Solutions to Problem Sheet 6

1. (a) Let

$$u = x \qquad \frac{dv}{dx} = \sin x$$
$$\frac{du}{dx} = 1 \qquad v = \cos x,$$

then use the by-parts formula. We obtain

$$\int x \sin x \mathrm{d}x = -x\cos x + \int \cos x \mathrm{d}x$$
$$= -x\cos x + \sin x + C.$$

(b) Let

$$u = x^2 \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = 2x \qquad v = e^x.$$

Then
$$\int x^2 e^x \mathrm{d}x = x^2 e^x - \int 2x e^x \mathrm{d}x,$$

which requires us to integrate by parts again! This time, let

$$u = 2x \qquad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = 2 \qquad v = e^x.$$

hence we have

$$\begin{aligned}\int 2xe^x \mathrm{d}x &= 2xe^x - \int 2e^x \mathrm{d}x \\ &= 2xe^x - 2e^x (+C).\end{aligned}$$

Putting everything together, we find that

$$\begin{aligned}\int x^2 e^x \mathrm{d}x &= x^2 e^x - [2xe^x - 2e^x] + C \\ &= x^2 e^x - 2xe^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C.\end{aligned}$$

(c) This question is a sneaky example of integrating by parts! This time you will want to choose

$$\begin{aligned} u &= \cos^{-1} x & \frac{dv}{dx} &= 1 \\ \frac{du}{dx} &= \frac{-1}{\sqrt{1-x^2}} & v &= x. \end{aligned}$$

Then the by-parts formula tells us that

$$\int 1 \cdot \cos^{-1} x \, \mathrm{d}x = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x.$$

But

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x &= \int (-2x) - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \mathrm{d}x \\ &= \int -\frac{1}{2} w^{-\frac{1}{2}} \mathrm{d}w \qquad [w = 1 - x^2] \\ &= -w^{\frac{1}{2}} + C \\ &= -\sqrt{1-x^2} + C,\end{aligned}$$

thus

$$\int \cos^{-1} x \, \mathrm{d}x = x \cos^{-1} x - \sqrt{1 - x^2} + C.$$

Note: To get the result for $\frac{du}{dx}$, note that

$$\cos u = x, \tag{1}$$

which can be differentiated implicitly, and the result is...

$$\begin{aligned} -\sin u \frac{du}{dx} &= 1 \\ \frac{du}{dx} &= \frac{-1}{\sin u} \\ &= \frac{-1}{\sqrt{1 - \cos^2 u}} \\ &= \frac{-1}{\sqrt{1 - x^2}}. \end{aligned}$$

2. (a) When we factorise the denominator, we can tell that it has two real roots...

$$\frac{x}{x^2 + x - 6} = \frac{x}{(x+3)(x-2)} \equiv \frac{A}{(x+3)} + \frac{B}{(x-2)}.$$

Then

$$x \equiv A(x-2) + B(x+3), \tag{2}$$

and if we let $x = 2$, we have

$$2 = 5B \quad \Rightarrow \quad B = \frac{2}{5}.$$

On the other hand, substituting $x = -3$ into (2), we yield

$$-3 = -5A \quad \Rightarrow \quad A = \frac{3}{5},$$

thus

$$\frac{x}{x^2 + x - 6} \equiv \frac{3}{5(x+3)} + \frac{2}{5(x-2)},$$

therefore the integral can be rewritten as

$$\begin{aligned}&\int \left[\frac{3}{5(x+3)} + \frac{2}{5(x-2)}\right] \mathrm{d}x \\ &= \frac{3}{5}\int \frac{1}{(x+3)}\mathrm{d}x + \frac{2}{5}\int \frac{1}{(x-2)}\mathrm{d}x \\ &= \frac{3}{5}\ln|x+3| - +\frac{2}{5}\ln|x-2| + C.\end{aligned}$$

(b) This denominator happens to have exactly one real root, hence it can be factorised too.

$$\frac{2x+2}{x^2+2x+1} = \frac{2x+2}{(x-1)^2} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2}.$$

Then

$$2x + 2 \equiv A(x-1) + B, \tag{3}$$

and comparing the x-coefficients reveals that $A = 2$. Meanwhile, if we let $x = 1$ in (3), we end up with $B = 4$. Hence

$$\frac{2x+2}{x^2+2x+1} \equiv \frac{2}{(x-1)} + \frac{4}{(x-1)^2},$$

and so our integral equals

$$\begin{aligned}
&\int \left[\frac{2}{(x-1)} + \frac{4}{(x-1)^2}\right] \mathrm{d}x \\
&= 2\int \frac{1}{(x-1)}\mathrm{d}x + 4\int \frac{1}{(x-1)^2}\mathrm{d}x \\
&= 2\ln|x-1| - \frac{4}{(x-1)} + C.
\end{aligned}$$

