Optimal Second-degree Price Discrimination under Collusion: On the Role of Asymmetric Information among Buyers*

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Abstract

The traditional theory of second-degree price discrimination tackles individual self-selection but does not address the possibility that buyers could form a coalition to coordinate their purchases and to reallocate the goods. In this paper, we design the optimal sale mechanism which takes into account both individual and coalition incentive compatibility when buyers can collude under asymmetric information. We show that the monopolist can achieve the same profit regardless of whether or not buyers can collude. Although, in the optimal sale mechanism, marginal rates of substitution are not equalized across buyers (hence there exists room for arbitrage), they fail to realize the gains from arbitrage because of the transaction costs in coalition formation generated by asymmetric information.

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1 Introduction

The theory of monopolistic screening[1] (second-degree price discrimination) studies a monopolist's optimal pricing scheme when she has incomplete information about buyers' individual preferences.[2] According to the theory, the monopolist can maximize her profit by using a menu of packages which induces each type of buyer to select the package designed for the type. While the theory tackles the self-selection issue at the individual level, it assumes away the possibility that price discrimination might induce buyers to form a coalition in order to coordinate their purchases and to reallocate the goods they bought, possibly at the expense of the seller. In other words, the theory is concerned with individual incentive compatibility but not with coalition incentive compatibility. In this paper, we study the optimal sale mechanism which takes into account both individual and coalition incentive compatibility.

The possibility that price discrimination might induce buyers to engage in arbitrage was pointed out in the context of third-degree price discrimination to explain an upstream monopolist's vertical integration as a response to arbitrage.[3] We think that arbitrage is an issue in the case of second-price discrimination as well since the standard optimal screening mechanisms create room for arbitrage. In reality, there exists much evidence of (illegal or legal) cooperation among buyers. On the one hand, bidders' collusive behavior in auctions is well documented and auction literature has been devoting an increasing attention to the topic.[4] On the other hand, buyers often form cooperatives to jointly purchase goods.[5] One central question regarding buyer coalitions is how asymmetric information among the buyers (about each other's willingness to pay) affects coalition formation. In our paper, we have two major goals: *i*) to identify the transaction costs in coalition formation generated by asymmetric information and *ii*) to design the optimal sale mechanism under collusion that exploits these transaction costs.

[1] See, for instance, Maskin and Riley (1984) and Mussa and Rosen (1978) for an introduction and Rochet and Stole (2002) for a recent contribution dealing with random participation.

[2] We use 'she' to represent the monopolist and 'he' to represent a buyer or the third-party.

[3] See pp. 141-142 in Tirole (1988).

[4] For examples, see Caillaud and Jehiel (1998), Graham and Marshall (1987), McAfee and McMillan (1992) and Brusco and Lopomo (2002).

[5] There exist various forms of supply cooperatives to purchase some products together. For instance, Heflebower (1980) describes three types of supply cooperatives: farmers's cooperatives, consumer cooperatives and those run by urban businesses.

Consider for example the situation in which an upstream monopolist sells her goods to two downstream firms operating in separate markets. Given a menu of quantity-transfer pairs offered by the monopolist, the two downstream firms can employ two instruments to increase their joint payoffs. First, they can jointly decide which pair each buyer should choose. In our paper, this is modeled by manipulation of the reports which the buyers send into the sale mechanism. Second, after choosing the manipulation of reports, they can reallocate among themselves the goods bought from the seller.

Our main result is that when buyers form a coalition under asymmetric information, the monopolist can do as well as when there is no coalition by fully exploiting the transaction costs in coalition formation. Although in the optimal sale mechanism the marginal rates of substitutions are not equalized across buyers with different types (hence there exists room for arbitrage), the buyers fail to realize any gain from arbitrage because of the incentive problem inside the coalition. We quantify the transaction costs generated by asymmetric information and show that they are larger than the gains from arbitrage. We also show that the allocation obtained by the optimal sale mechanism which deters buyer coalition at no cost can be implemented through a menu of two-part tariffs.

Consider, for simplicity, a two-buyer-two-type setting in which the seller can produce any positive amount of a homogeneous product at a constant marginal cost and a buyer has either high valuation (H-type) or low valuation (L-type) for the product. Types are independently and identically distributed and a buyer's type is his private information. In the optimal mechanisms without buyer coalition, the quantity sold to a buyer depends solely on his report on the type and is determined by equalizing the marginal cost to the type's marginal surplus evaluated with a "virtual valuation". As it is well known, L-type's virtual valuation is lower than his real valuation[6] and this results in a downward distortion in the quantity allocated to L-type compared to the first-best level. This downward distortion creates room for arbitrage when the state of nature is such that one buyer has H-type while the other has L-type, since L-type has a higher marginal surplus than H-type. In the absence of transaction costs in coalition formation, the buyers can successfully reallocate goods from H-type to L-type and Pareto increase their payoffs. Furthermore, this could alter buyers' incentive to report truthfully in the sale mechanism and modify the seller's expected profit. In this paper, we focus on how asymmetric information affects buyers' abilities to do arbitrage and how it ultimately

[6] This is because the payment the seller receives from H-type decreases in the quantity sold to L-type.

affects the seller's profit.

Drawing on Laffont and Martimort (1997, 2000), we model coalition formation under asymmetric information by a side-contract offered to the buyers by a third-party who maximizes the sum of buyers' payoffs. The side-contract specifies both the manipulation of the reports made into the sale mechanism and the reallocation of the goods obtained from the seller. The side-contract must satisfy budget balance constraints as well as participation and incentive constraints. The incentive constraints need to hold since the third-party ignores the buyers' types; the acceptance constraints are defined with respect to the utilities the buyers obtain when playing the sale mechanism non-cooperatively.

We first show that if the seller uses simple mechanisms in which the quantity that a buyer receives and his payment do not depend on the other buyer's report, buyers can realize strict gains at the seller's loss by coordinating their purchases and reallocating the goods. However, we also show that the seller can design sale mechanisms which deter manipulation of reports and reallocation of goods and yield the same profit as when there is no buyer coalition. In particular, the third party is not able to implement any efficient arbitrage between H-type and L-type because of the tension between incentive and participation constraints in the side-contract. The intuition can be provided as follows. Since the rent that H-type obtains by pretending to be L-type in the side mechanism is increasing in the quantity received by L-type, if the third-party reallocates some quantity from H-type to L-type then he is forced to concede H-type a higher rent in order to induce him to truthfully report his type.[7] This increase in the rent is defined as the transaction costs generated by asymmetric information. Since the transaction costs are larger than the gains from reallocating quantity from H-type to L-type, the arbitrage cannot be realized. Our main result extends to more general settings: when the marginal cost is increasing, when there are n buyers, when there are three types.[8]

The literature about consumer coalitions mostly addresses issues different from the one we consider in this paper.[9] Alger (1999) is one exception: She studies the optimal

[7] Lowering L-type's payoff is not feasible since it would induce L-type to reject the side-contract.

[8] These extensions are relatively straightforward except the one for the 3-type setting, in which a single crossing condition for coalitions (which is very useful to prove our result in the 2-type setting) holds only partially.

[9] For instance, Innes and Sexton (1993, 1994) analyze the case in which the monopolist is facing identical consumers who may form coalitions. They show that even though consumers' characteristics are homogeneous, the monopolist may price discriminate in order to deter the formation of coalitions,

menu of price-quantity pairs when (a continuum of) consumers are able to purchase multiple times or/and jointly in a two-type setting. She finds that with multiple purchases only, the monopolist offers strict quantity discounts while, with joint purchases only, discounts are infeasible. Her results are based on two following assumptions. First, consumer coalitions are formed under complete information among the consumers about each other's type.[10] Second, the set of mechanisms available to the seller is restricted by assuming that the quantity allocated to a consumer and his payment do not depend on the other consumers' choices. In contrast, in our model a coalition is formed under asymmetric information among buyers and the seller can use complete contracts such that the quantity sold to a buyer and his payment can depend on the others' choices.

Using a third-party to model collusion under asymmetric information was first introduced in auction literature.[11] While that literature studies the optimal auction in a restricted set of mechanisms (they usually find the optimal reserve price for a first or second price auction), Laffont and Martimort (1997, 2000) use a more general approach in that they characterize the set of collusion-proof mechanisms and optimize in this set. In their settings in which reallocation is infeasible,[12] they show that if the agents' types are independently distributed, then the second-best outcome can be implemented by a dominant-strategy mechanism which eliminates any gain from joint manipulation of reports;[13] furthermore, this mechanism does not exploit the transaction costs created by asymmetric information. We show that the dominant-strategy mechanism is not collusion-proof in our setting since the coalition owns the additional instrument of quantity reallocation and prove that the seller can still achieve the second-best profit by fully exploiting the transaction costs in coalition formation. We also note that Laffont and Martimort limit the analysis to the two-agent-two-type setting and do not consider implementation through non-direct mechanisms.

Our paper is to some extent related to the papers studying auctions with resale. For

whereas price discrimination is unprofitable in the absence of the coalitions.

[10]Furthermore, only consumers with the same type can form coalitions.

[11]See the first three papers mentioned in footnote 4

[12]In the first paper, they study collusion between two regulated firms producing complementary inputs. The firms have independently distributed types and collusion has bite since an exogenous restriction on the set of the principal's mechanisms is imposed. In the second paper, they consider collusion between consumers of a public good with correlated types. Consumers have incentives to collude since the principal will fully extract their rents if they behave non-cooperatively.

[13]See Proposition 11 and Proposition 6 in Laffont and Martimort (1997) and (2000), respectively.

instance, Ausubel and Cramton (2001) analyze the optimal auction when buyers can engage in resale after receiving goods from the auctioneer and the resale is (assumed to be) always efficient. They prove that the seller maximizes his profit by allocating goods efficiently. In contrast, in our setting, buyers sign a binding side-contract before each buyer chooses how much to buy. We show that they fail to achieve efficient reallocation because of the transaction costs in coalition formation.[14]

The rest of the paper is organized as follows. In Section 2, we introduce the model. In Section 3, after reviewing as a benchmark the optimal sale mechanisms without buyer coalition, we show that they exhibit room for arbitrage and this in turn can induce reports manipulations. In order to define the seller's optimization problem under collusion, in Section 4, we prove the (weakly) collusion-proofness principle and characterize the constraints that a collusion-proof mechanism must satisfy. In Section 5, we define and solve the seller's problem and prove our main result that these constraints can be satisfied without reducing the seller's profit. In Section 6, we extend the main result to more general settings. In Sections 4-6, we make some specific assumptions about buyers' off-the-equilibrium-path beliefs and behavior. In Section 7, we show that our main result is robust to changing these assumptions. Concluding remarks are given in Section 8. Most of the proofs are left to Appendix.

2 The model

2.1 Preferences, information and mechanisms

A seller (for instance, an upstream monopolist) can produce any amount $q \geq 0$ of homogeneous goods at cost $C(q)$ (with $C(0) = 0$, $C'(q) > 0$ and $C''(q) \geq 0$ for any $q \geq 0$) and sells the goods to $n \geq 2$ buyers (for instance, downstream firms operating in separate markets). The seller cannot monitor the quantity of the goods actually used by a buyer but can observe whether or not a buyer uses her goods.[15] In what follows,

[14] Zheng (2001) allows resale in a one-good auction with asymmetrically distributed buyers' values and proves that an equilibrium exists which induces the same payoffs as if resale can be costlessly banned.

[15] The assumption is from Rey and Tirole (1986). They introduce it to justify the use of two-part tariffs by an upstream monopolist. In our model, the assumption allows the seller to use a penalty in order to prohibit the resale from a buyer who bought goods from her to another who did not buy any good from her.

for expositional simplicity, we focus on the two-buyer-two-type setting with constant marginal cost $c(>0)$ but our main result holds for any convex cost function and for the n-buyer or three-type setting (see remark 3 after Proposition 7 and Section 6).

Buyer i $(i=1,2)$ obtains payoff $\theta^i u(q^i) - t^i$ with $\theta^i > 0$ from consuming quantity $q^i \geq 0$ of the goods and paying $t^i \in \mathbb{R}$ units of money to the seller. He privately observes his type $\theta^i \in \Theta \equiv \{\theta_L, \theta_H\}$, where $\Delta\theta \equiv \theta_H - \theta_L > 0$. The types θ^1 and θ^2 are identically and independently distributed with $\Pr\left\{\theta^i = \theta_L\right\} = p_L \in (0,1)$, $i=1,2$. The distribution of θ^1 and θ^2 is common knowledge. We suppose that u is twice differentiable, $u'(q) > 0 > u''(q)$ for any $q \geq 0$, $u(0)=0$ and $(\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(0) > c > \lim_{q\to+\infty}\theta_H u'(q)$, hence each type receives a positive and finite quantity in the optimal mechanism without buyer coalition.[16] The reservation utility of each type of buyer is normalized to zero.

The seller designs a sale mechanism to maximize her expected profit. A generic sale mechanism is denoted by M and, according to the revelation principle, we can restrict our attention to direct revelation mechanisms:

$$M = \left\{ q^i(\widehat{\theta}^1, \widehat{\theta}^2), t^i(\widehat{\theta}^1, \widehat{\theta}^2);\ i=1,2 \right\},$$

where $\widehat{\theta}^i \in \{\theta_L, \theta_H\}$ is buyer i's report, $q^i(\cdot)$ is the quantity he receives and $t^i(\cdot)$ is his payment to the seller. Since buyers are ex ante identical, without loss of generality we focus on symmetric mechanisms in which the quantity sold to a buyer and his payment depend only on the reports $(\widehat{\theta}^1, \widehat{\theta}^2)$ and not on his identity. Then, we can introduce the following notation to simplify the exposition: For quantities,

$$\begin{aligned} q_{HH} &= q^1(\theta_H,\theta_H) = q^2(\theta_H,\theta_H),\ q_{HL} = q^1(\theta_H,\theta_L) = q^2(\theta_L,\theta_H), \\ q_{LH} &= q^1(\theta_L,\theta_H) = q^2(\theta_H,\theta_L),\ q_{LL} = q^1(\theta_L,\theta_L) = q^2(\theta_L,\theta_L). \end{aligned}$$

$(t_{HH}, t_{HL}, t_{LH}, t_{LL}) \in \mathbb{R}^4$ are similarly defined. Let $\mathbf{q} \equiv (q_{HH}, q_{HL}, q_{LH}, q_{LL})$ denote the vector of quantities and $\mathbf{t} \equiv (t_{HH}, t_{HL}, t_{LH}, t_{LL})$ denote the vector of transfers.

2.2 Buyer coalition

Drawing on Laffont and Martimort (1997, 2000), we model buyers' coalition formation by a side-contract, denoted by S, offered by a benevolent third-party. This method

[16] Our results below extend to the case in which the seller refuses to serve L-type, which occurs if $(\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(0) \leq c$.

may appear unrealistic as it may seem more natural to model coalition formation by considering a specific bargaining model. However, we point out an important property of the coalition formation model we analyze: The revelation principle implies that, given a specific bargaining game G, any allocation achieved by a Bayesian equilibrium of G can be obtained by a side-contract offered by the third party. Since we let the third party maximize the sum of buyers' expected payoffs, we are describing the upper bound of what the coalition may achieve under asymmetric information. Furthermore, since we show that collusion does not hurt the seller, the property implies that specifying any particular bargaining game between the buyers would not change our main message.

