Implementation in Adaptive Better-Response Dynamics

Antonio Cabrales, Universidad Carlos III de Madrid
Roberto Serrano, Brown University and IMDEA

October 2007
Summary

- Introduction
- The model
- Results: complete information
- Results: incomplete information
MOTIVATION

- Implementation theory has produced many mechanisms.
  - Not easy to know which is more relevant.

- Dynamic approach to test their robustness and simplicity/learnability.

- Recent research (Cabrales 1999, Cabrales and Ponti 2000, Sandholm 2002) showed:
  - Canonical mechanism (when implementing in strict Nash) stable and learnable. Integer games nonessential
  - More “refined” mechanism (in iterative deletion of WD strategies) can stabilize “bad” equilibria.

- Are negative results purely mechanism-driven?
  - Negative (but qualified) answer in this paper.
RESULTS

- Quasimonotonicity necessary for implementation when all kinds of mutations are allowed.

- Quasimonotonicity plus 3 players and $\epsilon$–security also sufficient.

- More permissive sufficient conditions with other assumptions on mutations:
  - “Regret” makes more serious mistakes less likely.
  - Mutations are all same order of magnitude (and exploit myopy heavily).

- For incomplete information environments:
  - Bayesian quasimonotonicity plus incentive compatibility necessary (and sufficient with 3 players and $\epsilon$–security).
The model (1/4)

PRELIMINARIES

- $N = \{1, ..., n\}$: set of agents.
- Environment: exchange economy.
- $X_i$: $i$’s consumption set, grid in $\mathbb{R}_+^l$.
- $\omega_i \in X_i$: $i$’s initial endowment.
- Set of allocations:

$$Z = \left\{(x_i)_{i \in N} \in \prod_{i \in N} X_i : \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i \right\}.$$
PREFERENCES

• \( \theta_i \): \( i \)'s preference ordering.

• Assumptions:
  1. No externalities.
  2. 0 is worst bundle.
  3. Increasing preference: For all \( i \) and for all \( x_i \in X_i \), if \( y_i \gg x_i \), \( y_i \succ^\theta_i x_i \).

• \( \theta = (\theta_i)_{i\in N} \in \Theta \): preference profile.

• \( f : \Theta \rightarrow Z \): social choice function (SCF).
**MECHANISMS AND IMPLEMENTATION**

- $G = \left((M_i)_{i \in N}, g\right)$: mechanism, where $M_i$ is $i$’s message set and $g : \prod_{i \in N} M_i \to Z$ is the outcome function.

- Played simultaneously every period by boundedly rational agents.

- Better-response dynamics (unperturbed Markov process):
  - Let $m(t)$ message vector at time $t$.
  - $m_i(t + 1)$ (if chosen to update) puts positive probability on any $m'_i$ such that
    $$g\left(m'_i, m_{-i}(t)\right) \succ^\theta_i g(m(t))$$

- Better-response dynamics with mistakes (perturbed Markov process):
  - Irreducible and aperiodic perturbation of better-response dynamics.

- An SCF is **implementable in stochastically stable strategies** if there is a mechanism $G$ such that a perturbation of the better response dynamics applied to its induced game when the preference profile is $\theta$ has $f(\theta)$ as the unique outcome supported by stochastically stable message profiles.
PROPERTIES OF SCF

• An SCF is $\varepsilon$–secure if for each $\theta$, and for each $i \in N$, $f(\theta) \geq (\varepsilon, ..., \varepsilon)$.

• An SCF is quasimonotonic if, whenever it is true that for every $i \in N$, $f(\theta) \succ_i^\theta z$ implies that $f(\theta) \succ_i^\phi z$, we have that $f(\theta) = f(\phi)$ for all $\theta, \phi \in \Theta$. 
NECESSITY AND SUFFICIENCY

