Boundary Integral Equations on Complex Screens

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Workhop: Wave Propagation in Complex Domains
University College London
March 30, 2017

Screen Boundary Value Problems

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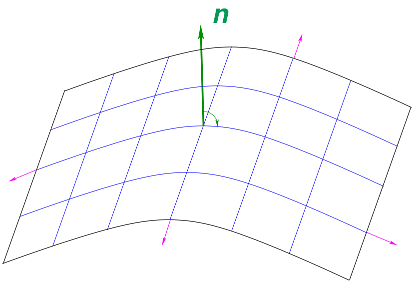
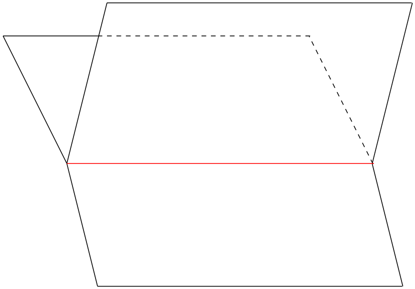
Examples: Screens in 3D





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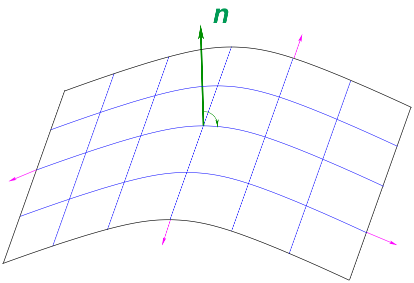
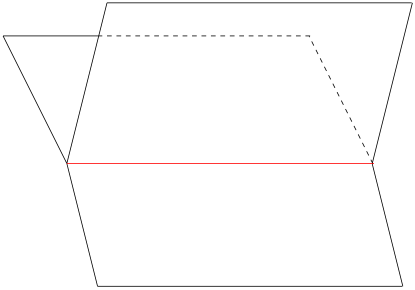


simple screen



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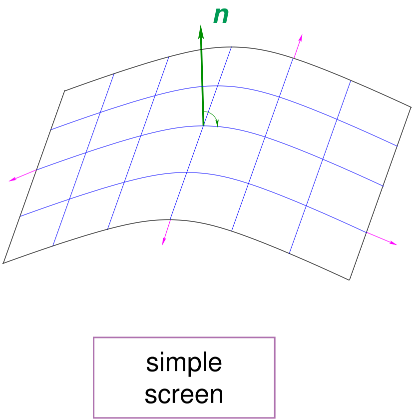

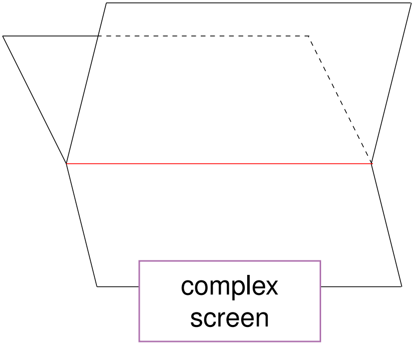

Simple Screens: BVP & BIE

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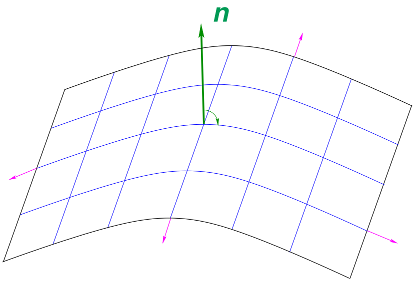

Exterior Dirichlet problem:

$$-\Delta u + u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma\,,$$
$$u = g \quad \text{on } \Gamma\,,$$
$$+ \text{ decay conditions at } \infty\,.$$

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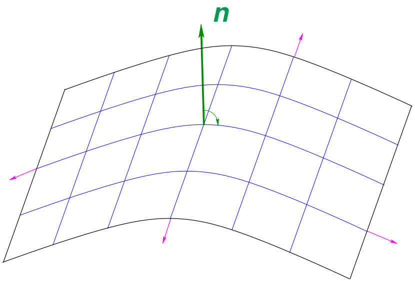

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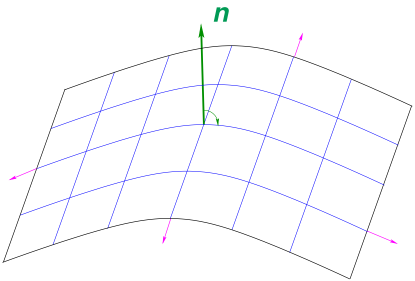

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- 1st-kind boundary integral equation: seek $\varphi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma) = H_{00}^{-\frac{1}{2}}(\Gamma)$

$$\langle \mathsf{V}\varphi, \varphi' \rangle := \int_\Gamma \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\varphi'(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) = -\int_\Gamma g(\boldsymbol{x})\varphi'(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})$$

for all $\varphi' \in \widetilde{H}^{-\frac{1}{2}}(\Gamma)$.