(c) The denominator for this question also has exactly one real root, so it can be factorised. We end up with

$$\frac{2x}{x^2+2x+1} = \frac{2x}{(x+1)^2} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2},$$

hence we would like

$$2x \equiv A(x+1) + B, \tag{4}$$

and comparing the x-coefficients gives us $A = 2$ straightaway. Next, put $x = -1$ in (4) to get $B = -2$, so we have

$$\frac{2x}{x^2+2x+1} \equiv \frac{2}{(x+1)} - \frac{2}{(x+1)^2},$$

so the integral turns out to be

$$\begin{aligned}
&\int \left[\frac{2}{(x+1)} - \frac{2}{(x+1)^2}\right] \mathrm{d}x \\
&= 2\int \frac{1}{(x+1)}\mathrm{d}x + 2\int \frac{-1}{(x+1)^2}\mathrm{d}x \\
&= 2\ln|x+1| + \frac{2}{(x+1)} + C.
\end{aligned}$$

3. (a) Note that the denominator is of the form $x^2 + a^2$ with $a = 1$, so we can substitute

$$x = \tan\theta \quad \Rightarrow \quad \mathrm{d}x = \sec^2\theta \mathrm{d}\theta.$$

Then

$$\begin{aligned} \int \frac{1}{x^2+1}\,\mathrm{d}x &= \int \frac{1}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\,\mathrm{d}\theta \\ &= \theta + C \\ &= \tan^{-1} x + C, \end{aligned}$$

indicating that

$$\begin{aligned} \int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x &= \lim_{b\to\infty}\left[\int_0^b \frac{1}{x^2+1}\,\mathrm{d}x\right] \\ &= \lim_{b\to\infty}\left[\tan^{-1} x\right]_0^b \\ &= \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} 0\right) \\ &= \lim_{b\to\infty}\left(\tan^{-1} b - 0\right) \\ &= \lim_{b\to\infty}\left(\tan^{-1} b\right). \end{aligned}$$

We note that $\tan\frac{\pi}{2}$ happens to be infinity. This tells us that the above limit will be simply $\frac{\pi}{2}$. Hence this improper integral does exist.

(b) We see that

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\mathrm{d}x &= \lim_{b\to\infty}\left[\int_1^b \frac{1}{x}\mathrm{d}x\right] \\ &= \lim_{b\to\infty}\,[\ln x]_1^b \\ &= \lim_{b\to\infty}\,(\ln b - \ln 1) \\ &= \lim_{b\to\infty}\,(\ln b - 0) \\ &= \lim_{b\to\infty}\,(\ln b) \\ &= \infty,\end{aligned}$$

which shows that this improper integral does not exist - it diverges!

4. By inspecting the sketch in Figure 1, we see that the area equals the following integral:

$$\int_0^{\frac{\pi}{4}} \left(\cos^2 x - \frac{1}{2}\right) \mathrm{d}x.$$

Recall the following double-angle formula:

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 \\ \Rightarrow \frac{1}{2}\cos 2x &= \cos^2 x - \frac{1}{2},\end{aligned}$$

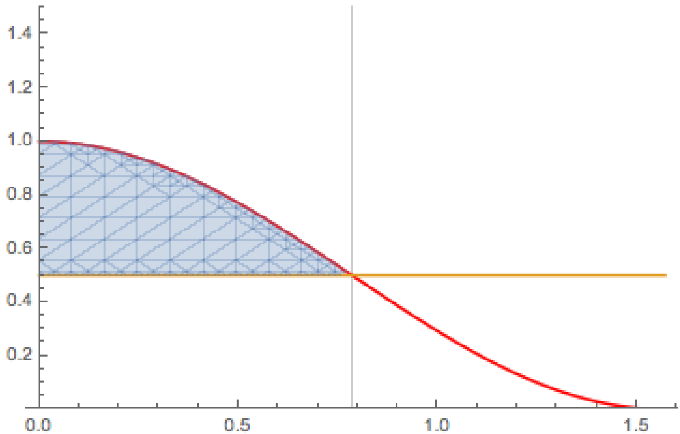

Figure 1: A sketch of the two curves $y = \cos^2 x$ and $y = \dfrac{1}{2}$. They just happen to intersect at $x = \frac{\pi}{4}$, which is the upper limit for the integral we want to calculate!

and so the area is

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \left(\cos^2 x - \frac{1}{2}\right) \mathrm{d}x &= \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin 0\right) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4}.\end{aligned}$$