**Theorem 1:** If \( f \) is implementable in SSS of any perturbed better-response dynamics, \( f \) is quasimonotonic.

**Proof:**

- Let true preference profile be \( \theta \).
- \( f \) implementable in SSS implies only \( f(\theta) \) is in set of recurrent classes.
- Let \( \phi \) such that for all \( i \), \( f(\theta) \succ_i^\theta z \) implies that \( f(\theta) \succ_i^\phi z \).
- Since \( f(\theta) \) is only outcome in recurrent class when preference is \( \theta \), when message profile gives \( \theta \):
  - Unilateral deviations for \( i \) must give either \( f(\theta) \) again,
  - or \( z \) with \( f(\theta) \succ_i^\theta z \).
- But this implies \( f(\theta) \) must also be in recurrent class when preferences are \( \phi \).
- And therefore \( f(\theta) = f(\phi) \), thus \( f \) is quasimonotonic.
Theorem 2: Let $n \geq 3$. If an SCF $f$ is $\varepsilon$-secure and quasimonotonic, it is implementable in SSS of any perturbed better-response dynamics.

Proof: Canonical mechanism

- **Message set:** $M_i = \Theta \times Z$.

- **Outcome function:**
  
  i. If $\forall i, m_i = (\theta, f(\theta))$, $g(m) = f(\theta)$.

  ii. If $\forall j \neq i, m_j = (\theta, f(\theta))$ and $m_i = (\phi, z) \neq (\theta, f(\theta))$:
    
    a. If $z \succeq_i f(\theta)$, $g(m) = (f_i(\theta) - \varepsilon, f_{-i}(\theta))$.
    
    b. If $f(\theta) \succ_i z$, $g(m) = z$.

  iii. In all other cases, $g(m) = 0$. 

Let $\theta$ be the true preference profile.

**Step 1** No message profile in rule (iii) is part of a recurrent class.

- W.l.o.g., suppose $m_1 = (\phi, z) \neq (\theta, f(\theta))$.
- Change one by one strategies of $i \neq 1$, to $(\theta, f(\theta))$.
- Outcome is still 0, so better response, until $(n-1)$ messages are $(\theta, f(\theta))$.
- Then outcome switches to either $z$ or $(f_1(\theta) - \beta, f_{-1}(\theta))$, both better-response.
- In last step agent 1 switches from $(\phi, z)$ to $(\theta, f(\theta))$. This yields $f(\theta)$, a better response and contradiction.

**Step 2** No message profile under rule (ii.a) is part of a recurrent class.

- $m_j = (\phi, f(\phi))$, for all $j \neq i$, and $m_i = (\phi', z')$ such that $z' \succeq_i f(\phi)$, leading to $f_i(\phi) - \beta$ for $i$.
- Agent $i$ switches to $(\phi, z)$, where $z_i = f_i(\phi) - \beta'$ (for $\beta' < \beta$) and $z_j = 0$ for every $j \neq i$, which yields outcome $z$.
- From here each $j \neq i$ can switch to $(\phi^j, z^j)$ (for some $(\phi^j, z^j) \neq (\phi, f(\phi))$), leading to rule (iii), contradiction.
Step 3  No recurrent class contains profiles under rule (ii.b).

- For all $j \neq i$ $m_j = (\phi, f(\phi))$, whereas $m_i = (\phi', z')$, satisfying that $f_i(\phi) \succ_i^\phi z'_i$. This implies outcome is $z'$.

- Agent $i$ switches, if necessary, to $(\phi', z)$, where $z_i = z'_i$ and for all $j \neq i$, $z_j = 0$, after which the outcome is $z$.

- As before, any of the other agents can switch to rule (iii), and contradiction.
Step 4  Only the truthful profile \((\theta, f(\theta))\) is a member of a recurrent class.

- Thus, all recurrent classes contain only profiles under rule (i). One cannot abandon rule (i) to get to another without passing through rule (ii). Thus, recurrent classes are singletons.

- Each recurrent class, a singleton under rule (i), must consist of a Nash equilibrium of the game when true preferences are \(\theta\), by better-response dynamics.

- One such Nash equilibrium is the truthful profile \((\theta, f(\theta))\) reported by every agent. Unilateral deviations lead to rule (ii.a) or rule (ii.b). Not possible under better-response dynamics.

- One may have other (non-truthful) Nash equilibria under rule (i). Let \((\phi, f(\phi))\) be such NE.