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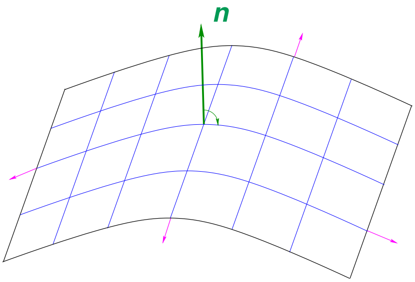

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Neumann jump: $\frac{\partial u}{\partial n}_+ - \frac{\partial u}{\partial n}_-$

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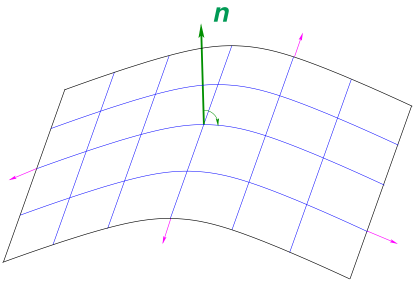

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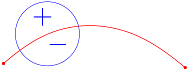

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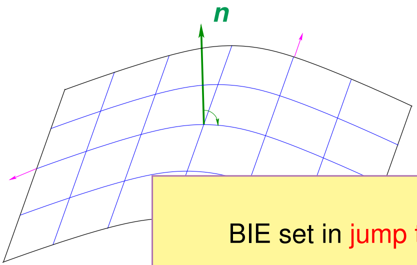

Exterior Dirichlet problem:

$\Delta u + u = 0$ in $\mathbb{R}^d \setminus \Gamma$,

,

at ∞ .

BIE set in jump trace space $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$

($H^{-\frac{1}{2}}$ with "zero boundary conditions")

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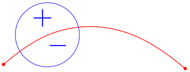

Complex Screens

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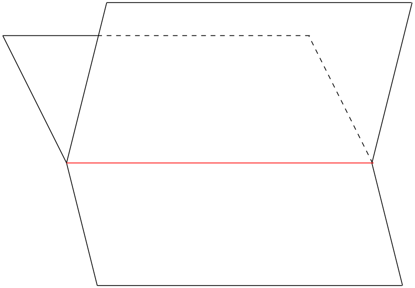

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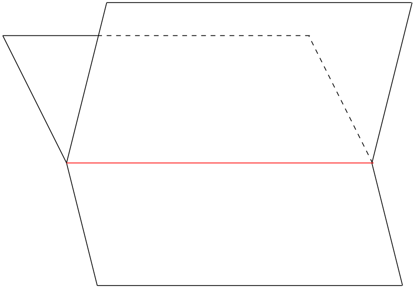

◁ Non-Lipschitz, non-orientable complex screen

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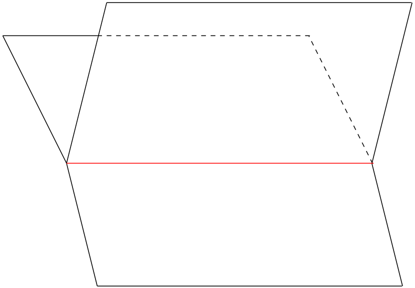

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(Lipschitz/orientable only locally away from "junction sets")

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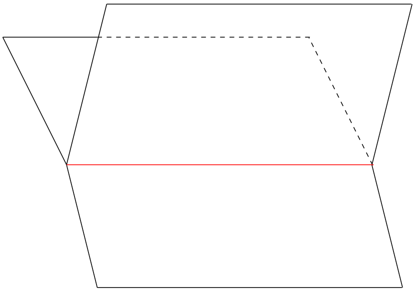

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Definition. Complex screen $\Gamma \subset \mathbb{R}^d$: $\exists$ mutually disjoint Lipschitz domains $\{\Omega_j\}_{j=1}^n$, such that

$$\Gamma \cap \partial\Omega_j \quad \text{is an orientable Lipschitz screen} \quad \forall j = 1, \ldots, n\,.$$

Coming up next

Going Quotient Space

Pradigm:

Get off Γ **!**

➢ Define trace spaces in $\mathbb{R}^d \setminus \Gamma$

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Recall: Ω bounded, Lipschitz ➥ trace spaces as quotient spaces:

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$\Gamma \doteq$ complex screen

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$\Gamma \triangleq$ complex screen

- Functions which jump across Γ : $H^1(\mathbb{R}^d \setminus \Gamma)$, $\boldsymbol{H}(\mathrm{div},\mathbb{R}^d \setminus \Gamma)$

