$

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"Global" and "local" spaces:

$$\widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}} \subset H^s \subset \mathcal{D}^*(\mathbb{R}^n) \quad \xrightarrow{|_\Gamma} \quad H^s_0(\Gamma) \subset H^s(\Gamma) \subset \mathcal{D}^*(\Gamma).$$

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There exist many works on Sobolev (Besov,...) spaces on rough sets; most use intrinsic definitions on d-sets.
Analogous to $W^s(\Gamma)$, based on $L^p(\Gamma, \mu_d)$.
Related to spaces in $\mathbb{R}^n$ by traces.
See: Jonsson–Wallin, Strichartz.

Our spaces are different, more suited for integral equations and BEM.

Duality

Theorem

Let Γ *be any open subset of* $\mathbb{R}^n$ *and let* $s \in \mathbb{R}$.
Then $(H^s(\Gamma))^* = \widetilde{H}^{-s}(\Gamma)$ *and* $(\widetilde{H}^s(\Gamma))^* = H^{-s}(\Gamma)$ *with equal norms and* $\langle u, w \rangle_{H^{-s}(\Gamma)\times\widetilde{H}^s(\Gamma)} = \langle U, w \rangle_{H^{-s}\times H^s}$ *for any* $U \in H^{-s}, U|_\Gamma = u$.

Well-known for Lipschitz but not in general case.

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Main ideas of proof:

- H Hilbert, $V \subset H$ closed ssp, $\mathcal{H}$ unitary realisation of H^*, then $(V^{a,\mathcal{H}})^\perp = \{\psi \in \mathcal{H}, \langle \psi, \phi \rangle = 0 \;\forall \phi \in V\}^\perp$ is unitary realisation of V^*
- $H^{-s}_{\Gamma^c} = \{u \in H^{-s} : u(\psi) = \langle u, \psi \rangle = 0 \;\forall \psi \in \mathcal{D}(\Gamma)\} = (\widetilde{H}^s(\Gamma))^{a,H^{-s}}$
- Restriction operator $|_\Gamma$ is unitary isomorphism $|_\Gamma : (H^{-s}_{\Gamma^c})^\perp \to H^{-s}(\Gamma)$ (from identification of $H^{-s}(\Gamma)$ with $H^{-s}/H^{-s}_{\Gamma^c}$)
- Choose $V = \widetilde{H}^s(\Gamma)$, $H = H^s$, $\mathcal{H} = H^{-s}$

Sobolev space questions

We address the following questions:

- When does $E \subset \mathbb{R}^n$ support non-zero $u \in H^s$?
- When is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?
- When is $H^s(\Gamma) = H^s_0(\Gamma)$?
- For which spaces is $|_\Gamma$ an isomorphism?
- When are $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ interpolation scales?
- What's the limit of a sequence of Galerkin solutions to a variational problem on prefractals?

Part II

s-nullity

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Definition

Given $s \in \mathbb{R}$ we say that a set $E \subset \mathbb{R}^n$ is s-null if there are no non-zero elements of H^s supported in E.

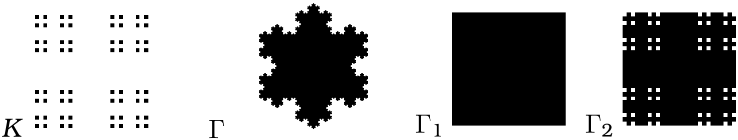
(I.e. if $H^s_F = \{0\}$ for every closed set $F \subset E$.)

Other terminology exists:
"$(-s)$-polar" (Maz'ya, Littman),
"set of uniqueness for H^s" (Maz'ya, Adams/Hedberg).

Relevance of s-nullity

For the screen scattering problem:

- For a compact screen K to be audible we need $H_K^{\pm 1/2} \neq \{0\}$.
- For the solution of the classical Dirichlet/Neumann BVP to be unique we need $H_{\partial\Gamma}^{\pm 1/2} = \{0\}$.
- Two screens Γ_1 and Γ_2 give the same scattered field for all incident waves if and only if $\Gamma_1 \ominus \Gamma_2$ is $\pm 1/2$-null.



s-nullity: basic results

- A subset of an s-null set is s-null.
- If E is s-null and $t > s$ then E is t-null.
- If E is s-null then has empty interior.
- If $s > n/2$ then E is s-null $\iff \mathrm{int}(E) = \emptyset$.
- For $s < -n/2$ there are no non-empty s-null sets.