- For this to be a NE, for all \(i \in N\), \(f(\phi) \succeq_i z\) implies that \(f(\phi) \succeq^\theta_i z\).

- Moreover, since profile is a absorbing state of the dynamics, we must also have for all \(i \in N\), \(f(\phi) \succeq^\phi_i z\) implies that \(f(\phi) \succeq^\theta_i z\).

- Thus, because \(f\) is quasimonotonic, we must have that \(f(\theta) = f(\phi)\).
PERMISSIVE RESULTS

1. REGRET DYNAMICS

- Suppose agent $i$ moves at time $t$.
- $z_i^0$: bundle at period $t$.
- $y_i$: bundle that $i$ proposes.
- $z_i$: bundle that he receives in new outcome.
- Resistance of such transition:
  \[
  [u_i(z_i^0) - u_i(z_i)] - \lambda [u_i(y_i) - u_i(z_i)],
  \]
  where $0 < \lambda < 1$ is small enough. Call these better-response regret dynamics.
**Theorem 3:** Let $n \geq 3$. Then, any $\varepsilon$-secure SCF $f$ is implementable in SSS of any perturbed better-response regret dynamics.

- Proof based on (modified) canonical mechanism of Theorem 2.
- Quasimonotonicity of $f$ implies again recurrent classes are singletons under rule (i).
- Let $\theta$ denote the true preferences.
- We classify recurrent classes of unperturbed process into:
  - $E_0$ truth-telling profile, for each $i \in N$, $m_i = (\theta, f(\theta))$.
  - $E_j$ for $j = 1, \ldots, J$ is coordinated lie on profile $\theta^j$: for each $i \in N$, $m_i = (\theta^j, f(\theta^j))$, a Nash equilibrium of the mechanism under $\theta$. These require that for all $i \in N$, $f(\theta^j) \succ^\theta_i z$ implies that $f(\theta^j) \succ^\theta_i z$. 
• Modify outcome function of proof of Theorem 2:

(ii.a’.) Replace $\beta$ with $(\Delta, 0, \ldots, 0)$, punishment is smallest unit of nummeraire.

• Profile in $E_0$ is only stochastically stable profile:

[a] To get out of $E_0$, through rule (ii.a’) paying $(1 + \lambda)\Delta$ or through (ii.b) paying no less than $(1 + \lambda)\Delta$.
- After that, a mistake to rule (iii), costs $K$, takes us to 0.
- From there for free to any equilibria in $E_j$.

[b] To get out of any $E_j$, two paths but cheapest under rule (ii.a’) again.
- In this case, resistance is strictly smaller than $(1 + \lambda)\Delta$, because of the relief term.
- After that, to rule (iii) paying also $K$, and from there for free to $E_0$. 
2. UNIFORM MUTATIONS

- An SCF \( f \) is (strongly) Pareto efficient if for all \( \theta \) and for all \( z \neq f(\theta) \), there exists an \( i(\theta, z) \) such that \( f(\theta) \succ \theta i(\theta, z) z \).

- For every \( \theta \) and \( \phi \), there is an \( j(\theta, \phi) \) and \( x(\theta, \phi) \) and \( y(\theta, \phi) \) such that

\[
\begin{align*}
x(\theta, \phi) & \succ \theta_{j(\theta, \phi)} y(\theta, \phi) \\
y(\theta, \phi) & \succeq \phi_{j(\theta, \phi)} x(\theta, \phi).
\end{align*}
\]

Denote by \( J(\theta, \phi) \) the set of agents \( j(\theta, \phi) \) for whom there exists a preference reversal between a pair of alternatives across states \( \theta \) and \( \phi \), as specified in (*)).

(5) For each \( \theta \) and \( \phi \), there is \( j(\theta, \phi) \in J(\theta, \phi) \) such that \( j(\theta, \phi) \neq i(\theta, x(\theta, \phi)) \), where \( x(\theta, \phi) \) is an alternative for which agent \( j(\theta, \phi) \) has a preference reversal as in (*).
Theorem 4. Suppose environment satisfies (1), (2) and (5). Let $n \geq 5$. Any $\epsilon$-secure and strongly Pareto efficient SCF $f$ is implementable in SSS, when mutations are uniform.