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Closed subspaces:

$$H^1_{0,\Gamma}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset H^1(\mathbb{R}^d \setminus \Gamma)\ ,$$
$$\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\mathbb{R}^d) \subset \boldsymbol{H}(\mathrm{div},\mathbb{R}^d) \subset \boldsymbol{H}(\mathrm{div},\mathbb{R}^d \setminus \Gamma)\ .$$

Multi-Trace Spaces

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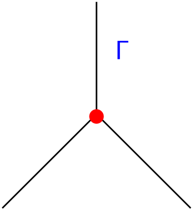
Mental picture:

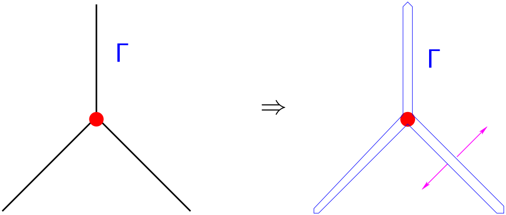


$\Rightarrow$

Multi-Trace Spaces

Mental picture:

thick screen



Multi-Trace Spaces

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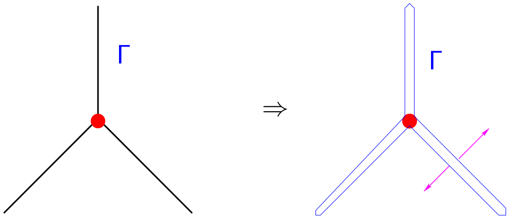
thick screen



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Multi-Trace Spaces

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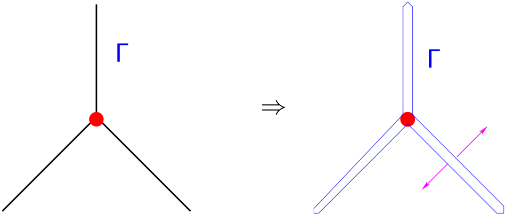

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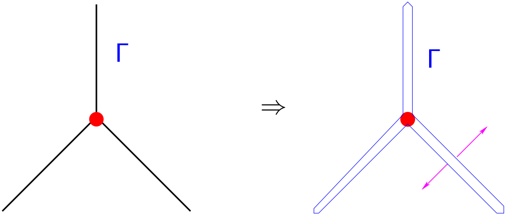

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Duality pairing: $\ll\ ,\ \gg : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{C}$

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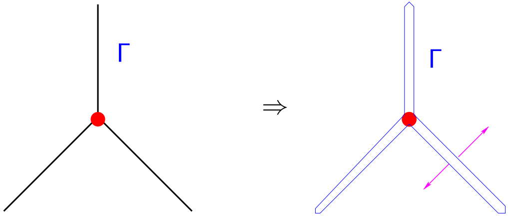

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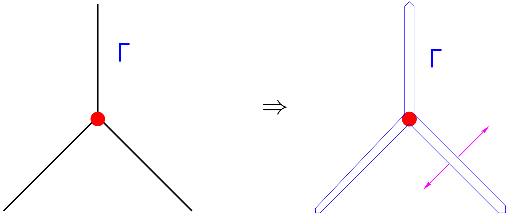

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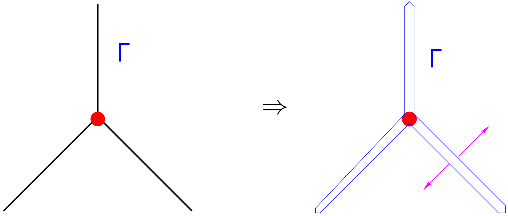

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$\updownarrow$

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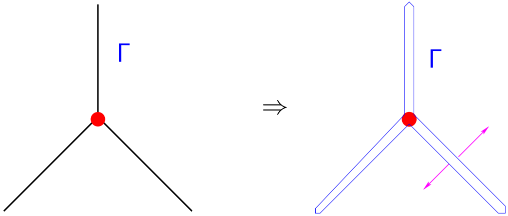

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$\ll\,,\,\gg$ induces isometric isomorphism: $(\mathbb{H}^{+\frac{1}{2}}(\Gamma))' \cong \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Single-Trace Spaces

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= single valued traces on Γ of globally defined functions

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Polarity by Green’s formula:

$$\begin{aligned} \dot{u} \in H^{+\frac{1}{2}}([\Gamma]) \quad &\Leftrightarrow \quad \ll \dot{u}, \dot{q} \gg = 0 \quad \forall \dot{q} \in H^{-\frac{1}{2}}([\Gamma])\,, \\ \dot{p} \in H^{-\frac{1}{2}}([\Gamma]) \quad &\Leftrightarrow \quad \ll \dot{v}, \dot{p} \gg = 0 \quad \forall \dot{v} \in H^{+\frac{1}{2}}([\Gamma])\,. \end{aligned}$$

Single-Trace Spaces

= single valued traces on Γ of globally defined functions

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“Natural trace spaces” through quotient spaces:

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Polarity by Green's formula:

no duality!