The third party designs S in order to maximize the sum of buyers' expected payoffs subject to incentive compatibility (since he ignores the types), participation and budget balance constraints. The participation constraints are written with respect to the utility that each type obtains when M is played non-cooperatively. Precisely, the game of seller's mechanism offer cum buyer coalition formation has the following timing.

Stage 1. Nature draws buyers' types (θ^1, θ^2); buyer i privately observes θ^i, $i = 1, 2$.

Stage 2. The seller proposes a sale mechanism M.

Stage 3. Each buyer simultaneously accepts or rejects M. If at least one buyer refuses M, then each buyer earns the reservation utility and the following stages do not occur.[17]

Stage 4. If both buyers accept to play M, then the third party proposes them a direct side-contract S in order to jointly manipulate their reports into M and to reallocate between themselves the goods bought from the seller.[18]

Stage 5. Each buyer simultaneously accepts or rejects S.

Stage 6. If at least one buyer refuses S, then M is played non-cooperatively. In this case, reports are directly made in M and stages 7 and 9 below do not occur. If instead

[17] We may rather assume that if one buyer (say, buyer 1) vetoes M, then the seller can serve buyer 2 by offering a one-buyer mechanism. Our results below are robust to this modification since the seller can prohibit buyer 2 from reselling to buyer 1 part of the goods he bought from the seller – recall that the seller can observe whether or not a buyer uses her goods, hence she can specify in her mechanism a high penalty that buyer 2 has to pay in case he resells goods to buyer 1.

[18] To be rigorous, the Revelation Principle applies to the third-party's design of S but does not apply to the seller's design of M. Thus, we should allow the seller to propose non-direct sale mechanisms. Nevertheless, as Proposition 3 in Laffont and Martimort (2000) establishes, any perfect Bayesian equilibrium outcome arising from a non-direct sale mechanism can be obtained as a perfect Bayesian equilibrium outcome induced by a direct sale mechanism.

S has been accepted by both buyers, then reports are made into S.

Stage 7. As a function of the reports in S, the third party enforces the manipulation of reports into M.

Stage 8. Quantities and transfers specified in M are enforced.

Stage 9. Quantity reallocation and side-transfers specified in S (if any) take place in the buyer coalition.

Formally, a side-contract takes the following form:

$$\{\phi(\widetilde{\theta}^1,\widetilde{\theta}^2), x^i(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi}), y^i(\widetilde{\theta}^1,\widetilde{\theta}^2);\ i=1,2\},$$

where $\widetilde{\theta}^i \in \{\theta_L,\theta_H\}$ is buyer i's report to the third-party. $\phi(\cdot)$ is the report manipulation function which maps any pair of reports made by the buyers to the third-party, i.e., $(\widetilde{\theta}^1,\widetilde{\theta}^2)$, into a pair of reports made to the seller. We assume that $\phi(\cdot)$ can specify stochastic manipulations, as this convexifies the third-party's feasible set. More precisely, let $\widetilde{\phi}\in\Theta^2$ denote an outcome of $\phi(\cdot)$. Then, $\phi(\cdot)$ specifies the probability $p^\phi(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi})$ that the third party, after receiving reports $(\widetilde{\theta}^1,\widetilde{\theta}^2)$, requires the buyers to report $\widetilde{\phi}$ to the seller. When the manipulation is deterministic, i.e., $p^\phi(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi})=1$ for some $\widetilde{\phi}\in\Theta^2$, we write $\phi(\widetilde{\theta}^1,\widetilde{\theta}^2)=\widetilde{\phi}$ with some abuse of notation. After the buyers bought goods from the seller, the third-party can reallocate them within the coalition. Let $x^i(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi})$ represent the quantity of goods that buyer i receives from the third-party when $\widetilde{\phi}$ is reported to the seller. Finally, $y^i(\widetilde{\theta}^1,\widetilde{\theta}^2)$ denotes the monetary transfer from buyer i to the third-party. Because of risk-neutrality and quasi linearity, we do not need to let y^i depend on $\widetilde{\phi}$. We impose the following ex post budget balance constraints for the reallocation of goods and for the side transfers, respectively

$$\sum_{i=1}^{2} x^i(\theta^1,\theta^2,\widetilde{\phi})=0 \quad \text{and} \quad \sum_{i=1}^{2} y^i(\theta^1,\theta^2)=0, \quad \text{for any } (\theta^1,\theta^2)\in\Theta^2 \text{ and any } \widetilde{\phi}\in\Theta^2.$$

After a side-contract S is proposed, a two-stage game is played by buyers: in its first stage (stage 5) each buyer accepts or rejects S; in the second stage (stage 6) the buyers report types either into M or into S depending on their decisions at the first stage. We are interested in (collusive continuation) equilibria in which both buyers accept S; thus, no learning about types occurs along the equilibrium path.[19] In Sections 4-6, we make

[19] Notice, however, that there also exists an equilibrium in which both buyers refuse any side mechanism: If buyer i is vetoing any side mechanism, then rejecting is a best reply for buyer j.

the following assumption:[20]

$$\left\{ \begin{array}{c} \textbf{Assumption WCP: } \text{Given an incentive compatible mechanism } M\text{, if buyer } i \\ \text{vetoes } S \text{ (which is an off-the-equilibrium-path event), then buyer } j \neq i \text{ still} \\ \text{has prior beliefs about } \theta^i \text{ and the truthful equilibrium is played in } M. \end{array} \right.$$

By definition, truthtelling is an equilibrium in M under prior beliefs if and only if M is incentive compatible. Let $U^M(\theta_j)$ $(j = L, H)$ denote the expected payoff of j-type in the truthful equilibrium in M. Then, $U^M(\theta_j)$ is the reservation utility for j-type when deciding whether to accept S or not. In Section 7, we relax this assumption WCP.

3 Do the optimal mechanisms without buyer coalition exhibit room for collusion?

In this section, we first analyze the optimal mechanisms in the absence of buyer coalition and then examine whether in such mechanisms there exists any room for collusion.

3.1 The optimal mechanisms without buyer coalition

In this subsection, we characterize the profit maximizing mechanisms when there is no buyer coalition. The seller's expected profit with mechanism $M = \{\mathbf{q}, \mathbf{t}\}$ is

$$\Pi \equiv 2p_L^2(t_{LL} - cq_{LL}) + 2p_L(1 - p_L)(t_{HL} + t_{LH} - cq_{HL} - cq_{LH}) + 2(1 - p_L)^2(t_{HH} - cq_{HH})$$

M should satisfy the following Bayesian incentive compatibility constraints: for H-type,

$$\begin{aligned} (BIC_H) \quad & p_L[\theta_H u(q_{HL}) - t_{HL}] + (1 - p_L)[\theta_H u(q_{HH}) - t_{HH}] \\ & \geq p_L[\theta_H u(q_{LL}) - t_{LL}] + (1 - p_L)[\theta_H u(q_{LH}) - t_{LH}]; \end{aligned} \tag{1}$$

for L-type,

$$\begin{aligned} (BIC_L) \quad & p_L[\theta_L u(q_{LL}) - t_{LL}] + (1 - p_L)[\theta_L u(q_{LH}) - t_{LH}] \\ & \geq p_L[\theta_L u(q_{HL}) - t_{HL}] + (1 - p_L)[\theta_L u(q_{HH}) - t_{HH}]. \end{aligned} \tag{2}$$

[20] WCP means weakly collusion-proof. The assumption makes us add the qualifier "weakly" in our definition of collusion-proof mechanisms: see Definition 2.

M should also satisfy the following individual rationality constraints: for H-type,

$$(BIR_H) \quad p_L[\theta_H u(q_{HL}) - t_{HL}] + (1-p_L)[\theta_H u(q_{HH}) - t_{HH}] \geq 0; \tag{3}$$

for L-type,

$$(BIR_L) \quad p_L[\theta_L u(q_{LL}) - t_{LL}] + (1-p_L)[\theta_L u(q_{LH}) - t_{LH}] \geq 0. \tag{4}$$

The seller designs M to maximize Π subject to (1) to (4). We characterize the optimal mechanisms in the next proposition:

Proposition 1 *The optimal mechanisms in the absence of buyer coalition are characterized as follows.*
(a) The optimal quantity schedule $\mathbf{q}^* = (q^*_{HH}, q^*_{HL}, q^*_{LH}, q^*_{LL})$ *is given by:*
(i) $q^*_{HH} = q^*_{HL} = q^*_H$, *where* $\theta_H u'(q^*_H) = c$;
(ii) $q^*_{LH} = q^*_{LL} = q^*_L$, *where* $(\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q^*_L) = c$.
(b) Transfers are such that the constraints (BIC_H) *and* (BIR_L) *are binding.*

Proof. The proof is standard and therefore it is omitted. ■

In Proposition 1, q^*_H (q^*_L) represents the optimal quantity allocated to H-type (L-type), when the seller faces only one buyer. In the one-buyer case, it is well known that L-type's virtual valuation is given by $\theta_L - \frac{1-p_L}{p_L}\Delta\theta$ since an increase in the quantity received by L-type reduces through (BIC_H) the payment the seller obtains from H-type. This makes her introduce a downward distortion in the quantity allocated to L-type with respect to the first-best level. Proposition 1 states that, in the optimal mechanisms for the two-buyer case, the quantity obtained by a buyer is equal to the quantity he would receive in the one-buyer setting, independently of the report of the other buyer.

Inspecting (1) to (4) and Π shows that the transfer scheme t matters only to determine the values of $p_L t_{LL} + (1-p_L)t_{LH}$ and of $p_L t_{HL} + (1-p_L)t_{HH}$. Therefore, the seller has two residual degrees of freedom in the choice of transfers and, in particular, he can use them to make each buyer's payment independent of the other buyer's report. Precisely, by setting $t_{LL} = t_{LH}$ and $t_{HL} = t_{HH}$, we obtain the optimal transfers in the one-buyer setting: $t^d_{HH} = t^d_{HL} = t^d_H \equiv \theta_H u(q^*_H) - (\Delta\theta)u(q^*_L)$ and $t^d_{LH} = t^d_{LL} = t^d_L \equiv \theta_L u(q^*_L)$. In what follows, we let $M^d \equiv \{\mathbf{q}^*, \mathbf{t}^d\}$ where $\mathbf{t}^d \equiv (t^d_{HH}, t^d_{HL}, t^d_{LH}, t^d_{LL})$. In M^d, each buyer's payoff is determined by his report only and, as a consequence, truthtelling is a dominant strategy. Basically, in the absence of buyer coalition, the seller can maximize her profit by dealing with each buyer separately. It is easy to see that the outcome achieved by M^d can be implemented by a menu of two-part tariffs such that each type of buyer chooses the tariff designed for his type and buys the quantity q_H^* or q_L^* according to his type. We note that the two-part tariff designed for L-type needs a kink in order to prevent H-type from choosing the tariff designed for L-type and buying more than q_L^*.[21]

3.2 Room for collusion

In this subsection, we investigate whether the mechanisms characterized by Proposition 1, and M^d in particular, exhibit any room for collusion. We say that room for collusion exists if buyers can realize some gain by coordinating their actions when there are no transaction costs in coalition formation. Therefore, this section identifies profitable cooperative actions in the absence of transaction costs and in Section 4 we verify whether these actions can be implemented under asymmetric information between the buyers.

We start by examining the case in which the buyers can use only one of the two instruments: either manipulation of reports or reallocation of quantity. Suppose first that they can do only manipulations of reports. Then, if the seller offers M^d, there exists no profitable joint manipulation of reports since a buyer's payoff is independent of the other buyer's report. Suppose now that they can do only reallocation of quantity. Given the quantity profile $\mathbf{q}^*$ characterized by Proposition 1, when the buyers have the same types there is no room for reallocation since they receive the same quantity: either q_H^* (if $\theta^1 = \theta^2 = \theta_H$) or q_L^* (if $\theta^1 = \theta^2 = \theta_L$). However, when one buyer has H-type and the other has L-type, the latter's marginal utility from consumption is strictly larger than the former's one since $\theta_H u'(q_H^*) = (\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q_L^*) = c$. Therefore, there exists an incentive to reallocate some quantity from H-type to L-type. Summarizing the above results, we have:

Proposition 2 *(a) Suppose that the buyers can manipulate their reports without transaction costs but cannot reallocate the goods. If the seller offers M^d, the buyers cannot*

[21] The two-part tariff for H-type takes the form $A_H + pq$ with $A_H = t_H^d - cq_H^*$ and $p = c$. Since the tariff for L-type needs a kink at the point $q = q_L^*$, the seller has some discretion in choosing the marginal price. For instance, she can use $A_L + pq$ with $A_L = t_L^d - cq_L^*$ and $p = c$ for $q \leq q_L^*$, $p = \theta_H u'(q_L^*)$ for $q > q_L^*$.

obtain any gain by manipulating their reports.
(b) Any optimal mechanism under no coalition leaves room for arbitrage: buyers in an HL-coalition have an incentive to reallocate some quantity from H-type to L-type.

Proposition 2(a) applies, for example, if q represents quality – as in Mussa and Rosen (1978) – instead of quantity, since in such a case buyers will hardly be able to reallocate q. We also notice that Proposition 2(a) is identical to the findings of Laffont and Martimort (1997, 2000) (Proposition 11 and Proposition 6, respectively). They show that when the agents' types are independently distributed, there exists a dominant-strategy optimal mechanism which eliminates any gain from joint manipulation of reports. Therefore, if buyers cannot reallocate the goods or if the screening is done in terms of quality, we conclude that by dealing each buyer separately, the seller achieves the same profit regardless of whether or not buyers can collude.

Regarding Proposition 2(b), we emphasize that the incentive for reallocation originates from the fact that the seller reduces the quantity consumed by L-type below the socially efficient level in order to extract more rent from H-type. In contrast, if he observed θ^1 and θ^2, there would be no room for quantity reallocation since the first-best quantity schedule (q_H^{FB}, q_L^{FB}) is characterized by $\theta_H u'(q_H^{FB}) = \theta_L u'(q_L^{FB}) = c$.

We now consider the case in which buyers can both manipulate their reports and reallocate the goods. Suppose that the seller offers M^d. Then, the coalition formed by two H-types has an incentive to report (θ_H, θ_L) to the seller and to (evenly) reallocate the goods since the following inequality holds:

$$2\theta_H u(q_H^*) - 2t_H^d < 2\theta_H u(\frac{q_H^* + q_L^*}{2}) - t_H^d - t_L^d.$$

Similarly, we can verify that HL-coalition has an incentive to report (θ_L, θ_L) to the seller and to reallocate the goods. Summarizing, we have:

Proposition 3 *If the seller offers M^d, the buyers have an incentive to manipulate their reports such that they report (θ_H, θ_L) if $\theta^1 = \theta^2 = \theta_H$ and (θ_L, θ_L) if $\theta^1 \neq \theta^2$; quantities are then reallocated within the coalition.*

We show in the next section that the manipulations and the reallocation stated in Proposition 3 can be implemented by a side mechanism even though coalition formation takes place under asymmetric information.

