Boundary element methods for scattering by fractal screens

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Abstract

We consider a flat screen \( \gamma \) that is a bounded subset of the plane \( \Gamma_\infty := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \). The case of 2-dimensional wave propagation \( \Gamma \subset \mathbb{R} \times \{0\} \subset \mathbb{R}^2 \) can easily be treated in the same way.

We use fractional (Bessel) Sobolev spaces. For \( s \in \mathbb{R}, \Omega \subset \mathbb{R}^2 \) open and \( F \subset \mathbb{R}^2 \) closed, let

\[
\begin{align*}
H^s(\mathbb{R}^2) := & \{ u \in S^s(\mathbb{R}^2) : \| u \|_{H^s(\mathbb{R}^2)} < \infty \} \\
\| u \|_{H^s(\mathbb{R}^2)} := & \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi, \\
H^s(\Omega) := & \{ u|_{\Omega} : u \in H^s(\mathbb{R}^2) \}, \\
\hat{H}^s(\Omega) := & C_0^\infty(\Omega)H^s(\mathbb{R}^2), \\
H^s_F := & \{ u \in H^s(\mathbb{R}^2) : \text{supp} u \subset F \}.
\end{align*}
\]

In general \( \hat{H}^s(\Omega) \subset H^s_F \); they coincide if \( \Omega \) is sufficiently regular but examples with \( \hat{H}^s(\Omega) \neq H^s_F \) can be constructed [3, Thm. 3.19]. We denote \( \gamma^\pm \) the traces \( \gamma^\pm : W^1(\mathbb{R}^3_\pm) \to H^{1/2}(\Gamma_\infty) \), where \( \mathbb{R}^3_\pm \) are the upper and lower half-spaces.

1 Notation and Sobolev spaces

2 Boundary value problems (BVP)

The classical sound-soft screen scattering BVP consists of looking for \( u \) satisfying the Helmholtz equation (1), the Sommerfeld condition (2) and the Dirichlet boundary condition (3):

\[
\begin{align*}
\Delta u + k^2 u = & 0 \quad \text{in } D := \mathbb{R}^3 \setminus \bar{\Gamma}, \quad (1) \\
\partial_t u(x) - iku(x) = & a(r^{-1}) \quad r := |x| \to \infty, \quad (2) \\
\quad u = & -u^i \quad \text{on } \Gamma, \quad (3)
\end{align*}
\]

where \( k > 0 \) is the wavenumber and \( u^i \) is a given incident wave. To formulate a well-posed BVP, one needs to be more precise about the sense in which the boundary condition (3) holds.

We first describe the case when \( \Gamma \) is a relatively open subset of \( \Gamma_\infty \).

Definition 1 (BVP \( D^{op}(\Gamma) \)) Let \( \Gamma \subset \Gamma_\infty \) be bounded and open and \( g \in H^{1/2}(\Gamma) \). Find \( u \in C^2(D) \cap W^{1, \text{loc}}(D) \) satisfying (1)–(2) and

\[
(\gamma^\pm u)|_{\Gamma} = g.
\]

Theorem 2 (Thm. 6.2 [2]) If \( \tilde{H}^{-1/2}(\Gamma) = H^{-1/2}_\Gamma \), then \( D^{op}(\Gamma) \) admits a unique solution \( u \).

Moreover, \( u \) satisfies the representation formula \( u(x) = -S_\Gamma \phi(x), \ x \in D \), where \( S_\Gamma \) is the single-layer potential and \( \phi = [\partial_\nu u] := \partial^n_{\nu} u - \partial^-_{\nu} u \in \tilde{H}^{-1/2}(\Gamma) \) is the unique solution of the BIE \( S_\Gamma \phi = -g \), with \( S_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) the single-layer operator.

The main assumption for the well-posedness of the BVP is \( \tilde{H}^{-1/2}(\Gamma) = H^{-1/2}_\Gamma \), equivalent to the density of \( C_0^\infty(\Gamma) \) in \( H^{-1/2}_\Gamma \). This is guaranteed if, e.g., (i) \( \Gamma \) is \( C^0 \) up to countably many points \( P \subset \partial \Gamma \) such that \( P \) has only finitely many limit points [3, Thm. 3.24], or (ii) \( \Gamma \) is “thick” in the sense of Triebel [1]. All Lipschitz \( \Gamma \), but also classical and exotic snowflakes with fractal boundaries, satisfy these conditions [1].