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Jump Spaces

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Definition. Spaces of Dirichlet- and Neumann jumps

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Example: Jump Spaces on Simple Screens

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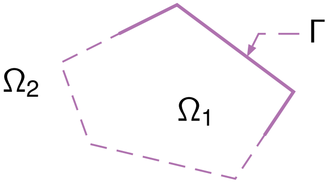

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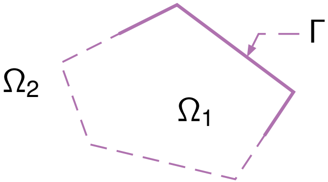

$\Gamma \subset \partial\Omega$ ("part of a boundary", simple screen)

$$\mathbb{H}^{+\frac{1}{2}}(\Gamma) \cong \left\{ \begin{array}{l} (v_1, v_2) \in H^{+\frac{1}{2}}(\Gamma) \times H^{+\frac{1}{2}}(\Gamma) : \\ v_1 - v_2 \in \widetilde{H}^{\frac{1}{2}}(\Gamma) \end{array} \right\}$$

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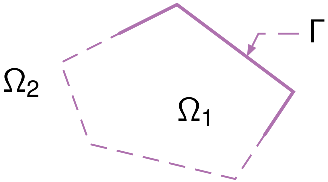

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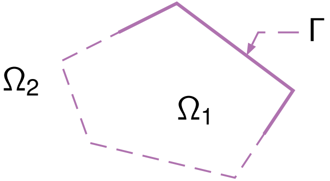

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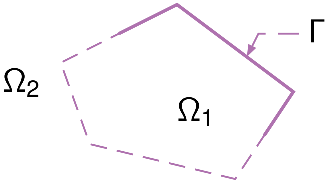

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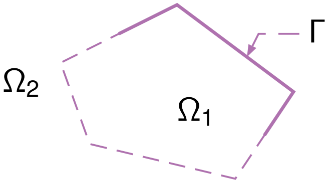

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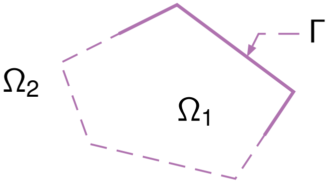

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➢ "Customary jump spaces" recovered

Coming up next

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By definition of duality pairing:

$$\int_{\mathbb{R}^3 \setminus \Gamma} \Delta u \, v - u \, \Delta v \, \mathrm{d}\boldsymbol{x} = \ll \gamma_N u, \gamma_D v \gg - \ll \gamma_D u, \gamma_N v \gg$$

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- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d \setminus \Gamma}\, u + u}) + \mathsf{DL}(\gamma_D\, u) + \mathsf{SL}(\gamma_N\, u)\ .$$

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Continuous potential operators:

$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D G(\boldsymbol{x} - \cdot), \dot{q} \gg ,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d \setminus \Gamma} u + u}) + \mathsf{DL}(\gamma_D u) + \mathsf{SL}(\gamma_N u) .$$

- Jump relations:

,

Potentials

Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$, $\gamma_D := \pi_{\mathbb{H}^{+\frac{1}{2}}(\Gamma)}$,

Neumann trace: $\gamma_N : H^1(\Delta, \mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$, $\gamma_N := \pi_{\mathbb{H}^{-\frac{1}{2}}(\Gamma)} \circ \mathbf{grad}$.

Continuous potential operators:

$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D\, G(\boldsymbol{x} - \cdot), \dot{q} \gg ,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N\, G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d\setminus\Gamma}\, u + u)} + \mathsf{DL}(\gamma_D\, u) + \mathsf{SL}(\gamma_N\, u) \;.$$

- Jump relations:

$$[\gamma_D]\,(\mathsf{SL}(\dot{p})) = 0 \;, \qquad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma) \;,$$

Potentials

Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$, $\gamma_D := \pi_{\mathbb{H}^{+\frac{1}{2}}(\Gamma)}$,

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Continuous potential operators:

$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D\, G(\boldsymbol{x} - \cdot), \dot{q} \gg ,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N\, G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \mathsf{N}\cancel{(-\Delta_{|\mathbb{R}^d \setminus \Gamma}\, u + u)} + \mathsf{DL}(\gamma_D\, u) + \mathsf{SL}(\gamma_N\, u)\ .$$

- Jump relations:

$$[\gamma_D]\,(\mathsf{SL}(\dot{p})) = 0\ , \qquad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma)\ ,$$

$$[\gamma_N]\,(\mathsf{DL}(\dot{u})) = 0 \qquad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma)\ .$$

Potentials

Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$, $\gamma_D := \pi_{\mathbb{H}^{+\frac{1}{2}}(\Gamma)}$,

Neumann trace: $\gamma_N : H^1(\Delta, \mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$, $\gamma_N := \pi_{\mathbb{H}^{-\frac{1}{2}}(\Gamma)} \circ \mathbf{grad}$.