4 Coalition under asymmetric information

From now on, we assume that buyers form a coalition under asymmetric information and can jointly manipulate reports and reallocate goods. In this section, by analyzing the third-party's design problem of S, we identify the constraints that buyer coalition imposes on the seller's optimization problem, which will allow us to define the seller's problem under collusion in Section 5. We start by introducing some definitions.

Let $p(\theta^1, \theta^2)$ (respectively, $p(\theta^i)$ with $i = 1, 2$) denote the probability of having $(\theta^1, \theta^2) \in \Theta^2$ (respectively, the probability of having $\theta^i \in \Theta$). We recall that $p^\phi(\widetilde{\theta}^1, \widetilde{\theta}^2, \widetilde{\phi})$ denotes the probability that, after receiving reports $(\widetilde{\theta}^1, \widetilde{\theta}^2)$, the third party requires the buyers to report $\widetilde{\phi} \in \Theta^2$ to the seller. When $\widetilde{\phi}$ is reported to the seller, buyer i receives quantity $q^i(\widetilde{\phi})$ from the seller and pays $t^i(\widetilde{\phi})$ to her.

Definition 1 *A side-contract $S^* = \{\phi^*(\cdot), x^{i*}(\cdot), y^{i*}(\cdot)\}$ is coalition-interim-efficient with respect to an incentive compatible mechanism M providing the reservation utilities $\{U^M(\theta_L), U^M(\theta_H)\}$ if and only if it solves the following program:*

$$\max_{\phi(\cdot), x^i(\cdot), y^i(\cdot)} \sum_{(\theta^1, \theta^2) \in \Theta^2} p(\theta^1, \theta^2)[U^1(\theta^1) + U^2(\theta^2)]$$

subject to

$$U^i(\theta^i) = \sum_{\theta^j \in \Theta} p(\theta^j) \left\{ \sum_{\widetilde{\phi} \in \Theta^2} p^\phi(\theta^i, \theta^j, \widetilde{\phi})[\theta^i u(q^i(\widetilde{\phi}) + x^i(\theta^i, \theta^j, \widetilde{\phi})) - t^i(\widetilde{\phi})] - y^i(\theta^i, \theta^j) \right\},$$

for any $\theta^i \in \Theta$ and $i, j = 1, 2$ with $i \neq j$;

$$(BIC^S)\ U^i(\theta^i) \geq \sum_{\theta^j \in \Theta} p(\theta^j) \left\{ \sum_{\widetilde{\phi} \in \Theta^2} p^\phi(\widetilde{\theta}^i, \theta^j, \widetilde{\phi})[\theta^i u(q^i(\widetilde{\phi}) + x^i(\widetilde{\theta}^i, \theta^j, \widetilde{\phi})) - t^i(\widetilde{\phi})] - y^i(\widetilde{\theta}^i, \theta^j) \right\},$$

for any $(\theta^i, \widetilde{\theta}^i) \in \Theta^2$ and $i, j = 1, 2$ with $i \neq j$;

$$(BIR^S)\ U^i(\theta^i) \geq U^M(\theta^i),\ \text{for any } \theta^i \in \Theta \text{ and } i = 1, 2;$$

$$(BB : x)\ x^1(\theta^1, \theta^2, \widetilde{\phi}) + x^2(\theta^1, \theta^2, \widetilde{\phi}) = 0,\ \text{for any } (\theta^1, \theta^2) \in \Theta^2 \text{ and any } \widetilde{\phi} \in \Theta^2;$$

$$(BB : y)\ y^1(\theta^1, \theta^2) + y^2(\theta^1, \theta^2) = 0,\ \text{for any } (\theta^1, \theta^2) \in \Theta^2.$$

In words, a side-contract is coalition-interim-efficient with respect to M if it maximizes the sum of the buyers' expected utilities subject to incentive, acceptance and budget balance constraints. Let $S^0 \equiv \{\phi(\cdot) = Id(\cdot), x^1(\cdot) = x^2(\cdot) = 0, y^1(\cdot) = y^2(\cdot) = 0\}$ denote the null side contract. In words, S^0 is such that no manipulation of reports, no reallocation of quantity and no side-transfer occurs. Hence, M is not affected by buyer coalition if the third-party proposes S^0. The next definition refers to this class of mechanisms.

Definition 2 *An incentive compatible mechanism M is weakly collusion-proof if S^0 is coalition-interim-efficient with respect to M.*

The next proposition shows that M^d is not weakly collusion-proof: Even though the coalition forms under asymmetric information, there exists a side-contract which makes the buyers better off with respect to playing M^d truthfully.

Proposition 4 *If the seller offers M^d, then there exists a side-contract S^d which increases (reduces) the payoff of each type of buyer (of the seller) compared to the payoff that he (she) obtains when M^d is played without collusion; S^d implements the reports manipulations described in Proposition 3 and then reallocates quantities.*

Proof. See Appendix. ■

According to Proposition 4, if the seller offers M^d then she earns a lower profit than under no coalition formation. It is natural, therefore, to inquire whether there exist mechanisms better than M^d. The following proposition simplifies our analysis, since in order to find the best mechanism for the seller we can restrict our attention to the set of weakly collusion-proof mechanisms.

Proposition 5 (weakly collusion-proofness principle) *There is no loss of generality in restricting the seller to offer weakly collusion-proof mechanisms in order to characterize the outcome of any perfect Bayesian equilibrium of the game of seller's mechanism offer cum coalition formation such that a collusive equilibrium occurs on the equilibrium path.*

Proof. The proof is omitted since it is a straightforward adaptation of the proof of Proposition 3 in Laffont and Martimort (2000). ■

The idea behind Proposition 5 is the following: since the third-party has no informational or instrumental advantage over the seller and is subject to the incentive, acceptance and budget balance constraints, any outcome that can be implemented by allowing coalitions to manipulate reports and/or to reallocate goods can be mimicked by the seller in a collusion-proof way without loss.

The next proposition characterizes the set of weakly collusion-proof mechanisms.[22] Before stating the proposition, it is useful to define the following variables θ_L^ϵ, $q_H^\epsilon(x)$ and $q_L^\epsilon(x)$, in which $\epsilon \in [0,1)$ and $x > 0$:

$$\theta_L^\epsilon \equiv \theta_L - \frac{1-p_L}{p_L}(\Delta\theta)\epsilon,$$

$$q_H^\epsilon(x) \equiv \arg\max_{z\in[0,x]} \theta_H u(z) + \theta_L^\epsilon u(x-z) \quad \text{and} \quad q_L^\epsilon(x) \equiv x - q_H^\epsilon(x) \tag{5}$$

We note that $q_H^\epsilon(x)$ is uniquely defined since $\theta_H u(z) + \theta_L^\epsilon u(x-z)$ is a strictly concave function of z. In particular, $(q_H^\epsilon(x), q_L^\epsilon(x))$ is the efficient allocation of a total quantity $x > 0$ between a buyer with valuation θ_H and a buyer with valuation θ_L^ϵ.

Proposition 6 *An incentive compatible sale mechanism $M = \{\mathbf{q}, \mathbf{t}\}$ is weakly collusion-proof if and only if there exists $\epsilon \in [0,1)$ such that*
(a) the following coalition incentive constraints are satisfied: for HH coalition,

$$(CIC_{HH,HL}) \quad 2\theta_H u(q_{HH}) - 2t_{HH} \geq 2\theta_H u(\frac{q_{HL}+q_{LH}}{2}) - t_{HL} - t_{LH}, \tag{6}$$

$$(CIC_{HH,LL}) \quad 2\theta_H u(q_{HH}) - 2t_{HH} \geq 2\theta_H u(q_{LL}) - 2t_{LL}; \tag{7}$$

for HL coalition,

$$(CIC_{HL,HH}) \quad \theta_H u(q_{HL}) + \theta_L^\epsilon u(q_{LH}) - t_{HL} - t_{LH} \geq \theta_H u(q_H^\epsilon(2q_{HH})) + \theta_L^\epsilon u(q_L^\epsilon(2q_{HH})) - 2t_{HH}, \tag{8}$$

$$(CIC_{HL,LL}) \quad \theta_H u(q_{HL}) + \theta_L^\epsilon u(q_{LH}) - t_{HL} - t_{LH} \geq \theta_H u(q_H^\epsilon(2q_{LL})) + \theta_L^\epsilon u(q_L^\epsilon(2q_{LL})) - 2t_{LL}; \tag{9}$$

for LL coalition,

$$(CIC_{LL,HH}) \quad 2\theta_L^\epsilon u(q_{LL}) - 2t_{LL} \geq 2\theta_L^\epsilon u(q_{HH}) - 2t_{HH}, \tag{10}$$

$$(CIC_{LL,HL}) \quad 2\theta_L^\epsilon u(q_{LL}) - 2t_{LL} \geq 2\theta_L^\epsilon u(\frac{q_{HL}+q_{LH}}{2}) - t_{HL} - t_{LH}; \tag{11}$$

[22] We here focus on weakly collusion-proof mechanisms where L-type's Bayesian individual incentive constraint is not binding. We prove in Section 5 that the seller is not going to offer a mechanism M such that L-type's incentive constraint binds in the side-contract which is optimal with respect to M.

(b) the following no arbitrage constraint is satisfied

$$q_{HL} = q_H^{\epsilon}(q_{HL} + q_{LH}), \tag{12}$$

(c) if $\epsilon > 0$, then H-type's incentive constraint in the side mechanism is binding.

Proof. See Appendix. ■

Notice that each coalition incentive constraint takes into account the reallocation of the goods: If both agents report the same types to the third party, each buyer receives half of the total quantity available (see (6)-(7) and (10)-(11)) while if the reports are different, the total quantity is allocated according to (5) (see (8)-(9)). When all the coalition incentive constraints are satisfied, the third-party does not manipulate the buyers' reports into M. Then, no room for reallocation exists if $\theta^1 = \theta^2$ since the seller allocates the same quantity to each buyer. If $\theta^1 \neq \theta^2$, then the third party will not reallocate the goods bought from the seller after making truthful reports if and only if the no-arbitrage constraint (12) is satisfied.

In (8)-(12), $\epsilon \in [0, 1)$ appears. Roughly speaking, ϵ is the Lagrange multiplier of (BIC_H^S), H-type's incentive constraint in the third-party's design problem of S, and it can be positive when (BIC_H^S) is binding.[23] The seller has some flexibility in choosing ϵ since S^0 is optimal for the third party if and only if it satisfies the necessary and sufficient conditions for optimality in the third party's problem for at least one ϵ in $[0, 1)$.

In the presence of complete information within the coalition, the side mechanism does not need to satisfy any individual incentive constraint. Therefore, the coalition incentive and the no-arbitrage constraints under complete information are obtained from (6)-(12) by taking ϵ equal to 0 and the third party realizes whatever gains from cooperative actions if there is any. When the coalition forms under asymmetric information, it may be costly to satisfy (BIC_H^S) because of a well-known tension between (BIC_H^S) and (BIR_L^S); ϵ measures how costly it is. The coalition incentive constraints under asymmetric information differ from the constraints under complete information since L-type's valuation θ_L is replaced by the virtual value θ_L^{ϵ}. The latter is smaller than θ_L for $\epsilon > 0$ since, as the quantity allocated to L-type (by the third party) increases, it is more difficult to satisfy (BIC_H^S). The value of θ_L^{ϵ} affects the coalition incentive constrains through two channels. First, given a quantity consumed by L-type, the third-party

[23] Precisely, $\epsilon = \frac{\delta}{a+\delta}$ where δ is the Lagrange multiplier of (BIC_H^S) and $a > 0$.

evaluates his surplus with θ_L^ϵ instead of θ_L. Second, this in turn affects the third-party's decision to reallocate the goods given a total quantity available to a coalition.

One might argue that the seller could ask the buyers for the information that they may have learned during the course of coalition formation. However, there is no loss in restricting the seller to use mechanisms such as those defined in subsection 2.1 since she can nevertheless deter buyer coalition at no cost.

5 The optimal weakly collusion-proof mechanisms

In this section, we analyze the optimal weakly collusion-proof mechanism. Observe that when the third party proposes S^0, (i) the Bayesian incentive constraints (BIC^S) in the side mechanism reduce to (BIC_H) and (BIC_L) introduced in subsection 3.1; (ii) the acceptance constraints (BIR^S) in the side mechanism are automatically satisfied with equality. Hence, the seller's maximization program under collusion- denoted by (P) - is defined as follows:

$$\max_{\{\mathbf{q},\mathbf{t},\epsilon\}} \quad \Pi \text{ subject to (1)-(4) and (6)-(12).}$$

Since (P) has more constraints than the seller's program without collusion, the seller cannot earn more profit in the presence of collusion than in its absence. However, the next proposition states that the profit level is the same in the two cases. More precisely, it provides a transfer schedule which, paired with the quantity profile $\mathbf{q}^*$ of Proposition 1, yields the seller the profit she obtains in the absence of collusion.

Proposition 7 *There exists a transfer scheme $\mathbf{t}^*$ such that*
(a) $M^* \equiv \{\mathbf{q}^*, \mathbf{t}^*\}$ is an optimal mechanism in the absence of buyer coalition;
(b) M^* is also weakly collusion-proof.

Proof. We basically prove that the seller can satisfy all the constraints imposed by weak collusion-proofness without any loss.
We first notice that $\mathbf{q}^*$ satisfies the no-arbitrage constraint (12) with $\varepsilon = 1$.[24] In fact, when $\varepsilon = 1$ both the seller and the third-party have the same virtual valuation of L-type,

[24] Although ϵ belongs to $[0, 1)$, we allow ϵ to take the value equal to one since we are interested in the Sup of the seller's profit.

$\theta_L - \frac{1-p_L}{p_L}\Delta\theta$; hence the third-party has no incentive to modify the quantity allocation q^* decided by the seller. Then, we can find a (unique) transfer profile t^* such that (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind when $\mathbf{q} = \mathbf{q}^*$ and $\varepsilon = 1$ (see the appendix). We remark that this is possible because satisfying (BIR_L) and (BIC_H) with equality absorbs only two degrees of freedom from the transfer schedule $\mathbf{t}$. By Proposition 1, $M^* = \{\mathbf{q}^*, \mathbf{t}^*\}$ is optimal in the absence of coalition since (BIR_L) and (BIC_H) bind.