Interesting cases: sets with empty interior and $-n/2 \leq s \leq n/2$.

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Non-trivial results:

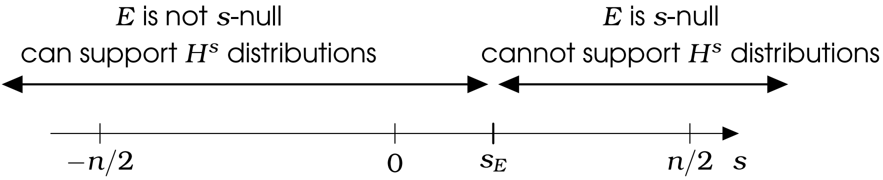
- The union of finitely many s-null closed sets is s-null.
- The union of countably many s-null Borel sets is s-null if $s \leq 0$.

Union of non-closed s-null sets for $s > 0$ is not s-null:
counterexample is $E_1 = \mathbb{Q}^n, E_2 = \mathbb{R}^n \setminus \mathbb{Q}^n, s > n/2$.

Nullity threshold

Definition

For every $E \subset \mathbb{R}^n$ with $\mathrm{int}(E) = \emptyset$ there exists $s_E \in [-n/2, n/2]$ such that E is s-null for $s > s_E$ and not s-null for $s < s_E$.
We call s_E the nullity threshold of E.



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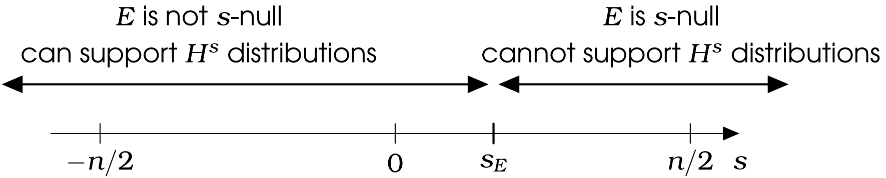

Q1: Given $E \subset \mathbb{R}^n$, can we determine s_E?

Q2: Given $s \in [-n/2, n/2]$, can we find some $E \subset \mathbb{R}^n$ for which $s_E = s$?

Q3: When is E s_E-null? (i.e. is the maximum regularity attained?)

We study separately sets with zero and positive Lebesgue measure.

Zero Lebesgue measure $\Rightarrow s_K \in [-n/2, 0]$

Let $K \subset \mathbb{R}^n$ be non-empty and compact. Then:

- $H^0_K = L^2(K) = \{0\} \iff m(K) = 0$.

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- If K is countable then $s_K = -n/2$ $\quad (H^s_K = \{0\} \Leftrightarrow s \geq -n/2)$.

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Theorem

If $m(K) = 0$, then $s_K = \dfrac{\dim_{\mathrm{H}} K - n}{2}$.

($\dim_{\mathrm{H}}$ =Hausdorff dimension, m =Lebesgue measure)

$$\dim_{\mathrm{H}} K = \inf\left\{d > 0 : H_K^{(d-n)/2} = \{0\}\right\}$$

This does not tell us if K is s_K-null; examples of both cases are possible.

Sharpens previous results by Littman (1967) and Triebel (1997).

Examples

Let $\Gamma \subset \mathbb{R}^n$ be non-empty and open.

- If Γ is C^0 then $s_{\partial\Gamma} \in [-1/2, 0]$.
- If Γ is $C^{0,\alpha}$ for some $0 < \alpha < 1$ then $s_{\partial\Gamma} \in [-1/2, -\alpha/2]$ (sharp).
- If Γ is Lipschitz then $s_{\partial\Gamma} = -1/2$ (and $H_{\partial\Gamma}^{-1/2} = \{0\}$).
- If Γ is Koch snowflake, $s_{\partial\Gamma} = \frac{\log 2}{\log 3} - 1 \approx -0.37$.

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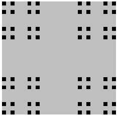
We can construct a "Cantor dust" $\mathcal{C}^n_\alpha \subset \mathbb{R}^n$ with given nullity threshold.
If prefractal edge lengths is $l_j = \alpha^j$, $0 < \alpha < 1/2$:



$$s_{\mathcal{C}^n_\alpha} = -\frac{n}{2}\left(1 + \frac{\log 2}{\log \alpha}\right) \in \left(-\frac{n}{2}, 0\right).$$

Choose $\alpha = 2^{-n/(2s+n)}$ to have $s_{\mathcal{C}^n_\alpha} = s$.