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$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D\, G(\boldsymbol{x} - \cdot), \dot{q} \gg \;,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N\, G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d \setminus \Gamma} u + u)} + \mathsf{DL}(\gamma_D u) + \mathsf{SL}(\gamma_N u)\;.$$

- Jump relations:

$$[\gamma_D]\,(\mathsf{SL}(\dot{p})) = 0\;, \qquad [\gamma_N]\,(\mathsf{SL}(\dot{p})) = [\dot{p}] \qquad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma)\;,$$

$$[\gamma_N]\,(\mathsf{DL}(\dot{u})) = 0 \qquad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma)\;.$$

Potentials

Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$, $\gamma_D := \pi_{\mathbb{H}^{+\frac{1}{2}}(\Gamma)}$,

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Continuous potential operators:

$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D G(\boldsymbol{x} - \cdot), \dot{q} \gg ,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d \setminus \Gamma} u + u)} + \mathsf{DL}(\gamma_D u) + \mathsf{SL}(\gamma_N u) .$$

- Jump relations:

$$[\gamma_D](\mathsf{SL}(\dot{p})) = 0 , \qquad [\gamma_N](\mathsf{SL}(\dot{p})) = [\dot{p}] \qquad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma) ,$$

$$[\gamma_D](\mathsf{DL}(\dot{u})) = [\dot{u}] , \qquad [\gamma_N](\mathsf{DL}(\dot{u})) = 0 \qquad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma) .$$

Potentials

Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \rightarrow \mathbb{H}^{+\frac{1}{2}}(\Gamma)$, $\gamma_D := \pi_{\mathbb{H}^{+\frac{1}{2}}(\Gamma)}$,

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$$\mathsf{SL} := \mathsf{N} \circ \gamma_D' : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{SL}(\dot{q}))(\boldsymbol{x}) = \ll \gamma_D\, G(\boldsymbol{x} - \cdot), \dot{q} \gg ,$$

$$\mathsf{DL} := -\mathsf{N} \circ \gamma_N' : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \rightarrow H^1(\Delta, \mathbb{R}^d \setminus \Gamma), \quad (\mathsf{DL}(\dot{v}))(\boldsymbol{x}) = \ll -\gamma_N\, G(\boldsymbol{x} - \cdot), \dot{v} \gg$$

- representation formula:

$$u = \cancel{\mathsf{N}(-\Delta_{|\mathbb{R}^d \setminus \Gamma}\, u + u)} + \mathsf{DL}(\gamma_D\, u) + \mathsf{SL}(\gamma_N\, u) \; .$$

- Jump relations:

Jumps here!

$$[\gamma_D]\,(\mathsf{SL}(\dot{p})) = 0 \,, \qquad [\gamma_N]\,(\mathsf{SL}(\dot{p})) = [\dot{p}] \qquad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma) \,,$$

$$[\gamma_D]\,(\mathsf{DL}(\dot{u})) = [\dot{u}] \,, \qquad [\gamma_N]\,(\mathsf{DL}(\dot{u})) = 0 \qquad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma) \,.$$

Boundary Integral Operators

Boundary Integral Operators

Single layer operator:

Boundary Integral Operators

Single layer operator: $\quad \mathsf{V} := \gamma_D \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \, \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator:

Boundary Integral Operators

Single layer operator: $\quad \mathsf{V} := \gamma_D \,\mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\quad \mathsf{W} := \gamma_N \,\mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \rightarrow \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \, .$$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \, \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \, \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels. **=** the single trace spaces!

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \; .$$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \, \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \, \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \; .$$

Ellipticity.