In order to prove that M^* satisfies all the coalition incentive constraints, let $V_m^\epsilon(x)$ denote the total virtual surplus[25] that a coalition having m number of buyers with H-type derives from consuming a total quantity $x > 0$; $m \in \{0, 1, 2\}$ is viewed as the "type" of the coalition. We regard each coalition as a consolidated agent and V_m^ϵ as the surplus function of type m. Then notice that (i) $2q_{HH}^* > q_{HL}^* + q_{LH}^* > 2q_{LL}^*$; (ii) the following single crossing condition holds: $\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^\epsilon(x)}{\partial x} > \frac{\partial V_0^\epsilon(x)}{\partial x}$ for any $x > 0$ and any $\epsilon \geq 0$ (see Lemma 1 in the appendix). These two properties, together with the fact that in M^* the local downward coalition incentive constraints bind, allow us to use a standard result from the theory of monopolistic screening [see Section 3 in Maskin and Riley (1984)] to conclude that all the coalition incentive constraints are satisfied. ■

Proposition 7 says that all the constraints generated by weak collusion-proofness can be satisfied at no cost. Hence, the seller can implement the quantity profile q^* as in the absence of buyer coalition and earn the same profit. Under asymmetric information, the possibility to form a coalition does not help the buyers to increase their payoffs. Even though the third party aims at maximizing the buyers' payoffs and there exists room for arbitrage within HL-coalition, no side mechanism implements a desirable reallocation when the seller proposes M^*.

Remark 1 (alternative approach): Our strategy to prove that the seller is not hurt by buyer coalition uses Propositions 5 and 6. The first implies that the profit under coalition is not larger than in its absence; by using the latter we show that M^* is weakly collusion-proof. It would have been possible to prove directly (without going through Propositions 5 and 6) that if the seller offers M^*, then S^0 is the best side-contract for the third-party. Nevertheless, (i) this alternative strategy would have required an argument very similar to the proof of Proposition 6; (ii) without Proposition 5, we could not prove that the seller's profit under collusion is upper bounded by the one without

[25] For instance, $V_1^\epsilon(x) \equiv \max_{z\in[0,x]} \theta_H u(z) + \theta_L^\epsilon u(x-z)$. In $V_0^\epsilon(x)$ and $V_1^\epsilon(x)$, a L-type's surplus is evaluated with θ_L^ϵ. $V_2^\epsilon(x)$ is independent of ϵ since there is no L-type in HH-coalition.

collusion; (iii) it would have provided considerably less insight into the design problem of the third party and, in particular, how the seller can induce the third-party to have L-type's virtual valuation equal to $\theta_L - \frac{1-p_L}{p_L}\Delta\theta$ (notice that the extensions in Section 6 build on these insights).

Remark 2 (multiple optimal transfers): There exist infinitely many transfer schemes $\mathbf{t}$ such that $(\mathbf{q}^*, \mathbf{t})$ is optimal under no coalition and weakly collusion-proof: $\mathbf{t}^*$ is just an example and it is possible to strictly satisfy all the coalition incentive constraints without reducing the profit.[26]

Remark 3 (general cost function): We can show that Proposition 7 holds for a general cost function $C(.)$ with $C(0) = 0$, $C'(q) > 0$ and $C''(q) \geq 0$ for any $q \geq 0$. First, in such a case the optimal mechanisms in the absence of buyer coalition are such that still (BIR_L) and (BIC_H) bind, while the optimal quantity profile $\mathbf{q}^*$ satisfies $2\theta_H u'(q^*_{HH}) = C'(2q^*_{HH})$, $\theta_H u'(q^*_{HL}) = (\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q^*_{LH}) = C'(q^*_{HL} + q^*_{LH})$ and $2(\theta_L - \frac{1-p_L}{p_L}\Delta\theta)u'(q^*_{LL}) = C'(2q^*_{LL})$. Second, although now $q^*_{HL} \neq q^*_{HH}$ and $q^*_{LL} \neq q^*_{LH}$, we still have $2q^*_{HH} > q^*_{HL} + q^*_{LH} > 2q^*_{LL}$; hence the proof of proposition 7 applies word by word to this setting. Furthermore, Proposition 7 holds in an auction setting in which the seller sells a single object: in this setting, q should be interpreted as the probability to win the object with $u(q) = q$.[27]

Remark 4 (when $\epsilon = 0$): Since we focus on the role of asymmetric information among buyers, we may ask what would happen if the third party owned a technology that allows credible reports from the buyers:[28] in this case the side mechanism would not need to satisfy any individual incentive constraint. It would still be possible to satisfy the coalition incentive constraints (which are (6)-(11) written with $\varepsilon = 0$) without any loss (the argument in the final part of the proof of Proposition 7 still holds). However, (12) would reduce to $\theta_H u'(q_{HL}) = \theta_L u(q_{LH})$ and imposing this constraint in the seller's

[26] In Section 7, we exploit this multiplicity to find an optimal weakly collusion-proof mechanism which is strategically more robust than M^*.

[27] In this environment the seller does not need to exploit the information asymmetry between the buyers (i.e., it is optimal to set $\varepsilon = 0$) since, under no buyer coalition, in the state of nature HL she already implements the first best allocation by selling the good to H-type. Hence, no room for arbitrage exists in HL-coalition. The reason is that the marginal surplus of each buyer is constant when $u(q) = q$ and therefore a corner solution achieves the first-best outcome and is optimal for the seller.

[28] For instance, Baron and Besanko (1999) assumes that the third-party who organizes an informational alliance can verify the private information of each agent forming the alliance.

problem results in a strict profit loss.

Remark 5 (correlation): Proposition 7 does not hold if θ^1 are θ^2 are correlated. Indeed, in a correlated environment the seller earns the first best profit in the absence of buyer coalition (see Crémer and McLean (1985)), but that is not anymore possible under coalition formation. We examined a specific case in which the payoff of type θ from consuming quantity $q \in [0, \theta]$ is $\theta q - \frac{1}{2}q^2$ and $\Pr\left\{\theta^1 = \theta^2 = \theta_H\right\} = \Pr\left\{\theta^1 = \theta^2 = \theta_L\right\}$, $\Pr\left\{\theta^1 = \theta_H,\, \theta^2 = \theta_L\right\} = \Pr\left\{\theta^1 = \theta_L,\, \theta^2 = \theta_H\right\}$. Now, a trade-off about the choice of ε arises. On the one hand, as in the case of independent types, a large ε helps to discriminate H-type from L-type. On the other hand, the constraint $(CIC_{HL,LL})$ binds and it is tightened as ε increases. For the case of small and positive correlation, it turns out that the trade-off is optimally resolved by setting ε strictly below 1. We also obtain that as in Laffont and Martimort (2000), the solution is continuous in the degree of correlation; furthermore, the optimal values of q_{HL} and q_{LH} are decreasing with respect to the degree of correlation, while q_{LL} is increasing.

To give an intuition of why the third-party fails to efficiently reallocate the goods, we notice that the rent which H-type obtains by pretending to be L-type to the third-party is increasing in the quantity received by L-type. Hence, a reallocation from H-type to L-type increases H-type's incentive to report L-type in the side mechanism and in order to induce truth-telling, the third-party has to give him a larger rent. We define this increase in H-type's rent as the transaction costs in coalition formation created by asymmetric information. In the next proposition we quantify both the gains from reallocation and the transaction costs and show that the latter is larger than the former.

Proposition 8 *Suppose that the seller offers $M^* = \{\mathbf{q}^*, \mathbf{t}^*\}$ and that the third-party does not manipulate reports but reallocates quantity $\Delta q \in (0, q_H^*]$ from H-type to L-type in HL coalition. Then*
(a) the expected gains from the reallocation are given by:

$$G \equiv 2p_L(1-p_L)\left\{\theta_L\left[u(q_L^* + \Delta q) - u(q_L^*)\right] - \theta_H\left[u(q_H^*) - u(q_H^* - \Delta q)\right]\right\},$$

(b) the transaction costs created by asymmetric information are given by:

$$TC \equiv 2(1-p_L)^2(\Delta\theta)\left[u(q_L^* + \Delta q) - u(q_L^*)\right], \tag{13}$$

(c) we have $TC - G > 0$ for any $\Delta q \in (0, q_H^*]$.

Proof. Appendix. ■

We can also describe this result by observing that L-type's virtual valuation, from the third-party's point of view, is $\theta_L - \frac{1-p_L}{p_L}\Delta\theta$ when the seller offers M^*. Hence, for the third-party, H-type's marginal surplus at q_H^* equals L-type's virtual marginal surplus at q_L^* and any reallocation from H-type to L-type results in an increase in H-type's rent larger than the efficiency gain.

We now examine the features of the transfers $\mathbf{t}^*$ in M^*. For this purpose, we start by investigating how the transfers look like when the coalition can manipulate reports but cannot reallocate goods. In the absence of reallocation, $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ with $\mathbf{q} = \mathbf{q}^*$ and $\epsilon = 1$ are written as follows:

$$2\theta_H u(q_H^*) - 2t_{HH} \geq \theta_H[u(q_H^*) + u(q_L^*)] - t_{HL} - t_{LH} \tag{14}$$

$$\theta_H u(q_H^*) + \theta_L^1 u(q_L^*) - t_{HL} - t_{LH} \geq (\theta_H + \theta_L^1)u(q_L^*) - 2t_{LL} \tag{15}$$

The transfer schedule which satisfies (BIC_H), (BIR_L), (14) and (15) with equality is given by: $t_{HL}^d = t_{HH}^d = \theta_H u(q_H^*) - (\Delta\theta)u(q_L^*)$ and $t_{LL}^d = t_{LH}^d = \theta_L u(q_L^*)$. In fact, these are exactly the transfers specified in mechanism M^d.

When reallocation is feasible, coalition becomes more powerful because of this additional instrument and the transfers are such that

$$t_{LH}^* < \theta_L u(q_L^*) < t_{LL}^* \quad \text{and} \quad t_{HH}^* < \theta_H u(q_H^*) - (\Delta\theta)u(q_L^*) < t_{HL}^* \tag{16}$$

This means that upon reporting a type, each buyer faces a lottery which determines his payment as a function of the report of the other buyer. In particular, facing an L-type is bad news because then the payment is higher than when facing an H-type. The intuition is as follows. As reallocation increases the gross payoff that HL-coalition obtains after manipulating its reports to LL, a t_{LL} larger than t_{LL}^d is needed to make such a manipulation less attractive. However, since (BIR_L) is binding, an increase in t_{LL} must be accompanied with a decrease in t_{LH}; thus $t_{LH}^* < \theta_L u(q_L^*) = t_{LH}^d = t_{LL}^d < t_{LL}^*$. A similar argument applies to $(CIC_{HH,HL})$: $t_{HL} > t_{HL}^d$ relaxes that constraint and this implies, since (BIC_H) binds, a smaller t_{HH}.

We now turn to the issue of implementing the outcome obtained from M^* by using a menu of two-part tariffs, which is a non-direct mechanism. We mentioned in Subsection 3.1 that when coalition formation is impossible, the optimal outcome can be implemented through a menu of two-part tariffs in which the tariff designed for L-type has a kink.

The next proposition states that a more complicated menu of two-part tariffs can be used to implement the optimal outcome when the buyers can form a coalition. Let the seller offer tariffs $T_H = \{(A_{HH}, p_{HH}), (A_{HL}, p_{HL})\}$ and $T_L = \{(A_{LH}, p_{LH}), (A_{LL}, p_{LL})\}$ where, for instance, A_{HL} and p_{HL} represent the fixed fee and the marginal price that the buyer choosing T_H pays if the other buyer chooses T_L. In particular, we consider the tariffs $\{T_H^*, T_L^*\}$ such that

$$A_{jk} = t_{jk}^* - cq_j^*, \text{ for } j,k \in \{H,L\},$$

$$p_{jk} = c \text{ for } q \leq q_j^* \text{ and } p_{jk} = \theta_H u'(q_L^*) \text{ for } q > q_j^* \text{ for } j,k \in \{H,L\}.$$

Proposition 9 *Suppose that the seller offers $\{T_H^*, T_L^*\}$ instead of M^*. Then, regardless of whether or not the buyers can form a coalition,*
(a) each buyer accepts the offer,
(b) j-type of buyer, with $j \in \{H,L\}$, chooses the tariff T_j^* and buys quantity q_j^*.

Proof. The proof is long but similar to those of Propositions 6 and 7, therefore it is omitted.[29] It requires to redefine the third-party's program taking into account that now he selects both tariffs and the quantity that each buyer will buy given a choice of tariffs. In this program, the optimal side mechanism for the third party is such that j-type of buyer ($j = H, L$) chooses T_j^* and buys quantity q_j^*. ■

Note that since we assume that the seller can observe whether or not a buyer uses her goods, it is impossible for a buyer to use a positive amount of the goods without paying any fixed fee to the seller as in Rey and Tirole (1986)[30]. The suitable menu of two-part tariffs is such that (i) the fixed fee a buyer pays depends on the two-part tariff chosen by the other buyer (which is necessary since t^* requires this sort of dependence) (ii) the tariff each buyer faces has a kink.[31] The kink is necessary because of the downward coalition incentive constraints $(CIC_{HH,HL})$, $(CIC_{HH,LL})$ and $(CIC_{HL,LL})$. Consider $(CIC_{HH,HL})$, for instance, and assume that there is no kink in T_H^*. Then, since $A_{HH} > A_{HL} + A_{LH}$ holds, an HH-coalition has an incentive to coordinate the buyers's purchases such that

[29] The proof can be received upon request from the authors.

[30] See also our footnote 17.

[31] Actually, no kink is necessary when both buyers choose T_H^*: we can have $p_{HH} = c$ for all $q \in R^+$. However, in this case, both the fixed fee and the marginal price paid by a buyer choosing T_H^* will depend on the tariff chosen by the other buyer.

only one buyer chooses T_H^*, he buys more than q_H^* and shares it with the other buyer who chooses T_L^*.[32] This deviation is prevented by the increase in the marginal price at $q = q_H^*$ - the kink - from c to $\theta_H u'(q_L^*)$.

6 Extensions

In the previous sections, we considered the two-buyer-two-type setting for simplicity. In this section, we show that our main result (Proposition 7) can be extended to the n-buyer-two-type setting and to the two-buyer-three-type setting. Once the notation is clearly defined, the first extension is rather straightforward since we can prove a single-crossing condition similar to the one used in the proof of Proposition 7. However, the second extension is more tricky since the single crossing condition holds only partially.

6.1 The case of $n > 2$ buyers

When the seller faces $n > 2$ buyers, we assume that the only feasible coalition is the grand coalition, the one including all the buyers. More precisely, we suppose that if at least one buyer rejects the side mechanism, then the sale mechanism is played non-cooperatively with prior beliefs (i.e., we keep assumption WCP). This assumption is justified when any attempt to organize a coalition - after the grand coalition was rejected - is sufficiently time consuming such that it is impossible for the third party to design a new side mechanism which is tailored for the buyers who accepted the original side mechanism. Clearly, this assumption is not needed if $n = 2$ but it makes the model quite tractable when $n > 2$.

