Second kind boundary integral equation for multi-subdomain diffusion problems

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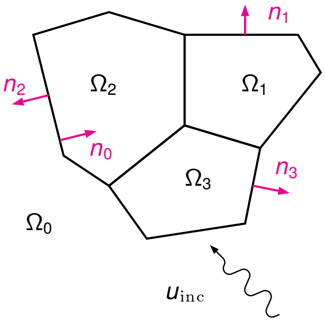
Multi-domain elliptic PDE



Geometry

$\mathbb{R}^d = \cup_{j=0}^{n} \overline{\Omega}_j, \quad \Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$

$\Sigma = \cup_{j=0}^{n} \Gamma_j, \;\; \Gamma_j := \partial\Omega_j$ Lipschitz

Material caracteristics

Piecewise constants $\mu, \epsilon : \mathbb{R}^d \to (0, +\infty)$

$\mu(\boldsymbol{x}) = \mu_j$ and $\epsilon(\boldsymbol{x}) = \epsilon_j$ for $\boldsymbol{x} \in \Omega_j$

Transmission problem (well posed) :

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ such that} \\ \mathrm{div}(\mu \nabla u) + \epsilon\omega^2 u = 0 \quad \text{in} \quad \mathbb{R}^d \\ u - u_{\mathrm{inc}} \text{ bounded/outgoing in } \Omega_0, \end{cases}$$

Concern :
Solution by means of **boundary integral equation** methods ?

Classification of boundary integral equations

Equation type	advantages	drawbacks
1st kind	• more accurate • more versatile	• ill conditionned • trickier quadrature • a posteriori estimates difficult
2nd kind	• easier discretisation • well conditionned • easier implementation	• requires stable discretisation

Multi-subdomain scattering
Boundary integral formulation of the first kind already exist for multi-scattering problems (Rumsey, BETI, MTF). Until recently though, no formulation of the 2nd kind was available.

Only recently, multi-subdomain (i.e. involving junctions) boundary integral equations of the second kind have been introduced for acoustic scattering problems with contrast in the wave number [Claeys, 2011], [Greengard & Lee, 2012], [Claeys, Hiptmair & Spindler, 2015].

Problem under study

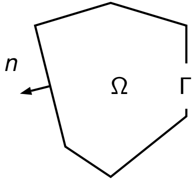
Here we focus on the case of contrasts coming into play in the principal part of the operator, see E.Spindler's thesis (2016) and [Claeys, Hiptmair & Spindler, 2017].

$$\begin{cases} \text{Find } u_{\mathrm{tot}} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ such that} \\ \mathrm{div}(\mu \nabla u_{\mathrm{tot}}) = 0 \quad \text{in } \mathbb{R}^3 \\ \lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}| \, |u_{\mathrm{tot}}(\boldsymbol{x}) - u_\infty(\boldsymbol{x})| < +\infty \end{cases}$$

where u_∞ is some smooth harmonic function.

Potential theory

Interior traces :
$\gamma_{\mathrm{D}}(u) := u|_{\partial\Omega}^{\mathrm{int}}\,, \quad \gamma_{\mathrm{N}}(u) := \partial_{\boldsymbol{n}} u|_{\partial\Omega}^{\mathrm{int}}$



Layer potentials :

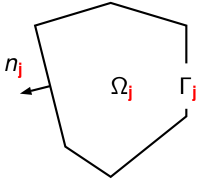
$$\mathrm{SL}\,(q)(\boldsymbol{x}) := \int_{\partial\Omega} \frac{q(\boldsymbol{y})d\sigma(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

$$\mathrm{DL}\,(v)(\boldsymbol{x}) := \int_{\partial\Omega} \frac{\boldsymbol{n}\,(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3} v(\boldsymbol{y})d\sigma(\boldsymbol{y})$$

Representation theorem :
For any $u \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega})$ satisfying $\Delta u = 0$ in Ω (+ decay condition)

$$\mathrm{SL}\,(\gamma_{\mathrm{N}}(u))(\boldsymbol{x}) + \mathrm{DL}\,(\gamma_{\mathrm{D}}(u))(\boldsymbol{x}) = \mathbf{1}_{\Omega}\,(\boldsymbol{x})u(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^3$$