Proof: Let $M_i = \Theta \times Z$, $m_i = (m^1_i, m^2_i)$, $m = (m^1, m^2)$.

(i.) If for every $i \in N$, $m^1_i = \theta$, $g(m) = f(\theta)$.

(ii.a.) If exactly $(n - 1)$ messages $m_i$ are such that $m^1_i = \theta$ and $m_i(\theta, x(\theta, \phi)) = (\phi, x(\theta, \phi))$, $g(m) = (x_i(\theta, x(\theta, \phi))(\theta, \phi), x_j(\theta, \phi)(\theta, \phi), 0, 0, \ldots, 0)$.

(ii.b.) If exactly $(n - 1)$ messages $m_i$ are such that $m^1_i = \theta$, but the odd man out, say agent $k$, does not satisfy the requirements of rule (ii.a), $g(m) = (f_k(\theta) - \beta, f_{-k}(\theta))$, where $f_k(\theta) \geq f_k(\theta) - \beta \geq (\epsilon, \ldots, \epsilon)$.

(iii.a.) If exactly $(n - 2)$ messages $m_i$ are such that $m^1_i = \theta$, $m_i(\theta, x(\theta, \phi)) = (\phi, x(\theta, \phi))$ and $m_j(\theta, \phi) = (\phi, y(\theta, \phi))$, $g(m) = (y_i(\theta, x(\theta, \phi))(\theta, \phi), y_j(\theta, \phi)(\theta, \phi), 0, 0, \ldots, 0)$.

(iii.b.) If exactly $(n - 2)$ messages $m_i$ are such that $m^1_i = \theta$, but we are not under rule (iii.a), for all $k \in N$, $g_k(m) = (\epsilon, \ldots, \epsilon)$.

(iv.) In all other cases, $g(m) = 0$. 
$E^j_0$ All $n$ agents report the true state $\theta$ as the first part of their announcement.

$E^j_1$ Agents' reported state is not $\theta$, the true state.

[a] To get out of $E^j_0$, $i(\theta, x(\theta, \phi))$

- imposes one reversal $x(\theta, \phi)$ – one mistake.
- Next, $j(\theta, \phi)$ imposes $y(\theta, \phi)$ – second mutation.
- Finally, anyone changes to (iv) where 0 is the outcome – third mutation.
- From 0, for free to any other absorbing state.

[b] To get out of an untruthful profile, say $m^1 = \phi$:

- $i(\phi, x(\phi, \theta))$ can impose $x(\phi, \theta)$. If $f(\phi) \succ^\theta_{i(\phi, x(\phi, \theta))} x(\phi, \theta)$, this requires a first mutation. If $x(\phi, \theta) \succeq^\theta_{i(\phi, x(\phi, \theta))} f(\phi)$, zero resistance.
- Next, $j(\phi, \theta)$ changes to $y(\phi, \theta)$ for free.
- Finally, someone changes to 0 under rule (iv), at most a second mutation.
- From there, for free to any other absorbing state.
ENVIRONMENT

- Each agent knows $\theta_i \in \Theta_i$.
- Let $\Theta = \prod_{i \in N} \Theta_i$ and $\Theta_{-i} = \prod_{j \neq i} \Theta_j$.
- We assume the set of states with ex-ante positive probability is $\Theta$.
- Let $q_i(\theta_{-i}|\theta_i)$ be type $\theta_i$'s interim probability over $\theta_{-i}$.
- An SCF is a mapping $f : \Theta \mapsto Z$.
- Let $A$ denote the set of SCFs.
- We shall $\theta_i$'s interim expected utility over an SCF $f$:
  \[ U_i(f|\theta_i) \equiv \sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i) u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})). \]
- $G = ((M_i)_{i \in N}, g), m_i : \Theta_i \rightarrow M_i$, and $g : \Theta \mapsto Z$. 