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \, \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \, \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \; .$$

Ellipticity. There is $C > 0$: for all $\dot{q} \in \widetilde{H}^{-\frac{1}{2}}([\Gamma])$, $\dot{v} \in \widetilde{H}^{+\frac{1}{2}}([\Gamma])$

$$\ll \mathsf{V}\dot{q}, \dot{q} \gg \geq C \, \|\dot{q}\|^2_{\widetilde{H}^{-\frac{1}{2}}([\Gamma])} \quad , \quad \ll \mathsf{W}\dot{v}, \dot{v} \gg \geq C \, \|\dot{v}\|^2_{\widetilde{H}^{+\frac{1}{2}}([\Gamma])} \; .$$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \,\mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \,\mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \; .$$

Ellipticity. There is $C > 0$: for all $\dot{q} \in \widetilde{H}^{-\frac{1}{2}}([\Gamma])$, $\dot{v} \in \widetilde{H}^{+\frac{1}{2}}([\Gamma])$

$$\ll \mathsf{V}\dot{q}, \dot{q} \gg \geq C \, \|\dot{q}\|^2_{\widetilde{H}^{-\frac{1}{2}}([\Gamma])} \quad , \quad \ll \mathsf{W}\dot{v}, \dot{v} \gg \geq C \, \|\dot{v}\|^2_{\widetilde{H}^{+\frac{1}{2}}([\Gamma])} \; .$$

Boundary Integral Operators

Single layer operator: $\mathsf{V} := \gamma_D \mathsf{SL} : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$

Hypersingular operator: $\mathsf{W} := \gamma_N \mathsf{DL} : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Kernels.

$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \; .$$

Ellipticity. There is $C > 0$: for all $\dot{q} \in \widetilde{H}^{-\frac{1}{2}}([\Gamma])$, $\dot{v} \in \widetilde{H}^{+\frac{1}{2}}([\Gamma])$

$$\ll \mathsf{V}\dot{q}, \dot{q} \gg \geq C \|\dot{q}\|^2_{\widetilde{H}^{-\frac{1}{2}}([\Gamma])} \quad , \quad \ll \mathsf{W}\dot{v}, \dot{v} \gg \geq C \|\dot{v}\|^2_{\widetilde{H}^{+\frac{1}{2}}([\Gamma])} \; .$$

► Isomorphisms: $\mathsf{V} : \widetilde{H}^{-\frac{1}{2}}([\Gamma]) \to H^{+\frac{1}{2}}([\Gamma])$, $\mathsf{W} : \widetilde{H}^{+\frac{1}{2}}([\Gamma]) \to H^{-\frac{1}{2}}([\Gamma])$

Coming up next

Tangential Trace Spaces

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space:

Tangential Trace Spaces

Domain spaces:

$$\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$$

Multi-trace space:

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)/\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$$

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space:

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)\subset\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)\subset\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma) := \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)/\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]) := \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)/\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)$

Tangential Trace Spaces

Domain spaces: $\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)/\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)/\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:
$$\begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C} \,, \\ \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x := \int_{\mathbb{R}^3 \setminus \Gamma} \mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \, \mathrm{d}\boldsymbol{x} \,. \end{cases}$$

Tangential Trace Spaces

Domain spaces:

$$\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$$

Multi-trace space:

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$$

Single-trace space:

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$$

Duality pairing:

$$\begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C} \,, \\ \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x := \int_{\mathbb{R}^3 \setminus \Gamma} \mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \, \mathrm{d}\boldsymbol{x} \,. \end{cases}$$

Self-duality!

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:
$$\begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C} \,, \\ \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x := \int_{\mathbb{R}^3 \setminus \Gamma} \mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \, \mathrm{d}\boldsymbol{x} \,. \end{cases}$$

Tangential jump space:

Tangential Trace Spaces

Domain spaces: $$\boxed{\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)\subset\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)\subset\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)}$$

Multi-trace space: $$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma):=\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)/\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)$$

Single-trace space: $$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]):=\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)/\boldsymbol{H}_0(\mathbf{curl},\mathbb{R}^3\setminus\Gamma)$$

Duality pairing: $$\begin{cases}\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)\times\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)\to\mathbb{C}\;, \\ \ll\dot{\mathbf{u}},\dot{\mathbf{v}}\gg_x:=\int_{\mathbb{R}^3\setminus\Gamma}\mathbf{curl}\,\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\boldsymbol{x}\;.\end{cases}$$

Tangential jump space: $$\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]):=(\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]))'$$

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:

$$\begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C} \,, \\ \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x := \int_{\mathbb{R}^3 \setminus \Gamma} \mathbf{curl}\,\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} \,\mathrm{d}\boldsymbol{x} \,. \end{cases}$$

Tangential jump space: $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := (\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]))'$

Jump operator:

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) / \boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:

$$\begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C} , \\ \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x := \int_{\mathbb{R}^3 \setminus \Gamma} \mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \, \mathrm{d}\boldsymbol{x} . \end{cases}$$

Tangential jump space: $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := (\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]))'$