Potential theory



Interior traces :

$\gamma_D(u) := u|^{\text{int}}_{\partial\Omega_j}, \quad \gamma_N(u) := \partial_{\boldsymbol{n}} u|^{\text{int}}_{\partial\Omega_j}$

Layer potentials :

$$\mathrm{SL}_j(q)(\boldsymbol{x}) := \int_{\partial\Omega_j} \frac{q(\boldsymbol{y})d\sigma(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

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Representation theorem :

For any $u \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega_j})$ satisfying $\Delta u = 0$ in Ω_j (+ decay condition)

$$\mathrm{SL}_j(\gamma_N(u))(\boldsymbol{x}) + \mathrm{DL}_j(\gamma_D(u))(\boldsymbol{x}) = \mathbf{1}_{\Omega_j}(\boldsymbol{x})u(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^3$$

Multi-domain trace spaces

Multi-trace space : $\mathbb{H}^s(\Sigma) := \mathrm{H}^s(\Gamma_0) \times \cdots \times \mathrm{H}^s(\Gamma_n)$ for $s \in (-1, +1)$.

Duality pairing : $\langle\langle \mathfrak{u}, \mathfrak{v} \rangle\rangle := \sum_{j=0}^{n} \int_{\Gamma_j} u_j v_j \, d\sigma$

Single-trace spaces :

$$\mathbb{X}_{\mathrm{D}}^{+s}(\Sigma) := \{\, (v|_{\Gamma_0}, \ldots, v|_{\Gamma_n}), \quad v \in \mathrm{H}^{\frac{1}{2}+s}(\mathbb{R}^3) \,\}$$

$$\mathbb{X}_{\mathrm{N}}^{-s}(\Sigma) := \{\, (\boldsymbol{n}_0 \cdot \boldsymbol{p}|_{\Gamma_0}, \ldots, \boldsymbol{n}_0 \cdot \boldsymbol{p}|_{\Gamma_n}), \quad \boldsymbol{p} \in \mathrm{H}^{\frac{1}{2}-s}(\mathrm{div}, \mathbb{R}^3) \,\} \quad \text{for } s \in (0, +1).$$

Polarity property :

For any $\mathfrak{u} \in \mathbb{H}^{-s}(\Sigma)$, we have $\mathfrak{u} \in \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma) \iff \langle\langle \mathfrak{u}, \mathfrak{v} \rangle\rangle = 0 \ \forall \mathfrak{v} \in \mathbb{X}_{\mathrm{D}}^{+s}(\Sigma)$

Multi-domain trace spaces

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There is another more explicit caracterisation of single trace spaces, consistent with the previous one, that holds for $s \in (-1/2, +1/2)$, namely

$$\mathbb{X}_{\mathrm{D}}^{s}(\Sigma) := \{\ (v_j)_{j=0}^{n} \in \mathbb{H}^{s}(\Sigma),\ v_j - v_k = 0 \text{ on } \Gamma_j \cap \Gamma_k\ \}$$

$$\mathbb{X}_{\mathrm{N}}^{s}(\Sigma) := \{\ (q_j)_{j=0}^{n} \in \mathbb{H}^{s}(\Sigma),\ q_j + q_k = 0 \text{ on } \Gamma_j \cap \Gamma_k\ \}$$

Decomposition :

$$\mathbb{H}^{s}(\Sigma) = \mathbb{X}_{\mathrm{D}}^{s}(\Sigma) \oplus \mathbb{X}_{\mathrm{N}}^{s}(\Sigma) \quad \text{for } s \in (-1/2, +1/2).$$

Multi-domain diffusion problem

$$\begin{cases} \text{Find } u_{\text{tot}} \in \mathrm{H}^1_{\text{loc}}(\mathbb{R}^3) \text{ such that} \\ \operatorname{div}(\mu \nabla u_{\text{tot}}) = 0 \quad \text{in } \mathbb{R}^3 \\ \limsup_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}| \, |u_{\text{tot}}(\boldsymbol{x}) - u_{\infty}(\boldsymbol{x})| < +\infty \end{cases}$$