Without loss of generality, we restrict our attention to symmetric sale mechanisms, which are now introduced. Let q_{Lm} $(m = 0, 1, ..., n-1)$ denote the quantity allocated to each L-type by the seller when the profile of reports $\hat{\theta} \equiv (\hat{\theta}^1, ..., \hat{\theta}^n) \in \Theta^n$ includes exactly m number of H-types. The variables q_{Hm}, t_{Hm} and t_{Lm} are defined similarly. Let $\mathbf{q}_n \equiv (q_{L0}, ..., q_{Ln-1}, q_{H1}, ..., q_{Hn})$ and $\mathbf{t}_n \equiv (t_{L0}, ..., t_{Ln-1}, t_{H1}, ..., t_{Hn})$, so that a sale mechanism is given by $M_n = \{\mathbf{q}_n, \mathbf{t}_n\}$. Any optimal mechanism $\{\mathbf{q}_n^*, \mathbf{t}_n\}$ without buyer coalition is such that $q_{Lm}^* = q_L^*$ and $q_{Hm}^* = q_H^*$ for any m and the expected payment of

[32] Likewise, if there were no kink in T_L^*, the buyer who pretended to be L-type may buy more than q_L^* and then share with the other buyer.

L-type and H-type is equal to $\theta_L u(q_L^*)$ and $\theta_H u(q_H^*) - (\Delta\theta)u(q_L^*)$, respectively.

Proposition 5, the weakly collusion-proofness principle, applies to this setting. Here we generalize Proposition 6 by describing the conditions under which an incentive compatible mechanism M_n is weakly collusion-proof. In order to do that, we need to investigate how goods are reallocated by the third party in an m-coalition – a coalition with m number of H-types and $n-m$ number of L-types – when $x(>0)$ is the total quantity available to the coalition. Since $u'' < 0$, in any $m-$coalition the third-party allocates the same quantity to each buyer of the same type. Precisely, if quantity z is allocated to each H-type, then each L-type receives $\frac{x-mz}{n-m}$; clearly, if $m = n$ (or $m = 0$) then each H-type (L-type) receives $\frac{x}{n}$. The quantity allocated to H-type is $q^{\epsilon}_{Hm}(x)$ defined as

$$q^{\epsilon}_{Hm}(x) \equiv \arg \max_{z \in [0, \frac{x}{m}]} m\theta_H u(z) + (n-m)\theta^{\epsilon}_L u(\frac{x-mz}{n-m}), \quad m = 1, ..., n-1$$

Hence, the no-reallocation condition for an m-coalition (if $q_{Lm} > 0$) is:

$$\theta_H u'(q_{Hm}) = \theta^{\epsilon}_L u'(q_{Lm}) \tag{17}$$

If (17) is satisfied by M_n, then an m-coalition which reports truthfully in M_n has no incentive to alter the allocation determined by the seller. Notice that

$$V^{\epsilon}_m(x) \equiv \max_{z \in [0, \frac{x}{m}]} m\theta_H u(z) + (n-m)\theta^{\epsilon}_L u(\frac{x-mz}{n-m}), \quad m = 1, ..., n-1$$

is the gross payoff for an m-coalition when it owns the total quantity x. For n-coalition and 0-coalition we have $V_n(x) = \theta_H u(\frac{x}{n})$ and $V^{\epsilon}_0(x) = \theta^{\epsilon}_L u(\frac{x}{n})$, respectively. As in the proof of Proposition 7, we regard each coalition as a consolidated agent and V^{ϵ}_m is the surplus function of type m. For an m-coalition, manipulating its reports is equivalent to reporting a number $m'(\neq m)$ of buyers with H-type. The next proposition summarizes the coalition incentive and the no-arbitrage constraints.

Proposition 10 *An incentive compatible sale mechanism M_n is weakly collusion-proof if and only if there exists $\epsilon \in [0, 1)$ such that*
(a) the following coalition incentive constraints are satisfied:

$$\begin{aligned} & V^{\epsilon}_m[mq_{Hm} + (n-m)q_{Lm}] - mt_{Hm} - (n-m)t_{Lm} \\ \geq \;& V^{\epsilon}_m[m'q_{Hm'} + (n-m')q_{Lm'}] - m't_{Hm'} - (n-m')t_{Lm'} \text{ for any } (m, m') \in \{0, 1, ..., n\}^2 \end{aligned}$$

(b) the no-arbitrage condition (17) holds for $m = 1, ..., n-1$.
(c) if $\epsilon > 0$, then H-type's incentive constraint in the side mechanism is binding.

The next proposition establishes that the buyer coalition does not create any loss to the seller, as in the case of $n = 2$.

Proposition 11 *Given the quantity schedule $\mathbf{q}_n^*$, there exists transfers $\mathbf{t}_n^*$ such that $M_n^* \equiv \{\mathbf{q}_n^*, \mathbf{t}_n^*\}$ is optimal under no buyer coalition and is also weakly collusion-proof.*

Proof. The proof is very similar to the one provided for $n = 2$, hence it is only sketched. First, the seller can choose $\epsilon = 1$ such that the third-party has the same virtual valuation as she has; therefore, (17) holds at $\mathbf{q}_n = \mathbf{q}_n^*$. Second, there exist transfers $\mathbf{t}_n^*$ satisfying with equality (BIC_H), (BIR_L) and local downward coalition incentive constraints (the ones preventing an $(m+1)-$coalition from reporting m for $m = 0, 1, ..., n-1$) written with $\mathbf{q}_n = \mathbf{q}_n^*$ and $\varepsilon = 1$. Third, the single crossing condition holds: $\frac{\partial V_{m+1}^{\epsilon}(x)}{\partial x} > \frac{\partial V_m^{\epsilon}(x)}{\partial x}$ for $m = 0, 1..., n-1$. Finally, since $(m+1)q_{Hm+1}^* + (n-m-1)q_{Lm+1}^* \geq m q_{Hm}^* + (n-m)q_{Lm}^*$ for $m = 0, ..., n-1$, we argue as in the proof to Proposition 7 to conclude that M_n^* satisfies all the coalition incentive constraints. ■

Remark 6 (transaction costs): We can compare the expected gains from arbitrage with the transaction costs generated by asymmetric information for the n-buyer case. Suppose for instance that the third-party reallocates quantity such that when there are m number of H-types, each L-type receives $\Delta q \in \left(0, \frac{m}{n-m} q_H^*\right]$. Then, the expected gains from arbitrage is given by:

$$\begin{aligned} G(n,m) &= \binom{n}{m} (p_L)^{n-m} (1-p_L)^m \{(n-m)\theta_L [u(q_L^* + \Delta q) - u(q_L^*)] \\ &\quad - m\theta_H \left[u(q_H^*) - u(q_H^* - \Delta q \frac{n-m}{m}) \right] \right\}. \end{aligned}$$

The transaction costs are given by:

$$TC(n,m) \equiv n \binom{n-1}{m} (p_L)^{n-m-1} (1-p_L)^{m+1} (\Delta\theta) [u(q_L^* + \Delta q) - u(q_L^*)] \, .$$

We have $TC(n,m) - G(n,m) > 0$ for any $\Delta q \in \left(0, \frac{m}{n-m} q_H^*\right]$. In particular, given $\Delta q > 0$, $TC(2m,m) = kG(2m,m)$ holds where k (> 1) does not depend on m.

6.2 The case of three types

Mechanism design problems under collusion often turn out to be qualitatively more complicated when there are more than two types than when there are only two types.

For instance, Laffont and Martimort (1997, 2000) limit their analysis to the two-type setting since it is difficult to determine the binding coalition incentive constraints when there are more than two types. However, we can show that in our model the main result – Proposition 7 – extends to the three-type setting. The main difficulty in performing such an extension comes from the fact that the single-crossing condition for coalitions which is used in the proof of Proposition 7 holds only partially; in particular, it does not provide an order between coalitions HL and MM.

We assume $n = 2$ for simplicity. Buyer i privately observes his type $\theta^i \in \Theta \equiv \{\theta_L, \theta_M, \theta_H\}$, where $\Delta_H \equiv \theta_H - \theta_M > 0$, $\Delta_M \equiv \theta_M - \theta_L > 0$ and $\theta_L > 0$. The types θ^1 and θ^2 are identically and independently distributed with $p_L \equiv \Pr\left\{\theta^i = \theta_L\right\} > 0$, $p_M \equiv \Pr\left\{\theta^i = \theta_M\right\} > 0$ and $p_H \equiv \Pr\left\{\theta^i = \theta_H\right\} > 0$; this distribution is common knowledge. In the absence of buyer coalition, the virtual valuations of M-type and L-type are given by:

$$\theta^v_M \equiv \theta_M - \frac{p_H}{p_M}\Delta_H \qquad \theta^v_L \equiv \theta_L - \frac{p_H + p_M}{p_L}\Delta_M$$

Clearly, $\theta_H > \max\{\theta^v_M, \theta^v_L\}$ but the order between θ^v_M and θ^v_L depends on the parameters of the model. If $\theta^v_M \geq \theta^v_L$, then virtual valuations are said to be monotonic. If $\theta^v_M < \theta^v_L$, then let $\bar{\theta}^v_{ML} \equiv \frac{p_L\theta^v_L + p_M\theta^v_M}{p_L + p_M}$. In any case, we assume that $\min\{\theta^v_M u'(0), \theta^v_L u'(0)\} > c > \lim_{q\to+\infty}\theta_H u'(q)$, so that each type receives a positive and bounded quantity in case of no coalition.

As in the previous sections, we can restrict our attention to symmetric direct revelation mechanisms. Hence we introduce the following notation:

$$q_{jk} \equiv q^1(\theta_j, \theta_k) = q^2(\theta_k, \theta_j), \qquad t_{jk} \equiv t^1(\theta_j, \theta_k) = t^2(\theta_k, \theta_j), \qquad j, k = L, M, H.$$

A sale mechanism M is given by $\{\mathbf{q}, \mathbf{t}\}$, where $\mathbf{q} \equiv \{q_{jk}\}_{j,k=L,M,H}$ and $\mathbf{t} \equiv \{t_{jk}\}_{j,k=L,M,H}$. Let $\bar{t}_j \equiv p_L t_{jL} + p_M t_{jM} + p_H t_{jH}$ and $\bar{u}_j \equiv p_L u(q_{jL}) + p_M u(q_{jM}) + p_H u(q_{jH})$ with $j = L, M, H$. Then, the expected profit is given by:

$$\begin{aligned}\Pi &= 2(p_L\bar{t}_L + p_M\bar{t}_M + p_H\bar{t}_H) - 2c[p_L^2 q_{LL} + p_L p_M(q_{LM} + q_{ML}) + p_L p_H(q_{HL} + q_{LH})] \\ &\quad -2c[p_M^2 q_{MM} + p_M p_H(q_{MH} + q_{HM}) + p_H^2 q_{HH}]\end{aligned}$$

The Bayesian incentive compatibility and participation constraints are

$$\begin{aligned}(BIC) \quad \theta_j\bar{u}_j - \bar{t}_j &\geq \theta_j\bar{u}_{j'} - \bar{t}_{j'}, \quad j, j' = L, M, H \\ (BIR) \quad \theta_j\bar{u}_j - \bar{t}_j &\geq 0, \quad j = L, M, H\end{aligned}$$

An optimal mechanism solves the problem $\max_{\{\mathbf{q},\mathbf{t}\}} \Pi$ s.t. (BIC) and (BIR). The next proposition characterizes the optimal mechanisms in the absence of buyer coalition.

Proposition 12 *The optimal mechanisms in the absence of buyer coalition are characterized by*

(a) The optimal quantity schedule $\mathbf{q}^* = \left\{q^*_{jk}\right\}_{j,k=L,M,H}$ *is characterized by:*

i) $q^*_{Hj} = q^*_H$ *for* $j = L, M, H$*, where* $\theta_H u'(q^*_H) = c$*;*

ii) If $\theta^v_M \geq \theta^v_L$*, then* $q^*_{Mj} = q^*_M$ *and* $q^*_{Lj} = q^*_L$ *for* $j = L, M, H$*, where* $\theta^v_M u'(q^*_M) = \theta^v_L u'(q^*_L) = c$*.*

If instead $\theta^v_M < \theta^v_L$*, then* $q^*_{Mj} = q^*_M = q^*_{Lj} = q^*_L$ *for* $j = L, M, H$*, where* $\bar{\theta}^v_{ML} u'(q^*_L) = c$*.*

iii) $q^*_H > q^*_M \geq q^*_L$*.*

(b) Transfers are such that constraints (BIC_{HM})*,* (BIC_{ML}) *and* (BIR_L) *bind.*

Proof. The proof is standard and therefore is omitted. ■

As in the two-type case, the weakly collusion-proofness principle holds. In order to characterize weakly collusion-proof mechanisms, it is useful to define i) the variables θ^ϵ_H, θ^ϵ_M and θ^ϵ_L; ii) the functions $q^\epsilon_j(x; jk)$ and $q^\epsilon_k(x; jk)$, $jk = HM, HL, ML$; iii) the functions $V^\epsilon_{jk}(x)$, $j, k = L, M, H$ as follows:

$$\theta^\epsilon_H \equiv \theta_H, \quad \theta^\epsilon_M \equiv \theta_M - \frac{p_H}{p_M}\Delta_H \epsilon_{HM}, \quad \theta^\epsilon_L \equiv \theta_L - \frac{p_H}{p_L}\Delta_M \epsilon_{ML},$$

$$q^\epsilon_j(x; jk) \equiv \arg\max_{z \in [0,x]} \theta^\epsilon_j u(z) + \theta^\epsilon_k u(x - z) \quad \text{and} \quad q^\epsilon_k(x; jk) \equiv x - q^\epsilon_j(x; jk)$$

$$V^\epsilon_{jk}(x) \equiv \max_{z \in [0,x]} \theta^\epsilon_j u(z) + \theta^\epsilon_k u(x - z), \quad j, k = L, M, H$$

where $\epsilon \equiv (\epsilon_{HM}, \epsilon_{ML}) \in [0, 1) \times [0, +\infty)$ and $x > 0$.

The next proposition characterizes weakly collusion-proof mechanisms.

Proposition 13 *An incentive compatible sale mechanism* M *is weakly collusion-proof if and only if there exists* $\epsilon \in [0, 1) \times [0, +\infty)$ *such that*

(a) the coalition incentive constraints are satisfied

$$V^\epsilon_{jk}(q_{jk} + q_{kj}) - t_{jk} - t_{kj} \geq V^\epsilon_{jk}(q_{j'k'} + q_{k'j'}) - t_{j'k'} - t_{k'j'}, \qquad \text{for any } j, k, j', k' = L, M, H. \tag{18}$$

(b) the no arbitrage constraints hold

$$q_{jk} = q^\epsilon_j(q_{jk} + q_{kj}; jk), \qquad \text{for } jk = HM, HL, ML. \tag{19}$$

(c) if $\epsilon_{HM} > 0$ *(resp.* $\epsilon_{ML} > 0$*), then* (BIC^S_{HM}) *[resp.* (BIC^S_{ML})*] binds.*

Proof. The proof is long and very similar to the proof of proposition 6), hence it is omitted.[33] ■

By exploiting Proposition 13 we can prove that the buyer coalition does not create any loss to the seller.