Jump operator:

$$[\,]_x : \begin{cases} \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) & \to & \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) , \\ \dot{\mathbf{u}} & \mapsto & \{\dot{\mathbf{v}} \to \ll \dot{\mathbf{u}}, \dot{\mathbf{v}} \gg_x\} . \end{cases}$$

Tangential Trace Spaces

Domain spaces: $\boxed{\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \subset \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)}$

Multi-trace space: $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)/\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Single-trace space: $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) := \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)/\boldsymbol{H}_0(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$

Duality pairing:

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isometry on factor space

NEW: Surface Differential Operators

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Domain sequences:

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❶: $$H^1(\mathbb{R}^3 \setminus \Gamma) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}, \mathbb{R}^3 \setminus \Gamma)\ ,$$

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\downarrow \pi_x & & \downarrow \pi_t & & \downarrow \pi_n \\
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D_Γ on Jump Spaces

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▶ $\mathbf{grad}_\Gamma^* = \mathbf{curl}_\Gamma \quad \leftrightarrow \quad \mathbf{curl}_\Gamma^* = \mathbf{grad}_\Gamma$

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- by duality:

$$\begin{array}{ccccc}
H^{\frac{1}{2}}([\Gamma]) & \xrightarrow{\mathbf{grad}_\Gamma} & \mathbf{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, [\Gamma]) & \xrightarrow{\mathbf{curl}_\Gamma} & H^{-\frac{1}{2}}([\Gamma]), \\
\| & & \| & & \| \\
\widetilde{H}^{-\frac{1}{2}}([\Gamma]) & \xleftarrow{\mathbf{grad}_\Gamma^* = \mathbf{curl}_\Gamma} & \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, [\Gamma]) & \xleftarrow{\mathbf{curl}_\Gamma^* = \mathbf{grad}_\Gamma} & \widetilde{H}^{\frac{1}{2}}([\Gamma]).
\end{array}$$

Summary: Relationships of Trace Spaces

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$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^{+\frac{1}{2}}([\Gamma]) & \xrightarrow{\nabla_\Gamma} & H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]) & \xrightarrow{\mathrm{curl}_\Gamma} & H^{-\frac{1}{2}}([\Gamma]) \\
\downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
\mathbb{H}^{+\frac{1}{2}}(\Gamma) & \xrightarrow{\nabla_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\Gamma) \\
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\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

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$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
H^{+\frac{1}{2}}([\Gamma]) \arrow[r, "\nabla_\Gamma"] \arrow[d, "\iota"] & H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \arrow[r, "\mathrm{curl}_\Gamma"] \arrow[d, "\iota"] & H^{-\frac{1}{2}}([\Gamma]) \arrow[d, "\iota"] \\
\mathbb{H}^{+\frac{1}{2}}(\Gamma) \arrow[r, "\nabla_\Gamma"] \arrow[d, "{[\cdot]}"] & \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \arrow[r, "\mathrm{curl}_\Gamma"] \arrow[d, "{[\cdot]}"] & \mathbb{H}^{-\frac{1}{2}}(\Gamma) \arrow[d, "{[\cdot]}"] \\
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0 & 0 & 0
\end{tikzcd}
$$

$\triangleleft$ Commuting diagram

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0 & & 0 & & 0
\end{array}
$$

◁ Commuting diagram

Horizontal:

Hilbert complexes

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0 & 0 & 0
\end{tikzcd}$$

◁ Commuting diagram

Horizontal:

Hilbert complexes

Vertical:

exact sequences

Looking Back and Forward

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Success of M. Costabel's paradigm:

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Success of M. Costabel's paradigm: "To understand BIE **get off** Γ"

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➤ Framework for analysis of 1st-kind BIE

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X. CLAEYS AND R. HIPTMAIR, *Integral equations on multi-screens*, Integral Equations and Operator Theory, 77 (2013), pp. 167–197.

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[EFIE, Hodge-type decompositions, fine structure of $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, [\Gamma])$]

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TODO: "Quotient space BEM" on complex screens

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Framework for **THANK YOU**

X. Claeys and R. Hiptmair, *Integral equations on multi-screens*, Integral Equations and Operator Theory, 77 (2013), pp. 167–197.

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[EFIE, Hodge-type decompositions, fine structure of $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, [\Gamma])$]

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Questions ?