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$$\Updownarrow$$

For $u := u_{\mathrm{tot}} - u_\infty$

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega_j, \; j = 0 \dots n \\ \limsup_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x} u(\boldsymbol{x})| < +\infty \end{cases}$$

$$\begin{cases} u|_{\Gamma_j} - u|_{\Gamma_k} = 0 \\ \mu_j \partial_{\boldsymbol{n}_j}(u + u_\infty)|_{\Gamma_j} + \mu_k \partial_{\boldsymbol{n}_k}(u + u_\infty)|_{\Gamma_k} = 0 \\ \qquad \text{on} \quad \Gamma_j \cap \Gamma_k \;\; \forall j, k = 0 \dots n \end{cases}$$

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$$u(\boldsymbol{x}) 1_{\Omega_k}(\boldsymbol{x}) = \mathrm{SL}_k(\gamma_{\mathrm{N}}^k(u))(\boldsymbol{x}) + \mathrm{DL}_k(\gamma_{\mathrm{D}}^k(u))(\boldsymbol{x}) \qquad \forall k = 0 \dots n,\ \forall \boldsymbol{x} \in \mathbb{R}^3$$

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$$u\, 1_{\Omega_k} = \qquad \mathrm{SL}_k(\gamma_{\mathrm{N}}^k(u)) + \mathrm{DL}_k(\gamma_{\mathrm{D}}^k(u)) \qquad \forall k = 0 \dots n$$

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$$\sum_{k=0}^{n} u\, 1_{\Omega_k} = \sum_{k=0}^{n} \mathrm{SL}_k(\gamma_{\mathrm{N}}^k(u)) + \mathrm{DL}_k(\gamma_{\mathrm{D}}^k(u))$$

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$$u \sum_{k=0}^{n} 1_{\Omega_k} = \sum_{k=0}^{n} \mathrm{SL}_k(\gamma_{\mathrm{N}}^k(u)) + \mathrm{DL}_k(\gamma_{\mathrm{D}}^k(u))$$

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$$(\gamma_{\mathrm{D}}^0(u), \dots, \gamma_{\mathrm{D}}^n(u)) \in \mathbb{X}_{\mathrm{D}}^{1/2}(\Sigma)$$

Proposition

$\sum_{j=0}^n \mathrm{DL}_j(v_j) = 0 \ \forall (v_j) \in \mathbb{X}_{\mathrm{D}}^{+s}(\Sigma)$

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$\sum_{j=0}^{n} \mathrm{SL}_j(q_j) = 0 \ \forall (q_j) \in \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma)$

$$u = \sum_{k=0}^{n} \mathrm{SL}_k(\gamma_{\mathrm{N}}^k(u) + \gamma_{\mathrm{N}}^k(u_\infty))$$

u_∞ smooth $\Rightarrow (\gamma_{\mathrm{N}}^0(u_\infty), \dots, \gamma_{\mathrm{N}}^n(u_\infty)) \in \mathbb{X}_{\mathrm{N}}^{-1/2}(\Sigma)$

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$$\gamma_{\mathrm{N}}^j (u + u_\infty) = \gamma_{\mathrm{N}}^j \sum_{k=0}^{n} \mathrm{SL}_k (\gamma_{\mathrm{N}}^k (u + u_\infty)) + \gamma_{\mathrm{N}}^j (u_\infty)$$

Multi-domain diffusion problem

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega_j,\ j = 0 \dots n \\ \limsup_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x} u(\boldsymbol{x})| < +\infty \end{cases}$$

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$$\mu_j^{-1} p_j - \sum_{k=0}^n \gamma_{\mathrm{N}}^j \mathrm{SL}_k(\mu_j^{-1} p_j) = \gamma_{\mathrm{N}}^j (u_\infty)$$

Multi-domain diffusion problem

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$$\begin{bmatrix} p_0/\mu_0 \\ \vdots \\ p_n/\mu_n \end{bmatrix} - \begin{bmatrix} \gamma_{\mathrm{N}}^0 \mathrm{SL}_0 & \cdots & \gamma_{\mathrm{N}}^0 \mathrm{SL}_n \\ \vdots & & \vdots \\ \gamma_{\mathrm{N}}^n \mathrm{SL}_0 & \cdots & \gamma_{\mathrm{N}}^n \mathrm{SL}_n \end{bmatrix} \cdot \begin{bmatrix} p_0/\mu_0 \\ \vdots \\ p_n/\mu_n \end{bmatrix} = \begin{bmatrix} \gamma_{\mathrm{N}}^0 (u_\infty) \\ \vdots \\ \gamma_{\mathrm{N}}^n (u_\infty) \end{bmatrix}$$

