Lecture 5: Subgame Perfect Equilibrium

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Osborne: ch 7

How do we analyze extensive form games where there are simultaneous moves?

Example:

Stage 1. Player 1 chooses between \{IN,OUT\}

If OUT, game ends, player 1 gets 2 and player 2 gets 3

If IN, the proceed to stage 2, where both players play a simultaneous move game, one-sided prisoners’ dilemma
In stage 2 we cannot use backwards induction.

But we know how to solve stage 2:

a Nash equilibrium must be played.

Unique NE is stage 2, \((B, R)\)

Since IN yields payoff of 1 to player 1, optimal to choose OUT in stage 1.
What if there is not a unique prediction in stage 2?

Eg. stage 2 game is Hawk-dove game.

<table>
<thead>
<tr>
<th></th>
<th>h</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1,1</td>
<td>4,1</td>
</tr>
<tr>
<td>D</td>
<td>1,4</td>
<td>2,2</td>
</tr>
</tbody>
</table>

In stage 2, a Nash equilibrium must be played.

Two pure strategy Nash equilibria, \((H, d)\) and \((D, h)\).
a) Suppose that \((H, d)\) will be played

\[\Rightarrow\] payoffs will be \((4,1)\) in stage 2

So in stage 1, player 1 must choose IN

b) Suppose that \((d, H)\) will be played

\[\Rightarrow\] payoffs will be \((1,4)\) in stage 2

So in stage 1, player 1 must choose OUT.

Two pure strategy subgame perfect equilibria.
i) Player 1 plays IN (in stage 1) and H in stage 2; player 2 plays d in stage 2.

(IN, H; d)

ii) Player 1 plays OUT (in stage 1) and d in stage 2; player 2 plays H in stage 2.

(OUT, d; H)
Extensive game with perfect information and simultaneous moves

1. Set of players $N$

2. set of nodes $X$ or histories

3. $E \subseteq X$ is a terminal node or history.

4. Every player has a payoff associated with every terminal node.

4. At any history $h \in X - E$, a subset of players has to choose an action. (player function)
at this node, every player knows what actions have been taken by other players before this.

5. at any $h$ where player $i$ has to choose an action $A_i(h)$ is the set of available actions.

6. Any non terminal node $h$ and the action profile chosen at $h$ lead to another node $h' \in X$
A (pure) strategy for a player in an extensive game $\Gamma$ of perfect information is a plan that specifies an action at each decision node that belongs to her.

$X_i$: set of decision nodes where player $i$ must choose

Now more than one player may choose at each decision node.

$A(x)$ set of actions available at decision mode $x$

A pure strategy for player $i$ is a function $s_i : \rightarrow A(x)$ satisfying $s_i(x) \in A(x)$

$S_i$ is the set of pure strategies for $i$

In example, $S_i = \{\text{OUT&T, OUT&B, IN&T, IN&B}\}$
Specifies actions even at nodes that are ruled out by own strategy.

If we fix a pure strategy for each player, this is a pure strategy profile \( s = (s_1, s_2, \ldots, s_n) \)

This profile fully determines what happens in the game.

If there is no randomness (chance moves), then there is determinate terminal node that results.

(if there are chance moves, then \( s \) determines a probability distribution over \( E \))

So \( s \) gives rise to a payoff for each player.
$(S_i, u_i)_{i \in I}$ is the strategic form of $\Gamma$

We can therefore analyze $\Gamma$ by analysing its strategic form

We can solve for the Nash equilibria of the strategic form
A subgame of an extensive game $\Gamma$ is the game starting from some node $x$, where one or more players move simultaneously.

Subgame Perfect Nash Equilibrium: a profile of strategies $s = (s_1, s_2, \ldots, s_n)$ is a subgame perfect Nash equilibrium if a Nash equilibrium is played in every subgame.

Example 1: (OUT&B, L) is a subgame perfect Nash equilibrium

Example 2: (IN, H; d) is one SPE

(OUT, d; H) is another SPE
Committee Decision making (ch 7, Osborne)

Example: 3 member committee \{A, B, C\}

3 alternatives \(X = \{x, y, z\}\)

\[
\begin{array}{ccc}
A & B & C \\
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

Strict preference orderings
Voting system: binary agenda

stage 1: members simultaneously vote whether to adopt $x$ or not

If $x$ is chosen, end of story, if not

stage 2: vote between $y$ and $z$.

majority vote at each stage.

members vote strategically – *sophisticated voting*

anticipate the effects of their choices in future
Analysis: Stage 2:

A & B strictly prefer $y$ to $z$

One Nash equilibrium at this stage where A & b vote for $y$, $y$ wins

(Another Nash equilibrium where all voters vote for same alternative (say $z$) since no one can make a difference (with two others voting for $z$, this is a NE

Nash equilibrium involves weakly dominated choice – not reasonable.

So $y$ is chosen in stage 2
In stage 1, choice between $x$ and $y$.

A & C strictly prefer $x$ to $y$

therefore $x$ will be chosen in the subgame perfect equilibrium where voters do not use dominated choices.
General model of voting in binary agendas

$n$ alternative.

odd number of committee members with strict preference orderings (no indifference)

majority vote in each stage

Binary agenda: sequential procedure

adopt $x_1$ or not; if not, adopt $x_2$ or not; & so on..

different binary agenda for different ordering of votes (e.g. $x_n, x_{n-1}, \ldots, x_1$)
What will be the result of sophisticated voting in binary agendas?

$x$ beats $y$ if a majority of the committee prefer $x$ to $y$

If $x$ beats every other alternative, $x$ is a Condorcet winner

\begin{tabular}{|c|c|c|}
\hline
A & B & C \\
\hline
$x$ & $y$ & $y$ \\
\hline
$y$ & $z$ & $x$ \\
\hline
$z$ & $x$ & $z$ \\
\hline
\end{tabular}

$y$ is a Condorcet winner
Condorcet winner $y$ need not Pareto dominate $x$

**Proposition:** If a Condorcet winner $x^*$ exists, then $x^*$ is the undominated subgame perfect equilibrium outcome of *any* binary agenda

Proof: By backwards induction, we can determine alternative that will result at any node.

At the node $h$ where $x^*$can be adopted:

Let $y$ be the alternative that will be chosen if $x^*$ is not chosen.

A majority prefers $x^*$ to $y$, so $x^*$ will be adopted at $h$
By backwards induction, every preceding alternative will be rejected so that we get to $h$

since $x^*$ beats every other alternative.
What happens if there is no Condorcet winner?

\( x \) \emph{indirectly} beats \( y \) if either

a) \( x \) beats \( y \), or

b) there is a sequence of alternatives \( u_1, u_2, \ldots, u_k \) such that

\( x \) beats \( u_1 \) \& \( u_1 \) beats \( u_2 \) \& \ldots \& \( u_{k-1} \) beats \( u_k \) \& \( u_k \) beats \( y \).

A set of alternatives \( S \subseteq X \) such that every \( x \in S \) indirectly beats every other alternative in \( y \) is a \emph{top cycle set}. 
**Proposition:** If \( x \) is the winner of some binary agenda, then \( x \) must belong to the top cycle set.

**Proposition:** If \( x \) belongs to the top cycle set, then there is some binary agenda such that \( x \) wins.