Results: incomplete information (1/5)
Results: incomplete information (2/5)

- Strategy revision using the interim better-response logic. That is, letting $m^t$ profile at period $t$, type $\theta_i$ switches from $m^t_i(\theta_i)$ to any $m'_i$ such that:

$$
\sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i) u_i(g(m'_i, m^t_{-i}(\theta_{-i})), (\theta_i, \theta_{-i})) \geq \sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i) u_i(m^t(\theta), \theta).
$$

- An SCF $f$ is implementable in asymptotically stable strategies if there exists $G$ such that interim better-response process has $f$ as unique outcome of the recurrent classes of the process.

- An SCF $f$ is implementable in stochastically stable strategies if there exists $G$ such that a perturbation of the interim better-response process has $f$ as unique outcome supported by stochastically stable strategy profiles.
NECESSITY

An SCF $f$ is strictly incentive compatible if for all $i$ and for all $\theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(f(\theta), \theta) > \sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i}))$$

for every $\theta_i' \neq \theta_i$.

**Theorem 5.** If $f$ is implementable in SSS of any perturbation of interim better-response dynamics, $f$ is incentive compatible. If at least one recurrent class is a singleton, $f$ is strictly incentive compatible.
Consider a mapping \( \alpha_i = (\alpha_i(\theta_i))_{\theta_i \in \Theta_i} : \Theta_i \mapsto \Theta_i \). A \textit{deception} \( \alpha = (\alpha_i)_{i \in N} \) is a collection of such mappings where at least one differs from the identity mapping.

Given an SCF \( f \) and a deception \( \alpha \), let \( [f \circ \alpha] \) denote the following SCF: \( [f \circ \alpha](\theta) = f(\alpha(\theta)) \) for every \( \theta \in \Theta \).

Finally, for a type \( \theta'_i \in \Theta_i \), and an arbitrary SCF \( y \), let \( y_{\theta'}(\theta) = y(\theta'_i, \theta_{-i}) \) for all \( \theta \in \Theta \).

An SCF \( f \) is \textit{Bayesian quasimonotonic} if for all deceptions \( \alpha \), for all \( i \in N \), and for all \( \theta_i \in \Theta_i \), whenever
\[
U_i(f \mid \theta_i) > U_i(y_{\theta'} \mid \theta_i) \forall \theta'_i \in \Theta_i \quad \text{implies} \quad U_i(f \circ \alpha \mid \theta_i) > U_i(y \circ \alpha \mid \theta_i), \quad (**)
\]
on one must have that \( f \circ \alpha = f \).

**Theorem 6.** If \( f \) is implementable in asymptotically stable strategies of an unperturbed interim better-response dynamic process, \( f \) is Bayesian quasimonotonic.
Theorem 7. Suppose the environments satisfy Assumptions (1) and (2) in each state. Let $n \geq 3$. If an SCF $f$ is $\epsilon$-secure, strictly incentive compatible and Bayesian quasimonotonic, $f$ is implementable in asymptotically stable strategies of interim better-response dynamics.

Proof: $G = ((M_i)_{i \in N}, g), M_i = \Theta_i \times A$. $m_i = (m^1_i, m^2_i)$. Outcome function $g$ is:

(i.) If for every agent $i \in N$, $m^2_i = f$, $g(m) = f(m^1)$.

(ii.) If for all $j \neq i$ $m^2_j = f$ and $m^2_i = y \neq f$, one can have two cases:

(ii.a.) If there exist types $\theta_i, \theta'_i \in \Theta_i$ such that $U_i(y_{\theta'_i} | \theta_i) \geq U_i(f | \theta_i)$, $g(m) = (f_i(m^1) - \beta, f_{-i}(m^1))$, where $f_i(m^1) \geq f_i(m^1) - \beta \in X_i$.

(ii.b.) If for all $\theta_i, \theta'_i \in \Theta_i$, $U_i(y_{\theta'_i} | \theta_i) < U_i(f | \theta_i)$, $g(m) = y(m^1)$.

(iii.) In all other cases, $g(m) = 0$. 
Implementation in Adaptive Better-Response Dynamics

Antonio Cabrales, Universidad Carlos III de Madrid
Roberto Serrano, Brown University and IMDEA

October 2007