Multi-domain diffusion problem

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$\mathfrak{p} = (p_0, \dots, p_n) \in \mathbb{X}_{\mathrm{N}}^{-1/2}(\Sigma)$

where $p_j := \mu_j \gamma_{\mathrm{N}}^j (u + u_\infty)$

$$\underbrace{\begin{bmatrix} p_0/\mu_0 \\ \vdots \\ p_n/\mu_n \end{bmatrix}}_{\mathrm{I}_\mu(\mathfrak{p}) =} - \underbrace{\begin{bmatrix} \gamma_{\mathrm{N}}^0 \mathrm{SL}_0 & \cdots & \gamma_{\mathrm{N}}^0 \mathrm{SL}_n \\ \vdots & & \vdots \\ \gamma_{\mathrm{N}}^n \mathrm{SL}_0 & \cdots & \gamma_{\mathrm{N}}^n \mathrm{SL}_n \end{bmatrix}}_{\mathrm{A} :=} \cdot \begin{bmatrix} p_0/\mu_0 \\ \vdots \\ p_n/\mu_n \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_{\mathrm{N}}^0 (u_\infty) \\ \vdots \\ \gamma_{\mathrm{N}}^n (u_\infty) \end{bmatrix}}_{\mathfrak{f} :=}$$

$\mathrm{I}_\mu := \mathrm{diag}(\mu_j^{-1})$

Continuity :

$\mathrm{A} : \mathbb{H}^{-s}(\Sigma) \to \mathbb{H}^{-s}(\Sigma),\ s \in (0, 1)$

Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_{\mathsf{N}}^{-s}(\Sigma) \text{ such that}$$

$$(\mathrm{Id} - \mathrm{A})\mathrm{I}_{\mu}(\mathfrak{p}) \quad = \quad \mathfrak{f}$$

Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_{\mathsf{N}}^{-s}(\Sigma) \text{ such that}$$

$$\langle\!\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu(\mathfrak{p}), \mathfrak{v} \rangle\!\rangle = \langle\!\langle \mathfrak{f}, \mathfrak{v} \rangle\!\rangle \qquad \forall \mathfrak{v} \in \mathbb{H}^{+s}(\Sigma)$$

Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma) \text{ such that}$$

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Proposition :
For any $s \in (0, 1)$ the operator $\mathrm{A} : \mathbb{H}^{-s}(\Sigma) \to \mathbb{H}^{-s}(\Sigma)$ is a continuous projector $\mathrm{A}^2 = \mathrm{A}$ such that $\ker(\mathrm{A}) = \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma)$.

As a consequence $\mathrm{range}(\mathrm{Id} - \mathrm{A}) = \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma)$ and, due to the polarity property, we have $\langle\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_{\mu}(\mathfrak{p}), \mathfrak{v} \rangle\rangle = 0 \ \forall \mathfrak{v} \in \mathbb{X}_{\mathrm{D}}^{+s}(\Sigma)$.

Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_{\mathsf{N}}^{-s}(\Sigma) \text{ such that}$$

$$\langle\!\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu(\mathfrak{p}), \mathfrak{v} \rangle\!\rangle = \langle\!\langle \mathfrak{f}, \mathfrak{v} \rangle\!\rangle \qquad \forall \mathfrak{v} \in \mathbb{X}_{\mathsf{N}}^{+s}(\Sigma) \oplus \mathbb{X}_{\mathsf{D}}^{+s}(\Sigma)$$

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Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_N^{-s}(\Sigma) \text{ such that}$$

$$\langle\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu(\mathfrak{p}), \mathfrak{v} \rangle\rangle = \langle\langle \mathfrak{f}, \mathfrak{v} \rangle\rangle \qquad \forall \mathfrak{v} \in \mathbb{X}_N^{+s}(\Sigma) \oplus \mathbb{X}_D^{+s}(\Sigma)$$