Proposition 14 *Given the quantity profile* $\mathbf{q}^* = \left\{q_{jk}^*\right\}_{j,k=L,M,H}$, *there exists a transfer scheme* $\mathbf{t}^* = \left\{t_{jk}^*\right\}_{j,k=L,M,H}$ *such that* $M^* \equiv \{\mathbf{q}^*, \mathbf{t}^*\}$ *is an optimal mechanism in the absence of buyer coalition and is also weakly collusion-proof.*

Proof. We only provide a brief sketch of the proof, since it mimics the proof of Proposition 7 but is considerably longer. As in the two-type case, the seller can choose $\epsilon^* = (1, \frac{p_H+p_M}{p_H})$ such that the third-party has the same virtual valuations as she has: $\theta_M^{\epsilon^*} = \theta_M^v$ and $\theta_L^{\epsilon^*} = \theta_L^v$. This implies that (19) is satisfied, hence the third-party will not reallocate goods conditional on that there is no manipulation of reports. Second, there remain some degrees of freedom in transfers in the optimal mechanisms under no coalition and the seller can use them to satisfy all the coalition incentive constraints (18). Since no order between HL and MM coalitions is provided by a single crossing condition, finding the right transfers is more tricky than in the two-type setting, although possible.[34] ■

We conjecture that our result will hold when there are more than three types as well.

7 Robustness to cheap-talk and multiplicity

In this section we eliminate the assumption WCP and examine two issues which arise after the third-party's proposal of S^0 in response to M^*: the first is about whether or not both buyers will accept S^0 and the second is about whether they will play the truthtelling equilibrium after accepting S^0. It turns out that under a mild condition on the function u (see Proposition 15 below), both buyers accept S^0 but in M^* truthtelling is iteratively weakly dominated for H-type although it is strictly dominant for L-type. This motivates us to find a robust mechanism M^R in the set of optimal weakly collusion-proof mechanisms such that if M^R is proposed by the seller, then both buyers accept S^0 and truthtelling is strictly dominant for L-type and iteratively weakly dominant for

[33] The proof can be received upon request from the authors.

[34] The proof can be received upon request from the authors.

H-type. In what follows, we first explain the two issues in more detail, present the results for M^* and then characterize M^R.

Let $\widetilde{M}$ be an optimal weakly collusion-proof mechanism offered by the seller. The first issue arises because, as we explained in Subsection 2.2, a two-stage game starts after the third party's proposal of S^0. First each buyer simultaneously announces whether he accepts or refuses S^0 and then buyers report either in S^0 if it was unanimously accepted, or in $\widetilde{M}$ otherwise. In any case, however, in the second stage $\widetilde{M}$ is actually played since S^0 is null. Therefore, buyer i's choice (veto or accept) in the first stage can be viewed as a preplay announcement which may signal some information about θ^i. In other words, the first stage is just a sort of cheap-talk stage in which a buyer may signal his type. We focussed above on the case in which each type of buyer accepts S^0, hence no learning occurs along the equilibrium path. Assume for a moment that it is common knowledge that buyers are going to play truthfully if S^0 is accepted (we deal with this issue below). Then, no type wishes to reject S^0 under the assumption WCP: in fact, buyers are indifferent between accepting and rejecting S^0. However, without the assumption, many off-the-equilibrium-path behavior and beliefs are possible. For instance, buyer 1 might expect that a non-truthful equilibrium of $\widetilde{M}$ (if any exists) will be played (possibly under non-prior beliefs of 2 about θ^1) in case he vetoes S^0. In other words, some type of buyer 1 might have the incentive to veto S^0 – which is a sort of out-of-equilibrium "message" – in order to manipulate buyer 2's beliefs about θ^1 and/or behavior such that he can reach a higher payoff for himself when playing $\widetilde{M}$ at the next stage.

The second issue arises when buyers have to report in S^0 after both of them accepted S^0. Reporting in S^0 is equivalent to playing non-cooperatively $\widetilde{M}$ with prior beliefs, since each buyer i has prior beliefs about θ^j $(j \neq i)$ after S^0 has been unanimously accepted. Although truthtelling is an equilibrium in $\widetilde{M}$, there may exist other equilibria which buyers may coordinate on.

The next proposition describes our results about the two issues when the seller offers M^*.

Proposition 15 *If $\frac{u''(x)}{u'(x)}$ is strictly increasing in x,*[35] *then in M^**

[35]When u is a Bernoulli utility function over money, this assumption on u is called "decreasing absolute risk-aversion".

(a) reporting L is strictly dominant for each L-type, while each H-type strictly prefers reporting H (L) if his opponent plays H (L);
(b) there is no belief of buyer i (after a rejection of S^0 by buyer $j(\neq i)$) which supports an equilibrium of M^* in which at least one type of buyer j is better off than in the truthtelling equilibrium;
(c) in the only non-truthful equilibrium, each type of buyer reports L. For buyers (and seller), the non-truthful equilibrium is strictly Pareto-dominated by the truthful one.

Proof. The proof is omitted for the sake of brevity. ■

Although Proposition 15 (b)-(c) deals with the two issues we introduced above for M^*, Proposition 15 (a) reveals that truthtelling is iteratively weakly dominated for H-type.

We avoid this problem by designing a mechanism M^R in which truthtelling is iteratively weakly dominant for H-type. For this purpose, it is useful to examine the payoff bimatrix of the symmetric 2×2 game played by two buyers with H-type – let 1_H and 2_H denote them - when the seller offers an optimal weakly collusion-proof mechanism and each L-type plays L:[36]

$1_H \backslash 2_H$	L		H	
L	$\theta_H u(q_L^*) - t_{LL}$	$\theta_H u(q_L^*) - t_{LL}$	$(\Delta\theta)u(q_L^*)$	$\theta_H u(q_H^*) - t_{HL}$
H	$\theta_H u(q_H^*) - t_{HL}$	$(\Delta\theta)u(q_L^*)$	$(\Delta\theta)u(q_L^*)$	$(\Delta\theta)u(q_L^*)$

We see that when his opponent H-type reports L, any H-type prefers reporting H to L if $\theta_H u(q_H^*) - t_{HL} > \theta_H u(q_L^*) - t_{LL}$. Therefore, we look for a robust mechanism M^R in the set of optimal weakly collusion-proof mechanisms which satisfies the following condition:

$$\theta_H u(q_H^*) - t_{HL} = \theta_H u(q_L^*) - t_{LL} + \alpha, \tag{20}$$

where α is strictly positive and small. Recall that there exists a continuum of optimal weakly collusion-proof mechanisms; hence, it might be the case that at least one of them satisfies (20) for some $\alpha > 0$. The next proposition characterizes M^R and describes some of its properties.

[36] That is the case when M^R is offered, as Proposition 16 below states.

Proposition 16 *Consider the mechanism* $M^R \equiv \{\mathbf{q}^R, \mathbf{t}^R\}$ *where* $\mathbf{q}^R = \mathbf{q}^*$ *and* $\mathbf{t}^R$ *solves the following linear system, in which* $\alpha > 0$ *and* $\beta > 0$ *are small numbers*[37]

(BIR_L), (BIC_H), $(CIC_{HH,HL})$ *and (20), all written with equality*	*if* $V_2(2q_H^*) - V_2(q_H^* + q_L^*) < V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$
(BIR_L), (BIC_H), $(CIC_{HL,LL}^{\beta})$ *and (20), all written with equality*	*if* $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$

Then
(a) M^R *is optimal under no coalition formation and weakly collusion-proof. The transfers are such that:* $t_{LL}^R > t_{LH}^R$ *and* $t_{HL}^R > t_{HH}^R$.
(b) there is no belief of buyer i *(after a rejection of* S^0 *by buyer* $j \neq i$*) which supports an equilibrium of* M^R *in which at least one type of buyer* j *is better off than in the truthtelling equilibrium;*
(c) in M^R*, reporting* L *is strictly dominant for each* L*-type, while each* H*-type strictly prefers reporting* H *(*L*) if his opponent plays* L *(*H*).*

Proof. See Appendix. ■

According to Proposition 16,[38] when the seller offers M^R, both buyers accept S^0 and truthtelling is strictly dominant for each L-type and serially weakly dominant for each H-type. Actually, M^R admits two (asymmetric) non-truthful equilibria: One in which buyer 1 reports truthfully and each type of buyer 2 reports L and the other in which buyer 2 reports truthfully and each type of buyer 1 reports L. However, it seems reasonable to discard them because they both involve the use of iteratively weakly dominated strategies and are Pareto dominated for buyers by the truthful equilibrium – the latter claim follows from Proposition 16 (a). We also note that there exist a continuum of robust mechanisms since we can find a robust one for each positive small α.

[37] $(CIC_{HL,LL}^{\beta})$ below is obtained by adding β to the right hand side of constraint $(CIC_{HL,LL})$.

[38] We note that the result (b) in Proposition 16 [and (b) in Proposition 15] is stronger than Proposition 9 in Laffont and Martimort (2000). Indeed, their result refers to the notion of ratifiability [see Cramton and Palfrey (1995)], which allows buyer i to have only "reasonable" or "consistent" beliefs about θ^j. In contrast, we do not need any "sophisticated" argument in order to make our point: simply no beliefs of i support buyer j's rejection of S^0.

8 Concluding remarks

We found that if the seller uses simple sale mechanisms in which the quantity sold to a buyer and his payment depend solely on his own report, buyers can realize strict gains at the seller's loss by coordinating their purchases and reallocating the goods. However, we showed that when the seller judiciously designs her mechanism(s) by exploiting the transaction costs in coalition formation, buyer coalition does not hurt her and, in particular, the buyers are unable to implement efficient arbitrage. We also showed that the optimal outcome can be implemented through a menu of two-part tariffs.

Some might find unnatural the feature of the optimal collusion-proof mechanisms that a buyer's payment depends on the other buyer's report while the quantity he receives is independent of the report. However, we point out that in a more general environment in which the marginal cost is not constant (i) our result still holds; (ii) even in the absence of buyer coalition, the quantity received by a buyer will depend on the other's report and therefore, under dominant strategy implementation, his payment will depend on the other's report too.

Our results suggest that buyer coalitions are likely to emerge either when they know each other's preference well or when the seller is constrained to use a restricted set of contracts such that a buyer's payment cannot depend on other buyers' actions. For instance, when there are a large number of buyers (in particular, a mass of buyers), the seller would not have complete information about the number and the identities of potential buyers and this might impose restrictions on the set of contracts available to the seller (see Alger (1999)). In this context, it would be interesting to study the case in which the seller can use only individual contracts: i.e., the quantity sold to a buyer and his payment do not depend on what other buyers do. In this setting, the collusion-proofness principle might not hold and the optimal mechanism might involve letting collusion to occur.[39]

[39] Another direction for extension is to consider different timing for buyer coalitions as Laffont and Martimort (1997) discuss. To focus on coordination of purchases and reallocation, we here adopted the timing chosen by Laffont and Martimort (1997, 2000). But the analysis can be extended to another timing in which buyers can form a coalition after receiving the seller's offer but before deciding whether to accept or reject the offer. Independently, Dequiedt (2002) recently studied collusion in an auction setting with this timing.

APPENDIX
Proof of Proposition 4

The side mechanism $S^d = \left\{\phi^d(\theta^1,\theta^2), x^{id}(\theta^1,\theta^2,\widetilde{\phi}), y^{id}(\theta^1,\theta^2)\right\}$ mentioned in the statement of Proposition 4 is formally defined as follows. For simplicity, let $\phi^d_{jk} = \phi^d(\theta_j,\theta_k)$, $x^{id}_{jk,\widetilde{\phi}} = x^{id}(\theta_j,\theta_k,\widetilde{\phi})$ and $y^{id}_{jk} = y^{id}(\theta_j,\theta_k)$ with $j,k \in \{H,L\}$.

Reports manipulations: $\phi^d_{HH} = (\theta_H,\theta_L)$, $\phi^d_{HL} = \phi^d_{LH} = \phi^d_{LL} = (\theta_L,\theta_L)$.[40]

Reallocation of goods[41]: $x^{1d}_{HH} = -\frac{q^*_H - q^*_L}{2}$, $x^{2d}_{HH} = \frac{q^*_H - q^*_L}{2}$; $x^{1d}_{HL} = \hat{x} > 0$, with $\hat{x}$ close to 0, $x^{2d}_{HL} = -\hat{x}$; $x^{2d}_{LH} = -x^{1d}_{HL} = \hat{x}$; $x^{1d}_{LL} = x^{2d}_{LL} = 0$.

Side transfers: $y^{1d}_{HH} = -\frac{t^d_H - t^d_L}{2}$, $y^{2d}_{HH} = \frac{t^d_H - t^d_L}{2}$; $y^{1d}_{HL} = y^{2d}_{LH} = \hat{y}$, $y^{2d}_{HL} = y^{1d}_{LH} = -\hat{y}$; $y^{1d}_{LL} = y^{2d}_{LL} = 0$, where $\hat{y} > 0$ is still to be defined.

In words, an $HH-$coalition reports HL; then goods and payments are equally shared between the buyers. A coalition HL or LH reports LL; then goods are slightly reallocated from L-type to H-type and H-type pays $\hat{y}$ to L-type.

We prove there exists a $\hat{y} > 0$ such that (BIR^S) and (BIC^S) are all satisfied (actually, they are slack) – $(BB : x)$ and $(BB : y)$ are satisfied by definition. This establishes that S^d is feasible and strictly increases the payoff of each buyer type with respect to playing M^d non-cooperatively.

Let $\widehat{q}_H \equiv q^*_L + \hat{x}$, $\widehat{q}_L \equiv q^*_L - \hat{x}$ and consider constraint (BIC^S_H):

$$\begin{aligned} & p_L[\theta_H u(\widehat{q}_H) - \theta_L u(q^*_L) - \hat{y}] + (1-p_L)\{\theta_H u(\frac{q^*_L + q^*_H}{2}) - \theta_L u(q^*_L) - \frac{\theta_H}{2}[u(q^*_H) - u(q^*_L)]\} \\ \geq\ & p_L(\Delta\theta)u(q^*_L) + (1-p_L)[\theta_H u(\widehat{q}_L) - \theta_L u(q^*_L) + \hat{y}] \end{aligned} \tag{21}$$

Let $\hat{y} = \tilde{y} \equiv \theta_H[u(q^*_L) - u(\widehat{q}_L)]$, so that (i) the right hand of (21) is equal to $U^{M^d}(\theta_H) = (\Delta\theta)u(q^*_L)$; (ii) if $\hat{x}$ were equal to 0, then (21) is strictly satisfied (because u is strictly concave) and therefore, when $\hat{x} > 0$ is close to 0, (21) is still strictly satisfied and (BIR^S_H) is strictly satisfied as well; (iii) (BIR^S_L) holds strictly. Now consider increasing $\hat{y}$ above $\tilde{y}$ until the point at which (21) binds. Then, (BIR^S_H) still holds strictly since the right hand side of (21) increased above $U^{M^d}(\theta_H)$; clearly, (BIR^S_L) holds strictly as well since $\hat{y}$ has been increased over $\tilde{y}$. In order to prove that (BIC^S_L) is satisfied, a standard argument can be used: sum (BIC^S_L) and (BIC^S_H) (which binds) and obtain

[40] We recall that when the manipulation is deterministic, i.e., $p^\phi(\widetilde{\theta}^1,\widetilde{\theta}^2,\widetilde{\phi}) = 1$ for some $\widetilde{\phi} \in \Theta^2$, we write $\phi(\widetilde{\theta}^1,\widetilde{\theta}^2) = \widetilde{\phi}$ (see Section 2.2).