Can also define "thin" Cantor dusts which have $s_K = -n/2$.

Capacity

Our proofs rely on the following equivalence.
This follows from results by Grusin 1962, Littman 1967, Adams and Hedberg 1996 and Maz'ya 2011.

Theorem

For $s > 0$, K compact, $H_K^{-s} = \{0\} \iff \mathrm{cap}_s(K) = 0$, *where*

$$\mathrm{cap}_s(K) := \inf\{\|u\|_{H^s}^2 : u \in C_0^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } K\}.$$

This allows us to apply well-known results relating $\mathrm{cap}_s(E)$ to $\dim_{\mathrm{H}}(E)$ (see e.g. Adams and Hedberg 1996).
Requires relating different set capacities.

Positive Lebesgue measure $\Rightarrow s_K \in [0, n/2]$

Theorem (Polking, 1972)

There exists a compact set K for which $s_K = n/2$. Also, $H_K^{n/2} \neq \{0\}$.

Maximal nullity threshold is achieved.
Proof is constructive: "Swiss cheese set".
Also "open minus countable-dense" (e.g. $\mathbb{R}^n \setminus \mathbb{Q}^n$) are not $n/2$-null.
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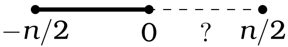
Open question: Do there exist sets K for which $s_K \in (0, n/2)$?



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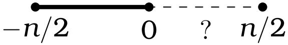

Our contribution:

Theorem

$\forall s_* \in (0, 1/2)$ the "fat" Cantor set $\mathcal{C}_{\alpha,\beta} \subset \mathbb{R}$ with

$$\alpha \in (0, 2^{-1/(1-2s_*)}), \quad \beta \in (0, 1-2\alpha), \quad l_j = \frac{1}{2^j}\left(1 - \beta\frac{1-(2\alpha)^j}{1-2\alpha}\right)$$

has nullity threshold $s_{\mathcal{C}_{\alpha,\beta}} \geq s_*$ (and $\chi_{\mathcal{C}_{\alpha,\beta}} \in H^{s_*}$).

Nullity of Cartesian products of sets

Let $n_1, n_2 \in \mathbb{N}$, and let $E_1 \subset \mathbb{R}^{n_1}$ and $E_2 \subset \mathbb{R}^{n_2}$ be Borel. Then

$$s_- \leq s_{E_1 \times E_2} \leq s_+, \quad \text{where}$$

$$s_- := \min\{s_{E_1},\ s_{E_2},\ s_{E_1} + s_{E_2}\},$$

$$s_+ := \begin{cases} \min\{s_{E_1},\ s_{E_2}\} & \text{if } m(E_1 \times E_2) = 0, \\ \min\{s_{E_1} + \frac{n_2}{2},\ s_{E_2} + \frac{n_1}{2}\} & \text{if } m(E_1 \times E_2) > 0. \end{cases}$$

$s_- \neq s_+$ is needed because s_{E_1}, s_{E_2} do not determine $s_{E_1 \times E_2}$:

$\exists E_j \subset \mathbb{R}$ such that $s_{E_1} = s_{E_2} = s_{E_3} = s_{E_1 \times E_2} = -1/2 \neq s_{E_3 \times E_3} = -1$.

Part III

Zero trace spaces

Comparison of the "zero trace" subspaces of $\mathbb{R}^n$

Recall definitions: for open $\Gamma \subset \mathbb{R}^n$

$$\widetilde{H}^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s}$$

$$H^s_{\overline{\Gamma}} := \{u \in H^s : \operatorname{supp} u \subset \overline{\Gamma}\}$$

$$\widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}} \subset H^s$$

When is $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$?

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Classical result (e.g. McLean)

Let $\Gamma \subset \mathbb{R}^n$ be C^0. Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

For smooth ($C^{k,1}$) domains and $s > 1/2$, $s - 1/2 \notin \mathbb{N}$, these spaces are kernel of trace operators.
Intuition fails for negative s: if $s < -n/2$, $\delta_{\mathbf{x}_0} \in \widetilde{H}^s(\Gamma)$ for any $\mathbf{x}_0 \in \partial\Gamma$.