Duality

Duality pairing: $\ll \,,\, \gg : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow \mathbb{C}$

$$\ll \dot{u}, \dot{p} \gg := \int_{\mathbb{R}^d \setminus \Gamma} \mathbf{p} \cdot \nabla u + u \operatorname{div}(\mathbf{p}) \, \mathrm{d}\boldsymbol{x} \; .$$

$$\begin{array}{ccl} \dot{u}, \dot{p} & \hat{=} & \text{equivalence classes} \\ & \updownarrow & \\ u, \mathbf{p} & \hat{=} & \text{representatives} \end{array}$$

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► $$-\Delta u + u = 0 \text{ in } \mathbb{R}^d \setminus \Gamma \qquad (*)$$

Duality

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► $$-\,\mathbf{grad}\,\mathrm{div}\,\mathbf{p} + \mathbf{p} = 0 \quad \Rightarrow \quad \ll \dot{p}, \dot{u} \gg = \|\dot{p}\|^2_{\mathbb{H}^{-\frac{1}{2}}(\Gamma)}$$

Multi-Trace Spaces: Examples

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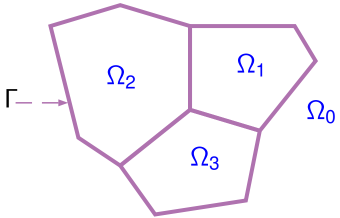

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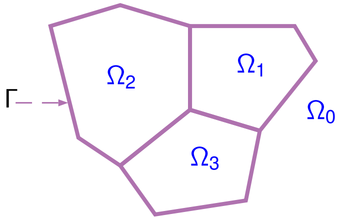

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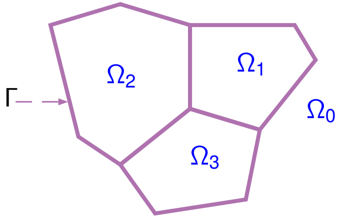

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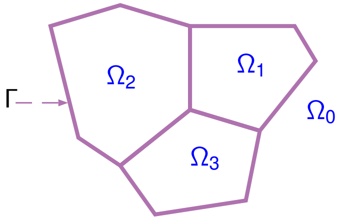

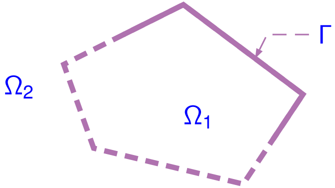

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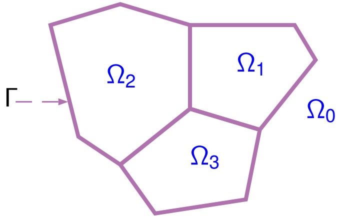

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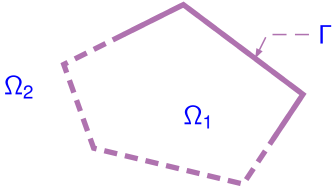

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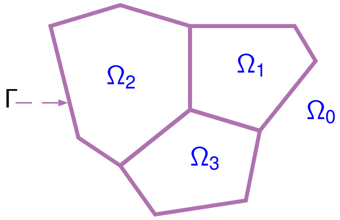

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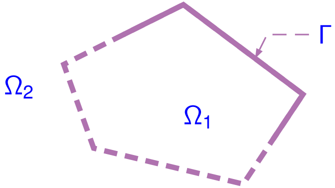

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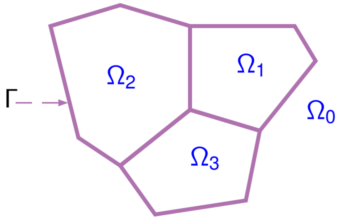

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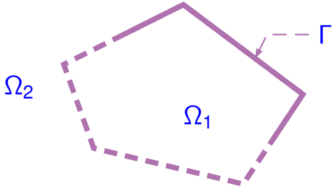

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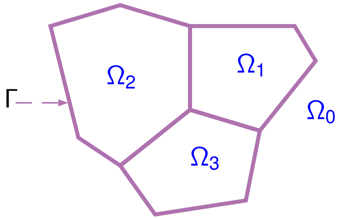

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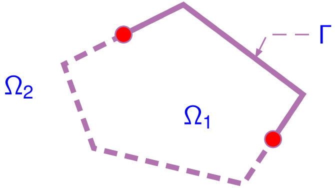

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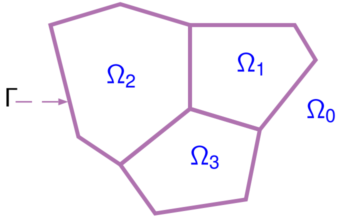

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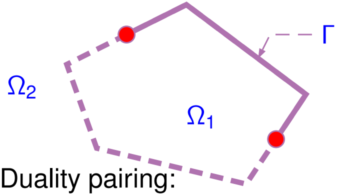

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$$\ll \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \gg = \langle v_1 - v_2, q_1 \rangle_\Gamma + \langle v_2, q_1 + q_2 \rangle_\Gamma = \langle v_1, q_1 \rangle_\Gamma + \langle v_2, q_2 \rangle_\Gamma \; .$$