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Boundary integral formulation

$$\text{Find } \mathfrak{p} \in \mathbb{X}_N^{-s}(\Sigma) \text{ such that}$$
$$\langle\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu(\mathfrak{p}), \mathfrak{v} \rangle\rangle = \langle\langle \mathfrak{f}, \mathfrak{v} \rangle\rangle \qquad \forall \mathfrak{v} \in \mathbb{X}_N^{+s}(\Sigma),\ s \in (1/2, 1/2 + \epsilon)$$

Proposition :
For any $s \in (0,1)$ the operator $\mathrm{A} : \mathbb{H}^{-s}(\Sigma) \to \mathbb{H}^{-s}(\Sigma)$ is a continuous projector $\mathrm{A}^2 = \mathrm{A}$ such that $\ker(\mathrm{A}) = \mathbb{X}_N^{-s}(\Sigma)$.

As a consequence $\mathrm{range}(\mathrm{Id} - \mathrm{A}) = \mathbb{X}_N^{-s}(\Sigma)$ and, due to the polarity property, we have $\langle\langle (\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu(\mathfrak{p}), \mathfrak{v} \rangle\rangle = 0\ \forall \mathfrak{v} \in \mathbb{X}_D^{+s}(\Sigma)$.

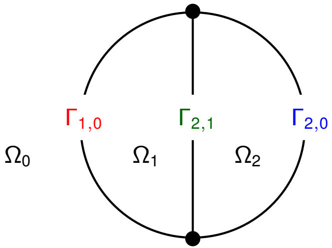
Theorem (well posedness) :
There exists $\epsilon > 0$ such that the operator $(\mathrm{Id} - \mathrm{A})\mathrm{I}_\mu$ isomorphically map $\mathbb{X}_N^{-s}(\Sigma)$ onto $\mathbb{X}_N^{-s}(\Sigma)$ for all $s \in (1/2 - \epsilon, 1/2)$.

There exist geometrical configurations involving junctions such that $\epsilon < 1/2$, so that well-posedness (and Fredholmness) does not hold in an L^2-setting. One might expect that Fredholmness is restored in a weighted L^2-setting though.

Rewriting the equation interface-wise

Interfaces : $\Sigma = \cup_{j=0}^{n} \Gamma_j = \cup_{\mathrm{J}\in\mathfrak{J}} \Gamma_{\mathrm{J}}$ with

$\mathfrak{J} := \{\mathrm{J} = (\mathrm{J}_+, \mathrm{J}_-) \mid \mathrm{J}_\pm = 0, \dots n, \mathrm{J}_+ > \mathrm{J}_- \text{ and } \Gamma_{\mathrm{J}} := \Gamma_{\mathrm{J}_+} \cap \Gamma_{\mathrm{J}_-}\}$



$\Sigma = \Gamma_{1,0} \cup \Gamma_{2,0} \cup \Gamma_{2,1}$
$\mathfrak{J} = \{\, (1,0),\ (2,0),\ (2,1)\,\}$

$$\langle\!\langle \mathfrak{u}, \mathfrak{v} \rangle\!\rangle = \sum_{j=0}^{n} \langle u_j, v_j \rangle_{\Gamma_j} = \sum_{\mathrm{J}\in\mathfrak{J}} \langle u_{\mathrm{J}_+}, v_{\mathrm{J}_+} \rangle_{\Gamma_{\mathrm{J}_+}} + \langle u_{\mathrm{J}_-}, v_{\mathrm{J}_-} \rangle_{\Gamma_{\mathrm{J}_-}}$$

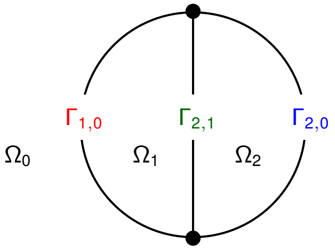
$$= \sum_{\mathrm{J}\in\mathfrak{J}} \frac{1}{2} \langle [u_{\mathrm{J}}], [v_{\mathrm{J}}] \rangle_{\Gamma_{\mathrm{J}}} + 2 \langle \{u_{\mathrm{J}}\}, \{v_{\mathrm{J}}\} \rangle_{\Gamma_{\mathrm{J}}} \quad \text{for } \mathfrak{u} \in \mathbb{H}^{-s}(\Sigma),\ \mathfrak{v} \in \mathbb{H}^{+s}(\Sigma)$$

where $\quad [u_{\mathrm{J}}] := u_{\mathrm{J}_+} - u_{\mathrm{J}_-}, \quad \{u_{\mathrm{J}}\} := (u_{\mathrm{J}_+} + u_{\mathrm{J}_-})/2.$