Comparison of the "zero trace" subspaces of $\mathbb{R}^n$

Recall definitions: for open $\Gamma \subset \mathbb{R}^n$

$$\widetilde{H}^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s} \qquad \widetilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}} \subset H^s$$

$$H^s_{\overline{\Gamma}} := \{u \in H^s : \operatorname{supp} u \subset \overline{\Gamma}\}$$

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Theorem (negative example)

For every $n \in \mathbb{N}$, there exists a bounded open set $\Gamma \subset \mathbb{R}^n$ such that,

$$\forall s \geq -n/2 \qquad \widetilde{H}^s(\Gamma) \subsetneqq H^s_{\overline{\Gamma}},$$

$$\forall s > 0 \qquad \widetilde{H}^s(\Gamma) \subsetneqq \{u \in H^s : u = 0 \text{ a.e. in } \Gamma^c\} \subsetneqq H^s_{\overline{\Gamma}}.$$

Set Γ constructed using Cantor and Polking sets.

Zero trace spaces and $\mathrm{int}(\overline{\Gamma}) \setminus \Gamma$

We consider two classes of open sets.
First, open Γ that is a “nice domain minus small holes”, i.e. $\mathrm{int}(\overline{\Gamma})$ is C^0.

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Suppose that $\Gamma \subsetneq \mathrm{int}(\overline{\Gamma})$ and that $\mathrm{int}(\overline{\Gamma})$ is C^0. Then

$\exists \widetilde{s}_\Gamma \in [-n/2, n/2]$ s.t. $\quad \widetilde{H}^{s_-}(\Gamma) = H^{s_-}_{\overline{\Gamma}}, \quad \widetilde{H}^{s_+}(\Gamma) \subsetneq H^{s_+}_{\overline{\Gamma}} \quad \forall s_- < \widetilde{s}_\Gamma < s_+$.

If $m(\mathrm{int}(\overline{\Gamma}) \setminus \Gamma) = 0$ then $\widetilde{s}_\Gamma = \frac{n - \dim_{\mathrm{H}}(\mathrm{int}(\overline{\Gamma}) \setminus \Gamma)}{2}$.

Sets with $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \leq 1$

Second, we want to understand whether $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ for Γ "regular except at a few points", e.g. prefractal  .

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Theorem

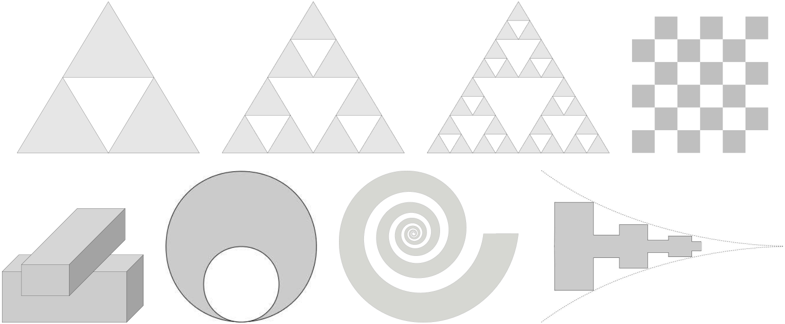
Fix $|s| \le 1$ if $n \ge 2$, $|s| \le 1/2$ if $n = 1$.
Let open $\Gamma \subset \mathbb{R}^n$ be C^0 except at countable $P \subset \partial\Gamma$, where P has at most finitely many limit points in every bounded subset of $\partial\Gamma$.
Then $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$.

E.g. union of disjoint C^0 open sets, whose closures intersect only in P.

Proof uses sequence of special Tartar's cutoffs (for $n = 2$, easier for $n \ge 3$) for $s = 1$, then duality and interpolation.

Examples of sets with $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \leq 1$

Examples of non-C^0 sets for which $\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$, $|s| \leq 1$:



Sierpinski triangle prefractal, (unbounded) checkerboard, double brick, inner and outer (double) curved cusps, spiral, Fraenkel's "rooms and passages".

Part IV

Relations between different spaces

When is $H_0^s(\Gamma) = H^s(\Gamma)$?

What about relation between spaces with and without "zero trace"?

Recall: $H_0^s(\Gamma) := \overline{\mathcal{D}(\Gamma)}^{H^s(\Gamma)} \subset H^s(\Gamma) := \{u|_\Gamma : u \in H^s\} \quad \subset \quad \mathcal{D}^*(\Gamma)$.