[41] Since the report manipulation is deterministic, we do not write $\widetilde{\phi}$ in $x^{id}_{jk,\widetilde{\phi}}$.

an inequality which is strictly satisfied because $\widehat{q}_H > q_L^*$ and $\frac{q_L^*+q_H^*}{2} > \widehat{q}_L$. Therefore, S^d satisfies (BIC^S) and (BIR^S) and the payoff of each type of buyer is strictly larger than from playing M^d non-cooperatively.

In this case, the buyer coalition strictly reduces the seller's profit because (i) in the states of nature in which the manipulations of reports occur, the quantity sold to the buyers is strictly reduced with respect to truthtelling, which reduces the surplus generated by the trade and (ii) each type of buyer obtains a higher payoff than under truthtelling.[42]

Proof of Proposition 6

We are interested in sale mechanisms such that L-type's incentive constraint is not binding. Since we are finding conditions under which S^0 is optimal for the third party, the incentive constraint of L-type will be slack in the side mechanism as well. In what follows, for the sake of brevity, let $x^i_{jk,\widetilde{\phi}}$ denote $x^i(\theta_j,\theta_k,\widetilde{\phi})$ with $j,k \in \{H,L\}$. Likewise, $p^{\phi}_{jk,\widetilde{\phi}}$ denotes $p^{\phi}(\theta_j,\theta_k,\widetilde{\phi})$.

The third-party maximizes the following objective,

$$\begin{aligned}
&(1-p_L)^2 \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) + \theta_H u(q^2(\widetilde{\phi}) + x^2_{HH,\widetilde{\phi}}) - t^2(\widetilde{\phi})] \\
&+p_L(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) + \theta_H u(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}}) - t^2(\widetilde{\phi})] \\
&+p_L(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) + \theta_L u(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}) - t^2(\widetilde{\phi})] \\
&+p_L^2 \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi}) + x^1_{LL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) + \theta_L u(q^2(\widetilde{\phi}) + x^2_{LL,\widetilde{\phi}}) - t^2(\widetilde{\phi})]
\end{aligned}$$

subject to the following constraints.

- Budget balance constraints: for the quantity reallocation

$$\sum_{i=1}^{2} x^i(\theta^1,\theta^2,\widetilde{\phi}) = 0, \text{ for any } (\theta^1,\theta^2)\in\Theta^2 \text{ and any } \widetilde{\phi}\in\Theta^2;$$

[42] Actually, S^d may not be the optimal side mechanism against M^d. In particular, goods are not efficiently reallocated within HL-coalition since otherwise we are not sure of whether (BIR^S) and (BIC^S) can all be satisfied. However, if the third party chooses the optimal side mechanism against M^d, then the profit is still smaller than if M^d is played non-cooperatively.

for the side transfers

$$\sum_{i=1}^{2} y^i(\theta^1,\theta^2) = 0, \text{ for any } (\theta^1,\theta^2) \in \Theta^2,$$

- H-type's Bayesian incentive constraint for buyer 1: $(BIC_1^S(\theta_H))$

$$\begin{aligned} & p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{HL}] \\ & +(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{HH}] \\ \geq\ & p_L \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{LL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{LL}] \\ & +(1-p_L) \sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{LH}], \end{aligned}$$

- H-type's acceptance constraint for buyer 1: $(BIR_1^S(\theta_H))$

$$\begin{aligned} & p_L \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HL,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{HL}] \\ & +(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{HH,\widetilde{\phi}}[\theta_H u(q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{HH}] \geq U^M(\theta_H), \end{aligned}$$

- L-type's acceptance constraint for buyer 1: $(BIR_1^S(\theta_L))$

$$\begin{aligned} & p_L \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LL,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi}) + x^1_{LL,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{LL}] \\ & +(1-p_L) \textstyle\sum_{\widetilde{\phi}\in\Theta^2} p^{\phi}_{LH,\widetilde{\phi}}[\theta_L u(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) - t^1(\widetilde{\phi}) - y^1_{LH}] \geq U^M(\theta_L), \end{aligned}$$

- H-type's Bayesian incentive constraint for buyer 2 : $(BIC_2^S(\theta_H))$
- H-type's acceptance constraint for buyer 2: $(BIR_2^S(\theta_H))$
- L-type's acceptance constraint for buyer 2: $(BIR_2^S(\theta_L))$,

where $(BIC_2^S(\theta_H))$, $(BIR_2^S(\theta_H))$, $(BIR_2^S(\theta_L))$ are in the same way as $(BIC_1^S(\theta_H))$, $(BIR_1^S(\theta_H))$, $(BIR_1^S(\theta_L))$ are defined.

We introduce the following multipliers:

- $\rho^x(\theta^1,\theta^2,\widetilde{\phi})$ for the budget-balance constraint for the quantity reallocation in state $(\theta^1,\theta^2,\widetilde{\phi})$,
- $\rho^y(\theta^1,\theta^2)$ for the budget-balance constraint for the side-transfers in state (θ^1,θ^2),
- δ^i for the H-type's Bayesian incentive constraint concerning buyer i,

- v_H^i for the H-type's acceptance constraint concerning buyer i,
- v_L^i for the L-type's acceptance constraint concerning buyer i.

We define the Lagrangian function as follows:

$$L = E(U_1 + U_2) + \sum_{i=1,2} \delta^i (BIC_i^S)(\theta_H) + \sum_{i=1,2} v_H^i (BIR_i^S)(\theta_H) + \sum_{i=1,2} v_L^i (BIR_i^S)(\theta_L)$$

$$+ \sum_{(\theta^1,\theta^2)\in\Theta^2} \sum_{\widetilde{\phi}\in\Theta^2} \rho^x(\theta^1,\theta^2,\widetilde{\phi})(BB:x)(\theta^1,\theta^2,\widetilde{\phi}) + \sum_{(\theta^1,\theta^2)\in\Theta^2} \rho^y(\theta^1,\theta^2)(BB:y)(\theta^1,\theta^2)$$

Step 1: Optimizing with respect to $y^i(\theta^1,\theta^2)$

After optimizing with respect to y_{HH}^i, we have:

$$\rho_{HH}^y - \delta^i(1-p_L) - v_H^i(1-p_L) = 0, \text{ for } i = 1,2.$$

After optimizing with respect to y_{HL}^1 and y_{HL}^2 respectively, we have:

$$\begin{aligned} \rho_{HL}^y - \delta^1 p_L - v_H^1 p_L &= 0; \\ \rho_{HL}^y + \delta^2(1-p_L) - v_L^2(1-p_L) &= 0 \end{aligned}$$

After optimizing with respect to y_{LH}^1 and y_{LH}^2 respectively, we have:

$$\begin{aligned} \rho_{LH}^y + \delta^1(1-p_L) - v_L^1(1-p_L) &= 0; \\ \rho_{LH}^y - \delta^2 p_L - v_H^2 p_L &= 0 \end{aligned}$$

After optimizing with respect to y_{LL}^i, we have:

$$\rho_{LL}^y + \delta^i p_L - v_L^i p_L = 0, \text{ for } i = 1,2.$$

In what follows, without loss of generality, we restrict our attention to symmetric multipliers:

$$\delta \equiv \delta^1 = \delta^2, \quad v_H \equiv v_H^1 = v_H^2, \quad v_L \equiv v_L^1 = v_L^2$$

From the above equations, we have:

$$p_L(\delta + v_H) = (1-p_L)(v_L - \delta)$$

Step 2: Optimizing with respect to $x^i(\theta^1, \theta^2, \widetilde{\phi})$ given $p^\phi(\theta^1, \theta^2, \widetilde{\phi})$

For simplicity, let $\rho^x_{jk,\widetilde{\phi}} = \rho^x(\theta_j, \theta_k, \widetilde{\phi})$.

After optimizing with respect to $x^i_{HH,\widetilde{\phi}}$, we have:[43]

$$\rho^x_{HH,\widetilde{\phi}} + p^\phi_{HH,\widetilde{\phi}}(1 - p_L + \delta + v_H)(1 - p_L)\theta_H u'(q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}}) = 0, \text{ for } i = 1, 2, \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply that $q^1(\widetilde{\phi}) + x^1_{HH,\widetilde{\phi}} = q^2(\widetilde{\phi}) + x^2_{HH,\widetilde{\phi}}$ for any $\widetilde{\phi} \in \Theta^2$. Since $x^1_{HH,\widetilde{\phi}} + x^2_{HH,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i(\widetilde{\phi}) + x^i_{HH,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$ for each $\widetilde{\phi}$. Hence, any total quantity which is available to $HH-$coalition is always split equally between the two buyers. We will see that the same result holds for $LL-$coalition.

After optimizing with respect to $x^1_{HL,\widetilde{\phi}}$ and $x^2_{HL,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}}(1 - p_L + \delta + v_H)p_L\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2, \\ \rho^x_{HL,\widetilde{\phi}} + p^\phi_{HL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)(1 - p_L)u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2. \end{aligned}$$

By using $p_L(\delta + v_H) = (1 - p_L)(v_L - \delta)$, we obtain from the two above equations:

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) = \left(\theta_L - \frac{1 - p_L}{p_L}(\Delta\theta)\epsilon\right) u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2,$$

where $\epsilon \equiv \frac{\delta}{1-p_L+\delta+v_H}$. Since $\theta^\epsilon_L = \theta_L - \frac{1-p_L}{p_L}(\Delta\theta)\epsilon$, any total quantity available to $HL-$coalition is split according to the following condition:

$$\theta_H u'(q^1(\widetilde{\phi}) + x^1_{HL,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^2(\widetilde{\phi}) + x^2_{HL,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2.$$

After optimizing with respect to $x^1_{LH,\widetilde{\phi}}$ and $x^2_{LH,\widetilde{\phi}}$ respectively, we have:

$$\begin{aligned} \rho^x_{LH,\widetilde{\phi}} + p^\phi_{LH,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)(1 - p_L)u'(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2, \\ \rho^x_{LH,\widetilde{\phi}} + p^\phi_{LH,\widetilde{\phi}}(1 - p_L + \delta + v_H)p_L\theta_H u'(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}}) &= 0, \text{ for any } \widetilde{\phi} \in \Theta^2. \end{aligned}$$

From the two above equations, we obtain:

$$\theta_H u'(q^2(\widetilde{\phi}) + x^2_{LH,\widetilde{\phi}}) = \theta^\epsilon_L u'(q^1(\widetilde{\phi}) + x^1_{LH,\widetilde{\phi}}), \text{ for any } \widetilde{\phi} \in \Theta^2.$$

[43] In homogeneous coalitions, HH and LL, the reallocation cannot lead to corner solutions. In $HL-$coalition, instead, this is conceivable but it is not going to occur when the seller designs the sale mechanism optimally. Hence, we only consider interior solutions for the reallocation problem.

After optimizing with respect to $x^i_{LL,\widetilde{\phi}}$, we have:

$$\rho^x_{LL,\widetilde{\phi}} + p^{\phi}_{LL,\widetilde{\phi}}(p_L\theta_L - \delta\theta_H + v_L\theta_L)p_L u'(q^i(\widetilde{\phi}) + x^i_{LL,\widetilde{\phi}}) = 0, \text{ for } i = 1, 2 \text{ and any } \widetilde{\phi} \in \Theta^2.$$

The above equations imply that $q^1(\widetilde{\phi}) + x^1_{LL,\widetilde{\phi}} = q^2(\widetilde{\phi}) + x^2_{LL,\widetilde{\phi}}$. Since $x^1_{LL,\widetilde{\phi}} + x^2_{LL,\widetilde{\phi}} = 0$ from the budget balance constraint, we have $q^i(\widetilde{\phi}) + x^i_{LL,\widetilde{\phi}} = \frac{q^1(\widetilde{\phi})+q^2(\widetilde{\phi})}{2}$.

Step 3: Optimizing with respect to $\phi(\theta^1, \theta^2)$

Recall that we want to find conditions under which the third party optimally requires any coalition with $(\theta^1, \theta^2) = (\theta_j, \theta_k)$ to report (θ_j, θ_k), i.e., $\phi(\theta_j, \theta_k) = (\theta_j, \theta_k)$.

- HH coalition:

$$(\theta_H, \theta_H) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ 2\theta_H u(\frac{q^1(\tilde{\phi}) + q^2(\tilde{\phi})}{2}) - t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \right\}.$$

- HL coalition:

$$(\theta_H, \theta_L) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} \theta_H u\left[q^{\epsilon}_H(q^1(\tilde{\phi}) + q^2(\tilde{\phi}))\right] + \theta^{\epsilon}_L u\left[q^{\epsilon}_L(q^1(\tilde{\phi}) + q^2(\tilde{\phi}))\right] \\ -t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \end{array} \right\}.$$

- LH coalition:

$$(\theta_L, \theta_H) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ \begin{array}{c} \theta^{\epsilon}_L u\left[q^{\epsilon}_L(q^1(\tilde{\phi}) + q^2(\tilde{\phi}))\right] + \theta_H u\left[q^{\epsilon}_H(q^1(\tilde{\phi}) + q^2(\tilde{\phi}))\right] \\ -t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \end{array} \right\}.$$

- LL coalition:

$$(\theta_L, \theta_L) \in \arg\max_{\tilde{\phi}\in\Theta^2} \left\{ 2\theta^{\epsilon}_L u(\frac{q^1(\tilde{\phi}) + q^2(\tilde{\phi})}{2}) - t^1(\tilde{\phi}) - t^2(\tilde{\phi}) \right\}.$$

Finally, notice that the above conditions are equivalent to (6)-(11).