If the screen \( \Gamma \) is a compact set, we substitute the restriction operator \( r \) in the boundary conditions with the orthogonal projection \( P_r : H^{1/2}(\mathbb{R}^2) \to (H^{1/2}(\Gamma^c))^\perp \), where \( \Gamma^c = \mathbb{R}^2 \setminus \Gamma \).
Definition 3 (BVP $\mathbb{D}^{\alpha}\left(\Gamma\right)$) Let $\Gamma \subset \Gamma_\infty$ be compact and $g \in \left(\widetilde{H}^{1/2}\left(\Gamma^c\right)\right)^\perp$. Find $u \in C^2\left(\mathbb{D}\right) \cap \mathcal{W}^{1,\text{loc}}\left(\mathbb{D}\right)$ satisfying (1)–(2) and

$$P_\Gamma \gamma^\pm u = g.$$ 

This choice of $P_\Gamma$ ensures that if $\Omega \subset \Gamma_\infty$ is bounded, open, and $H^{-1/2}\left(\Omega\right) = H_{\Gamma_\infty}^{-1/2}$, then the problems $\mathbb{D}^{\text{op}}\left(\Omega\right)$ and $\mathbb{D}^{\alpha}\left(\Omega\right)$ are equivalent.

Theorem 4 (Thm. 6.4 [2]) Problem $\mathbb{D}^{\alpha}\left(\Gamma\right)$ admits a unique solution $u$.

Moreover, $u$ satisfies the representation formula $u(x) = -S_\Gamma \phi(x)$, $x \in \mathbb{D}$, with $\phi = \left[\delta_n u\right]$ the solution of the BIE $S_\Gamma \phi = -g$ for the single-layer operator $S_\Gamma : H_{\Gamma_\infty}^{-1/2} \to \left(\widetilde{H}^{1/2}\left(\Gamma^c\right)\right)^\perp$.

3 Prefractal to fractal convergence

To study the scattering by a fractal screen $\Gamma$, we approximate it with simpler prefractal shapes $(\Gamma_j)_{j \in \mathbb{N}}$. The BVP $\mathbb{D}^\varepsilon\left(\Gamma_j\right)$ (for $\varepsilon \in \{\text{op}, \alpha\}$) is correctly approximated by a sequence of BVPs $\mathbb{D}^\varepsilon\left(\Gamma_j\right)$ if the corresponding sequence of subspaces (either $\widetilde{H}^{-1/2}\left(\Gamma_j\right)$ or $H_{\Gamma_j}^{-1/2}$) of $H^{-1/2}\left(\mathbb{R}^2\right)$ converges in the sense of Mosco [4]. In [4] we show:

Theorem 5 The solution $\phi_j$ of the BIE on $\Gamma_j$ converges in $H^{-1/2}\left(\mathbb{R}^2\right)$ to the solution $\phi$ of the BIE on $\Gamma$ and $S\phi_j \to S\phi$ in $W^{1,\text{loc}}\left(\mathbb{R}^2\right)$ if, e.g.,

- $\Gamma$ and $\Gamma_j$ are bounded, open with $\Gamma_j \subset \Gamma_{j+1}$ and $\Gamma = \bigcup_{j \in \mathbb{N}} \Gamma_j$, or
- $\Gamma_j$ are compact, $\Gamma_j \subset \Gamma_{j+1}$ and $\Gamma = \bigcap_{j \in \mathbb{N}} \Gamma_j$.

We also show convergence for a class of non-nested prefractals such as those in Figure 1.

![Figure 1: Non-nested prefractals $\Gamma_j$ for the square snowflake $\Gamma$, for which $\phi_j \to \phi$ in $H^{-1/2}$](image)

4 BEM discretisation

We approximate the solution of scattering problems posed on a non-Lipschitz screen $\Gamma$ (fractal or with fractal boundary) using a piecewise-constant boundary element methods (BEM) on prefractal screens $\Gamma_j$. In [4] we give general criteria on the mesh to guarantee convergence of the Galerkin BEM: the key is the Mosco convergence of the discrete spaces on $\Gamma_j$ to the desired Sobolev space, either $\widetilde{H}^{-1/2}(\Gamma)$ or $H_{\Gamma_j}^{-1/2}$.

E.g., if $\Gamma_j$ are the classical prefractal approximation of the Koch snowflake, or the square snowflake prefractals of Figure 1, then any sequence of meshes $\mathcal{T}_j$ on $\Gamma_j$ with meshsize $h_j \searrow 0$ provides a provably convergent BEM scheme.

If $\Gamma$ is a Cantor dust (the Cartesian product of two identical Cantor sets) then its scattered field is non-zero (for almost every incident plane waves) if and only if the Hausdorff dimension of $\Gamma$ is larger than 1. [2]. We verify this numerically in Figure 2. For details and more extensive numerical tests, see [4].

![Figure 2: On-screen (○), near-field (□) and far-field (*) norms of the field scattered by Cantor dust prefractals $\Gamma_6, \ldots, \Gamma_6$ computed with BEM. When the prefractal level is refined, for Hausdorff dimension $d = \frac{\log 4}{\log 3} > 1$ (left), the norms provably converge to a positive value, while for $d = \frac{\log 4}{\log 10} < 1$ (right) they converge to 0.](image)

References


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