Solutions to Problem Sheet 1

1. (a) First, let $y = \ln(\sin(x^{2016}))$. Put

$$f(u) = \ln u, \qquad u(x) = \sin(x^{2016}),$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{1}{u}\frac{d}{dx}\left(\sin(x^{2016})\right).$$

Next, substitute for u in terms of x. This gives

$$\frac{dy}{dx} = \frac{1}{\sin(x^{2016})}\frac{d}{dx}\left(\sin(x^{2016})\right).$$

Now apply the Chain Rule again to compute

$$\frac{d}{dx}\left(\sin(x^{2016})\right),$$

this time with

$$f(u) = \sin u, \qquad u(x) = x^{2016}.$$

Then

$$\begin{aligned}\frac{d}{dx}\left(\sin(x^{2016})\right) &= (\cos u)(2016\,x^{2015}).\\ &= 2016\,x^{2015}\cos(x^{2016}).\end{aligned}$$

Finally, putting everything together, one deduces that

$$\begin{aligned}\frac{dy}{dx} &= 2016\,x^{2015}\frac{\cos(x^{2016})}{\sin(x^{2016})}\\ &= 2016\,x^{2015}\cot(x^{2016}).\end{aligned}$$

(b) The easiest route for $y = (\cos x)^x$ would be to start with logarithmic differentiation, so take logs on both sides of this equation to get

$$\ln y = x\,\ln(\cos x).$$

Differentiating both sides w.r.t. x gives the following...

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left(x\,\ln(\cos x)\right)\\ &= 1\cdot\ln(\cos x) + x\cdot\frac{d}{dx}\left(\ln(\cos x)\right)\\ &= \ln(\cos x) + x\left(\frac{-\sin x}{\cos x}\right)\end{aligned}$$

where the Product and then the Chain Rule have been applied. Therefore

$$\frac{1}{y}\frac{dy}{dx} = \ln(\cos x) - x\tan x,$$

and so

$$\begin{aligned}\frac{dy}{dx} &= y\left[\ln(\cos x) - x\tan x\right]\\ &= (\cos x)^x\left[\ln(\cos x) - x\tan x\right].\end{aligned}$$

(c) First of all, note that $\cos^3(\exp x)$ is the same as $\cos^3(e^x)$. Then to compute the derivative of $y = \cos^3(e^x)$ by direct use of the Chain Rule, start by taking

$$f(u) = u^3, \qquad u(x) = \cos\left(e^x\right),$$

then

$$\begin{aligned}\frac{dy}{dx} &= 3u^2 \frac{d}{dx}\left(\cos\left(e^x\right)\right)\\ &= 3\cos^2(e^x)\frac{d}{dx}\left(\cos\left(e^x\right)\right).\end{aligned}$$

Next, use the Chain Rule a second time, this time with

$$f(u) = \cos u, \qquad u(x) = e^x$$

hence

$$\frac{d}{dx}\left(\cos\left(e^x\right)\right) = -\sin u \frac{d}{dx}(e^x) = -\sin\left(e^x\right)e^x.$$

Putting everything together, one has that

$$\frac{dy}{dx} = -3e^x\cos^2(e^x)\sin\left(e^x\right).$$

(d) The easiest method for computing the derivative of $y = \dfrac{\sin x}{x^2 + \sin x}$ is by using the Quotient Rule with

$$u = \sin x, \qquad v = x^2 + \sin x,$$

$$\frac{du}{dx} = \cos x, \qquad \frac{dv}{dx} = 2x + \cos x.$$

This gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2 + \cancel{\sin x})\cdot\cos x - \sin x\cdot(2x + \cancel{\cos x})}{(x^2 + \sin x)^2}\\ &= \frac{x^2\cos x - 2x\sin x}{(x^2 + \sin x)^2}\\ &= \frac{x(x\cos x - 2\sin x)}{(x^2 + \sin x)^2}.\end{aligned}$$

(e) Use the Product Rule with

$$u = e^{7x}, \qquad v = 2\sin(10x) + \cos(10x),$$

$$\frac{du}{dx} = 7e^{7x}, \qquad \frac{dv}{dx} = 20\cos(10x) - 10\sin(10x).$$

Thus

$$\begin{aligned}\frac{dy}{dx} &= e^{7x}\left[20\cos(10x) - 10\sin(10x)\right] \\ &+ 7e^{7x}\left[2\sin(10x) + \cos(10x)\right] \\ &= e^{7x}\left[(14-10)\sin(10x) + (20+7)\cos(10x)\right].\end{aligned}$$