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Lemma

For open $\Gamma \subset \mathbb{R}^n$, $s \in \mathbb{R}$, $H^s_0(\Gamma) = H^s(\Gamma) \iff \widetilde{H}^{-s}(\Gamma) \cap H^{-s}_{\partial\Gamma} = \{0\}$.

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Corollary

For any open $\emptyset \neq \Gamma \subsetneq \mathbb{R}^n$, *there exists* $0 \leq s_0(\Gamma) \leq n/2$ *such that*

$H_0^{s_-}(\Gamma) = H^{s_-}(\Gamma)$ *and* $H_0^{s_+}(\Gamma) \subsetneq H^{s_+}(\Gamma)$ *for all* $s_- < s_0(\Gamma) < s_+$.

- $s_0(\Gamma) \geq -s_{\partial\Gamma}$ (nullity threshold), with equality if Γ is C^0.
- $s_0(\Gamma) \geq (n - \dim_{\mathrm{H}}\partial\Gamma)/2$.
- If Γ is C^0, then $0 \leq s_0(\Gamma) \leq 1/2$.
- If Γ is $C^{0,\alpha}$ then $\alpha/2 \leq s_0(\Gamma) \leq 1/2$.
- If Γ is Lipschitz, then $s_0(\Gamma) = 1/2$.
- If $\Gamma = \mathbb{R}^n \setminus F$, F countable, $s_0(\Gamma) = n/2$.

All bounds on s_0 can be achieved. Improvement on Caetano 2000.

Relations between "global" and "local" spaces

The relations between subspaces of $\mathcal{D}^*(\mathbb{R}^n)$ and $\mathcal{D}^*(\Gamma)$ are described by the restriction operator $|_\Gamma : \mathcal{D}^*(\mathbb{R}^n) \to \mathcal{D}^*(\Gamma)$.

- $|_\Gamma : H^s(\mathbb{R}^n) \to H^s(\Gamma)$ is continuous with norm one;
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- For $s \geq 0$, $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is injective and has dense image; if $s \in \mathbb{N}_0$ then it is isomorphism;
- If Γ is finite union of disjoint Lipschitz open sets, $\partial\Gamma$ is bounded, $s > -1/2$, $s + 1/2 \notin \mathbb{N}$, then $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is isomorphism;

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- If Γ is bounded, or Γ^c is bounded with non-empty interior, then $|_\Gamma : \widetilde{H}^s(\Gamma) \to H^s_0(\Gamma)$ is a unitary isomorphism $\iff$ $s \in \mathbb{N}_0$ (equivalent to say that H^s norm is local only for $s \in \mathbb{N}_0$);
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(If one defines $H^s_{00}(\Gamma)$, $s \geq 0$, from interpolation of $H^k_0(\Gamma)$, $k \in \mathbb{N}_0$, then for sufficiently smooth Γ (e.g. Lipschitz) $H^s_{00}(\Gamma) = \widetilde{H}^s(\Gamma)|_\Gamma$.)

A warning about interpolation

It is well-known that, for $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$, and $s = s_0(1-\theta) + s_1\theta$,

$$\big(H^{s_0}(\mathbb{R}^n), H^{s_1}(\mathbb{R}^n)\big)_\theta = H^s(\mathbb{R}^n) \qquad \text{with equal norms.}$$

In McLean's book *Strongly Elliptic Systems and Boundary Integral Equations* it is claimed (in Theorem B.8) that the same holds for $H^s(\Gamma) := \{u|_\Gamma : u \in H^s(\mathbb{R}^n)\}$, for arbitrary open sets $\Gamma \subset \mathbb{R}^n$.

THIS RESULT IS FALSE!
The interpolation result only holds for Γ sufficiently smooth (e.g. Lipschitz) and even then, equality of norms does not hold in general.

Simple counterexamples:
for a cusp domain in $\mathbb{R}^2$, $\{H^s(\Gamma), 0 \leq s \leq 2\}$ is not interpolation scale;
for open interval in $\mathbb{R}$, no normalisation of $(\widetilde{H}^0(\Gamma), \widetilde{H}^1(\Gamma))_{1/2}$ can give norm equal to $\widetilde{H}^{1/2}(\Gamma)$.

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We have studied (classical, fractional, Bessel-potential, Hilbert) Sobolev spaces on general open and closed subset of $\mathbb{R}^n$.

In particular we contributed to the questions:

- What are the duals of these spaces?
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Thank you!

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