Rewriting the equation interface-wise

Interfaces : $\Sigma = \cup_{j=0}^{n} \Gamma_j = \cup_{\mathrm{J}\in\mathfrak{I}} \Gamma_{\mathrm{J}}$ with

$\mathfrak{I} := \{\mathrm{J} = (\mathrm{J}_+, \mathrm{J}_-) \mid \mathrm{J}_\pm = 0, \dots n, \mathrm{J}_+ > \mathrm{J}_- \text{ and } \Gamma_{\mathrm{J}} := \Gamma_{\mathrm{J}_+} \cap \Gamma_{\mathrm{J}_-}\}$



$\Sigma = \Gamma_{1,0} \cup \Gamma_{2,0} \cup \Gamma_{2,1}$

$\mathfrak{I} = \{\,(1,0),\ (2,0),\ (2,1)\,\}$

$$\langle\!\langle \mathfrak{u}, \mathfrak{v} \rangle\!\rangle = \sum_{j=0}^{n} \langle u_j, v_j \rangle_{\Gamma_j} = \sum_{\mathrm{J}\in\mathfrak{I}} \langle u_{\mathrm{J}_+}, v_{\mathrm{J}_+} \rangle_{\Gamma_{\mathrm{J}_+}} + \langle u_{\mathrm{J}_-}, v_{\mathrm{J}_-} \rangle_{\Gamma_{\mathrm{J}_-}}$$

$$= \sum_{\mathrm{J}\in\mathfrak{I}} \frac{1}{2} \langle [u_{\mathrm{J}}], [v_{\mathrm{J}}] \rangle_{\Gamma_{\mathrm{J}}} + 2\langle \{u_{\mathrm{J}}\}, \{v_{\mathrm{J}}\} \rangle_{\Gamma_{\mathrm{J}}} \quad \text{for } \mathfrak{u} \in \mathbb{H}^{-s}(\Sigma),\ \mathfrak{v} \in \mathbb{H}^{+s}(\Sigma)$$

where $[u_{\mathrm{J}}] := u_{\mathrm{J}_+} - u_{\mathrm{J}_-}$, $\{u_{\mathrm{J}}\} := (u_{\mathrm{J}_+} + u_{\mathrm{J}_-})/2 = 0$ if $\mathfrak{u} \in \mathbb{X}_{\mathrm{N}}^{-s}(\Sigma)$.

Rewriting the equation interface-wise

Interfaces : $\Sigma = \cup_{j=0}^{n} \Gamma_j = \cup_{\mathrm{J}\in\mathfrak{J}} \Gamma_{\mathrm{J}}$ with

$\mathfrak{J} := \{\mathrm{J} = (\mathrm{J}_+, \mathrm{J}_-) \mid \mathrm{J}_\pm = 0, \dots n, \mathrm{J}_+ > \mathrm{J}_- \text{ and } \Gamma_{\mathrm{J}} := \Gamma_{\mathrm{J}_+} \cap \Gamma_{\mathrm{J}_-}\}$

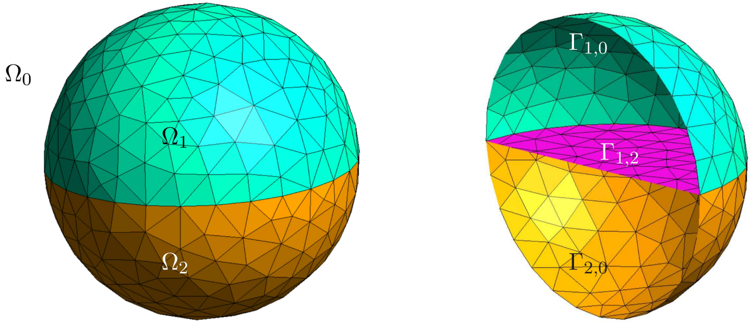
Considering our BIE posed in $\mathbb{X}_{\mathrm{N}}^{-s}(\Sigma)$ with $s \in (1/2 - \epsilon, 1/2)$, traces can be decomposed traces according to interfaces. Taking $p_{\mathrm{J}} = [u_{\mathrm{J}}]$ as unknowns, our integral equation can be re-arranged as