So

$$\frac{dy}{dx} = e^{7x}\left[4\sin(10x) + 27\cos(10x)\right].$$

(f) Start with the Chain Rule where

$$f(u) = \sin u, \qquad u(x) = \cos\left(e^x \ln\, x\right),$$

then

$$\begin{aligned}\frac{dy}{dx} &= \cos u \frac{du}{dx} \\ &= \cos(\cos\left(e^x \ln\, x\right))\frac{d}{dx}\left(\cos\left(e^x \ln\, x\right)\right).\end{aligned}$$

Next, use the Chain Rule again, this time with

$$f(u) = \cos u, u(x) = e^x \ln\, x,$$

to yield

$$\begin{aligned}\frac{d}{dx}\left(\cos\left(e^x \ln\, x\right)\right) &= -\sin u \frac{du}{dx} \\ &= -\sin\left(e^x \ln\, x\right)\frac{d}{dx}\left(e^x \ln\, x\right) \\ &= -\sin\left(e^x \ln\, x\right)\left(e^x \frac{1}{x} + e^x \ln\, x\right),\end{aligned}$$

where the Product Rule has been used for the last step.
Hence

$$\frac{dy}{dx} = -\cos(\cos(e^x \ln x))\sin(e^x \ln x)\left(e^x \frac{1}{x} + e^x \ln x\right)$$
$$= -\frac{e^x}{x}(1 + x \ln x)\sin(e^x \ln x)\cos(\cos(e^x \ln x)).$$

2. If $w(x) = \frac{1}{v(x)}$, then the Product Rule states that:

$$f'(x) = u'(x)w(x) + u(x)w'(x) \tag{1}$$
$$= u'(x) \cdot \frac{1}{v(x)} + u(x) \cdot \frac{d}{dx}\left(\frac{1}{v(x)}\right). \tag{2}$$

But by the Chain Rule,

$$\frac{d}{dx}\left(\frac{1}{v(x)}\right) = \frac{-1}{[v(x)^2]} \cdot v'(x) = \frac{-v'(x)}{[v(x)^2]}.$$

Thus

$$f'(x) = \frac{u'(x)}{v(x)} + u(x) \cdot \left(\frac{-v'(x)}{[v(x)^2]}\right) \tag{3}$$
$$= \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{[v(x)^2]} \tag{4}$$
$$= \frac{v(x)u'(x)}{[v(x)^2]} - \frac{u(x)v'(x)}{[v(x)^2]}. \tag{5}$$

Therefore

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2},$$

which happens to be the Quotient Rule!

3. (a) The Quotient Rule will be used here...

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \qquad [\because \ \cos^2 x + \sin^2 x \equiv 1] \\
&= \sec^2 x, \qquad [\because \ \sec x \equiv \frac{1}{\cos x}]
\end{aligned}$$

which is the desired result.

(b) Put $y = \tan^{-1} x$. Then

$$\tan y = x. \tag{6}$$

Next, both sides can be differentiated implicitly (with the aid of the result from part a). The result is that...

$$\begin{aligned}
1 &= \sec^2 y \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y}.
\end{aligned}$$

But $\sec^2 y \equiv 1 + \tan^2 y$, hence

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}.$$

Finally, one can substitute in Equation (6) to rewrite the derivative in terms of x. This gives the expected result of

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

4. Implicitly differentiate both sides of the equation w.r.t. x, i.e. compute

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(3xy^2) + \frac{d}{dx}(2y) - \frac{d}{dx}(e^x) + \frac{d}{dx}(1) = \frac{d}{dx}(\cos y).$$

This reduces to

$$3x^2 - 3y^2 - 6xy\frac{dy}{dx} + 2\frac{dy}{dx} - e^x + 0 = -\sin y\frac{dy}{dx}.$$

Next, the reader is advised to collect all the $\frac{dy}{dx}$ terms on the right hand side as follows:

$$3x^2 - 3y^2 - e^x = (6xy - 2 - \sin y)\frac{dy}{dx},$$

which rearranges to

$$\frac{dy}{dx} = \frac{3x^2 - 3y^2 - e^x}{6xy - 2 - \sin y}.$$

5. (a) Look closely at the function

$$R(p) = p \cdot f(p).$$

This is a product of two functions, thus the Product Rule comes in handy here. One finds that

$$\begin{aligned} R'(p) &= 1 \cdot f(p) + p \cdot f'(p) \\ &= f(p) + p \cdot f'(p). \end{aligned}$$

Next, let $p = 120$.

$$R'(120) = f(120) + 120f'(120).$$

Thankfully, we are given that $f(120) = 9000$ and $f'(120) = -60$. If we substitute in these numbers, one sees that:

$$\begin{aligned} R'(120) &= 9000 - (120 \times 60) \\ &= 9000 - 7200 \\ &= 1800. \end{aligned}$$

(b) Because $R'(120) > 0$, a small increase in price will result in an increase of the manufacturer's revenue.