$$\begin{cases} \text{Find } p_{\mathrm{J}} \in \mathrm{H}^{-s}(\Gamma_{\mathrm{J}}),\ \mathrm{J} \in \mathfrak{J} \text{ such that} \\ p_{\mathrm{J}} + \sum_{\mathrm{Q}\in\mathfrak{J}} 2\dfrac{\mu_{\mathrm{J}_+} - \mu_{\mathrm{J}_-}}{\mu_{\mathrm{J}_+} + \mu_{\mathrm{J}_-}} \mathrm{A}_{\mathrm{J},\mathrm{Q}}(p_{\mathrm{Q}}) = f_{\mathrm{J}} \quad \forall \mathrm{J} \in \mathfrak{J}. \end{cases}$$

where

$$\mathrm{A}_{\mathrm{J},\mathrm{Q}}(p)(\boldsymbol{x}) := \lim_{\delta\to 0_+} \int_{\Gamma_{\mathrm{Q}}\setminus \mathrm{B}_\delta(\boldsymbol{x})} \frac{\mathbf{n}_{\mathrm{J}}(\boldsymbol{x}) \cdot (\boldsymbol{y} - \boldsymbol{x})}{4\pi|\boldsymbol{y} - \boldsymbol{x}|^3} p(\boldsymbol{y}) d\sigma(\boldsymbol{y})$$

Numerical experiment



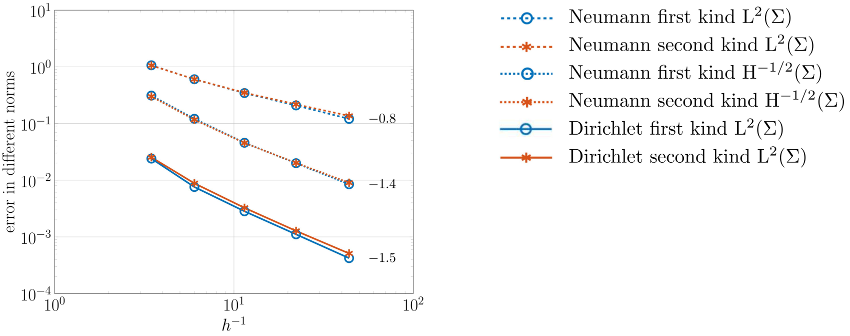
Geometry : $\Omega_0 = \mathbb{R}^3 \setminus \overline{\mathrm{B}}(0, 1/2)$,

Backrground field : $u_\infty(\boldsymbol{x}) = \sin(x_1)\sinh(x_2)$

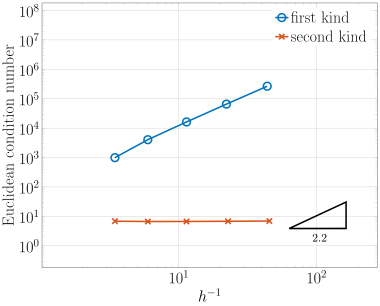
Material carac : $\mu_0 = 5, \mu_1 = 1, \mu_2 = 7$

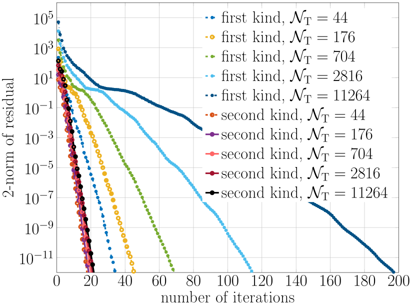
Discretisation : piecewise constants.

Numerical experiment



Numerical experiment





IABEM 2018

Symposium of the International Association for Boundary Element Methods

What ?
International conference focused on boundary integral equations
Both theory and application oriented

Where ?
University Pierre-et-Marie Curie (Paris 6)

When ?
June 26-28, 2018

Website
`https://project.inria.fr/iabem2018/`

IABEM

Thank you
for your attention