The transfers $\mathbf{t}^*$ in M^* (Proposition 7) are given as follows:

$$\begin{aligned} t^*_{HL} &= \frac{(1+p_L)\theta_L - (3-p_L^2)\theta_H}{2} u(q^*_L) + \theta_H \frac{p_L(3-p_L)}{2} u(q^*_H) \\ &\quad + (1-p_L)(2-p_L)\theta_H u(\frac{q^*_H + q^*_L}{2}) + \frac{p_L(1-p_L)}{2} V^1_1(2q^*_L), \end{aligned}$$

$$
\begin{aligned}
t_{LH}^* &= \frac{(p_L+3)\theta_L + (2p_L + p_L^2 - 1)\theta_H}{2} u(q_L^*) + \theta_H \frac{p_L(1-p_L)}{2} u(q_H^*) \\
&\quad - p_L(1-p_L)\theta_H u\left(\frac{q_H^* + q_L^*}{2}\right) - \frac{p_L(1+p_L)}{2} V_1^1(2q_L^*),
\end{aligned}
$$

$$
\begin{aligned}
t_{HH}^* &= \frac{(p_L+2)\theta_L - (1-p_L)(2+p_L)\theta_H}{2} u(q_L^*) + \theta_H \frac{2+2p_L - p_L^2}{2} u(q_H^*) \\
&\quad - p_L(2-p_L)\theta_H u\left(\frac{q_H^* + q_L^*}{2}\right) - \frac{p_L^2}{2} V_1^1(2q_L^*),
\end{aligned}
$$

$$
\begin{aligned}
t_{LL}^* &= \frac{(p_L^2 + 2p_L - 1)\theta_L^1}{2} u(q_L^*) - \theta_H \frac{(1-p_L)^2}{2} u(q_H^*) + (1-p_L)^2 \theta_H u\left(\frac{q_H^* + q_L^*}{2}\right) \\
&\quad + \frac{1-p_L^2}{2} V_1^1(2q_L^*).
\end{aligned}
$$

Lemma for the proof of Proposition 7

Lemma 1 *A single crossing condition for coalitions holds:*

$$
\frac{\partial V_2(x)}{\partial x} > \frac{\partial V_1^\epsilon(x)}{\partial x} > \frac{\partial V_0^\epsilon(x)}{\partial x} \text{ for any } x > 0 \text{ and } \epsilon \geq 0.
$$

Proof. We have $V_2(x) = 2\theta_H u(\frac{x}{2})$ and $V_0^\epsilon(x) = 2\theta_L^\epsilon u(\frac{x}{2})$; hence $\frac{\partial V_2(x)}{\partial x} = \theta_H u'(\frac{x}{2})$ and $\frac{\partial V_0^\epsilon(x)}{\partial x} = \theta_L^\epsilon u'(\frac{x}{2})$. For an HL-coalition, let us consider for simplicity interior allocations (but the proof is easily adapted to the non-interior case). Then $q_H^\epsilon(x)$ and $q_L^\epsilon(x)$ are such that $\theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)]$ and the envelope theorem implies $\frac{\partial V_1^\epsilon(x)}{\partial x} = \theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)]$. Since u' is strictly decreasing and $\theta_H > \theta_L^\epsilon$, we have $q_H^\epsilon(x) > \frac{x}{2} > q_L^\epsilon(x)$; hence $\frac{\partial V_2(x)}{\partial x} = \theta_H u'(\frac{x}{2}) > \theta_H u'[q_H^\epsilon(x)] = \theta_L^\epsilon u'[q_L^\epsilon(x)] > \theta_L^\epsilon u(\frac{x}{2}) = \frac{\partial V_0^\epsilon(x)}{\partial x}$. ■

Proof of Proposition 8

Since it is straightforward to compute the gains from reallocation, we focus on the computation of the transaction costs. Suppose that buyer 2 reports his type truthfully in S and compute the payoff that H-type of buyer 1 obtains by pretending to be L-type to the third-party. The H-type's expected surplus from consumption is given by $\theta_H\left[(1-p_L)u(q_L^* + \Delta q) + p_L u(q_L^*)\right]$ and his expected payment is equal, from the binding

L-type's participation constraint, to $\theta_L\left[(1-p_L)u(q_L^*+\Delta q)+p_L u(q_L^*)\right]$. Hence, in order to implement the reallocation, the third-party has to give an H-type a rent equal to $(\Delta\theta)\left[(1-p_L)u(q_L^*+\Delta q)+p_L u(q_L^*)\right]$, which is larger than $(\Delta\theta)u(q_L^*)$, an H-type's rent in the absence of reallocation. This increase in H-type's rent is the transaction costs in coalition formation due to asymmetric information. From the ex ante point of view, the transaction costs are given by (13). Finally, observe that (i) $TC-G$ is a strictly convex function of Δq which has the value 0 at $\Delta q=0$ and (ii) its derivative $2p_L(1-p_L)\left[\theta_H u'(q_H^*-\Delta q)-(\theta_L-\frac{1-p_L}{p_L}\Delta\theta)u'(q_L^*+\Delta q)\right]$ vanishes at $\Delta q=0$. Therefore, $TC-G$ attains a strict minimum at $\Delta q=0$ and $TC-G>0$ for any $\Delta q\in(0,q_H^*]$.

Proof of Proposition 16

(a) M^R is optimal under no coalition formation since $\mathbf{q}^R=\mathbf{q}^*$ and (BIC_H) and (BIR_L) bind. In order to show that M^R is weakly collusion-proof, notice that no reallocation occurs if $\epsilon=1$ since $\mathbf{q}^R=\mathbf{q}^*$, hence we need to prove that all coalition incentive constraints are satisfied by M^R when $\epsilon=1$.

First observe that we need to take care only of local (upward and downward) coalition incentive constraints. Indeed, both $(CIC_{HH,LL})$ and $(CIC_{LL,HH})$ are automatically satisfied if all the other coalition incentive constraints hold, thanks to the single crossing condition. To prove this claim, suppose that $(CIC_{HH,HL})$, $(CIC_{HL,HH})$, $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$ are all satisfied. Then, add up $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ to find $V_2(2q_H^*)-2t_{HH}\geq V_2(q_H^*+q_L^*)-V_1^1(q_H^*+q_L^*)+V_1^1(2q_L^*)-2t_{LL}$; since $V_2(q_H^*+q_L^*)-V_1^1(q_H^*+q_L^*)+V_1^1(2q_L^*)>V_2(2q_L^*)$ by single crossing, we obtain $V_2(2q_H^*)-2t_{HH}>V_2(2q_L^*)-2t_{LL}$. Thus, $(CIC_{HH,LL})$ is satisfied. About $(CIC_{LL,HH})$, add up $(CIC_{LL,HL})$ and $(CIC_{HL,HH})$ to obtain $V_0^1(2q_L^*)-2t_{LL}\geq V_0^1(q_H^*+q_L^*)-V_1^1(q_H^*+q_L^*)+V_1^1(2q_H^*)-2t_{HH}>V_0^1(2q_H^*)-2t_{HH}$ by single crossing,,hence $(CIC_{LL,HH})$ is satisfied. Therefore, we take care only of $(CIC_{HH,HL})$, $(CIC_{HL,HH})$, $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$.

From (BIR_L), (BIC_H) and (20) written with equality we obtain

$$
\begin{aligned}
t_{LL} &= \theta_H[u(q_L^*)-u(q_H^*)]+\alpha+t_{HL} \qquad t_{HH}=\frac{\theta_H u(q_H^*)-(\Delta\theta)u(q_L^*)-p_L t_{HL}}{1-p_L}\\
t_{LH} &= \frac{\theta_L u(q_L^*)+p_L\theta_H[u(q_H^*)-u(q_L^*)]-p_L\alpha-p_L t_{HL}}{1-p_L}
\end{aligned}
$$

We substitute these expressions into the local coalition incentive constraints – after letting $K\equiv(2-p_L)\theta_H u(q_H^*)+[\theta_L-(2-p_L)\theta_H]u(q_L^*)$ – to find that $(CIC_{HH,HL})$ and

$(CIC_{HL,HH})$ are equivalent to

$$\begin{aligned} &K + p_L\alpha - (1-p_L)[V_2(2q_H^*) - V_2(q_H^* + q_L^*)] \\ &\le t_{HL} \le K + p_L\alpha - (1-p_L)[V_1^1(2q_H^*) - V_1^1(q_H^* + q_L^*)] \end{aligned} \tag{22}$$

while $(CIC_{HL,LL})$ and $(CIC_{LL,HL})$ are equivalent to

$$\begin{aligned} &K - (2-p_L)\alpha - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\ &\le t_{HL} \le K - (2-p_L)\alpha - (1-p_L)[V_0^1(q_H^* + q_L^*) - V_0^1(2q_L^*)] \end{aligned} \tag{23}$$

If $V_2(2q_H^*) - V_2(q_H^* + q_L^*) < V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$, then we set t_{HL} so that $(CIC_{HH,HL})$ binds. We want to prove that the other local coalition incentive constraints hold if $\alpha > 0$ is small. For this purpose, first we show that they are strictly satisfied when $\alpha = 0$ and then argue by continuity. $(CIC_{HL,HH})$ is strictly satisfied because of single crossing [see (22)], while $(CIC_{HL,LL})$ is equivalent to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \le V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$ – which strictly holds by hypothesis – and $(CIC_{LL,HL})$ reduces to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) - [V_0^1(q_H^* + q_L^*) - V_0^1(2q_L^*)] \ge 0$. In order to establish that the latter inequality holds strictly, define $g(z) \equiv V_2(q_H^* + q_L^* + z) - V_2(q_H^* + q_L^*) - [V_0^1(2q_L^* + z) - V_0^1(2q_L^*)]$; we want to prove that $g(q_H^* - q_L^*) > 0$. Observe that $g(0) = 0$ and $g'(z) = \theta_H u'(\frac{q_H^* + q_L^* + z}{2}) - \theta_L^1 u'(q_L^* + \frac{z}{2}) > 0$ because $\theta_H u'(\frac{q_H^* + q_L^* + z}{2}) > c > \theta_L^1 u'(q_L^* + \frac{z}{2})$ for any $z \in [0, q_H^* - q_L^*)$. Here transfers are found by solving the linear system made up of (BIR_L), (BIC_H), $(CIC_{HH,HL})$ and (20), all written with equality:

$$\begin{aligned} t_{HL}^R &= (p_L\theta_L^1 - \theta_H)u(q_L^*) + p_L\theta_H u(q_H^*) + 2(1-p_L)\theta_H u(\frac{q_H^* + q_L^*}{2}) + p_L\alpha \\ t_{LH}^R &= (\theta_L + p_L\theta_H)u(q_L^*) + p_L\theta_H u(q_H^*) - 2p_L\theta_H u(\frac{q_H^* + q_L^*}{2}) - \frac{p_L(1+p_L)}{1-p_L}\alpha \\ t_{HH}^R &= p_L\theta_L^1 u(q_L^*) + (1+p_L)\theta_H u(q_H^*) - 2p_L\theta_H u(\frac{q_H^* + q_L^*}{2}) - \frac{p_L^2}{1-p_L}\alpha \\ t_{LL}^R &= p_L\theta_L^1 u(q_L^*) - (1-p_L)\theta_H u(q_H^*) + 2(1-p_L)\theta_H u(\frac{q_H^* + q_L^*}{2}) + (1+p_L)\alpha \end{aligned}$$

If $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \ge V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$, then we set

$$t_{HL}^R = K - (2-p_L)\alpha - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] + \beta$$

with $\beta > 0$ and small so that $(CIC_{HL,LL})$ is slightly slack. We now show that the other local coalition incentive constraints are strictly satisfied when $\alpha = 0$, hence they are still so if $\alpha > 0$ is small. $(CIC_{LL,HL})$ is strictly satisfied because of single crossing [see (23)], while $(CIC_{HH,HL})$ is equivalent to $V_2(2q_H^*) - V_2(q_H^* + q_L^*) + \frac{\beta}{1-p_L} \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$ – which holds strictly by assumption – and $(CIC_{HL\ HH})$ reduces to $V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*) - [V_1^1(2q_H^*) - V_1^1(q_H^* + q_L^*)] - \frac{\beta}{1-p_L} \geq 0$. The latter inequality holds strictly because of the following argument. Define $g(z) \equiv V_1^1(2q_L^* + z) - V_1^1(2q_L^*) - [V_1^1(q_H^* + q_L^* + z) - V_1^1(q_H^* + q_L^*)]$ and notice that $g(0) = 0$. Moreover, $g'(z) = \theta_H u'[q_H^1(2q_L^* + z)] - \theta_H u'[q_H^1(q_H^* + q_L^* + z)] > 0$ because $q_H^1(2q_L^* + z) < q_H^1(q_H^* + q_L^* + z)$ for any $z \in [0, q_H^* - q_L^*]$. Hence $g(q_H^* - q_L^*) > \frac{\beta}{1-p_L}$ since $\beta > 0$ is small. In this case transfers are found by solving the linear system made up of (BIR_L), (BIC_H), $(CIC_{HL,LL}^{\beta})$ and (20), all written with equality:

$$\begin{aligned}
t_{HL}^R &= (2-p_L)\theta_H u(q_H^*) + [\theta_L - (2-p_L)\theta_H]u(q_L^*) - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\
&\quad -(2-p_L)\alpha + \beta \\
t_{LL}^R &= (1-p_L)\theta_H u(q_H^*) + [\theta_L - (1-p_L)\theta_H]u(q_L^*) - (1-p_L)[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\
&\quad -(1-p_L)\alpha + \beta \\
t_{HH}^R &= p_L\theta_L^1 u(q_L^*) + (1-p_L)\theta_H u(q_H^*) + p_L[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\
&\quad +\frac{p_L(2-p_L)}{1-p_L}\alpha - \frac{p_L}{1-p_L}\beta \\
t_{LH}^R &= (\theta_L + p_L\theta_H)u(q_L^*) - p_L\theta_H u(q_H^*) + p_L[V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \\
&\quad +p_L\alpha - \frac{p_L}{1-p_L}\beta
\end{aligned}$$

Properties of transfers $\mathbf{t}^R$ Let $\alpha = 0$. When $V_2(2q_H^*) - V_2(q_H^* + q_L^*) < V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$ we have

$$t_{LL}^R - t_{LH}^R = t_{HL}^R - t_{HH}^R = \theta_H[2u(\frac{q_H^* + q_L^*}{2}) - u(q_H^*) - u(q_L^*)] > 0 \text{ since } u \text{ is strictly concave.}$$

When $V_2(2q_H^*) - V_2(q_H^* + q_L^*) \geq V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)$, we have

$$t_{LL}^R - t_{LH}^R = t_{HL}^R - t_{HH}^R = \theta_H[u(q_H^*) - u(q_L^*)] - [V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] + \frac{\beta}{1-p_L}.$$

Here, by hypothesis $\theta_H[u(q_H^*) - u(q_L^*)] - [V_1^1(q_H^* + q_L^*) - V_1^1(2q_L^*)] \geq \theta_H[u(q_H^*) - u(q_L^*)] - [V_2(2q_H^*) - V_2(q_H^* + q_L^*)]$ and $\theta_H[u(q_H^*) - u(q_L^*)] - [V_2(2q_H^*) - V_2(q_H^* + q_L^*)] > 0$ is equivalent to $\theta_H[2u(\frac{q_H^* + q_L^*}{2}) - u(q_H^*) - u(q_L^*)] > 0$, which holds since u is strictly concave.

We conclude that $t^R_{LL} > t^R_{LH}$ and $t^R_{HL} > t^R_{HH}$ when $\alpha = 0$, hence these strict inequalities still hold if $\alpha > 0$ is small.

(b) Since $t^R_{LL} > t^R_{LH}$ and $t^R_{HL} > t^R_{HH}$, for instance, buyer 1 (regardless of his type) has a chance to be better off with respect to the truthtelling equilibrium only if his opponent plays H more often than under truthtelling. However, this cannot occur in any equilibrium of M^R – regardless of buyer 2's beliefs about θ^1 - since reporting L is strictly dominant for L-type of buyer 2. Hence, in any equilibrium of M^R the probability that 2 reports H is at most equal to the probability that 2 reports H under truthtelling.

(c) Consider $\mathbf{t}^R$ with $\alpha = 0$. Then, by (20) and since (BIC_H) binds, each H-type is indifferent between reporting H or L, regardless of the report of the opponent. If $\alpha > 0$ is small, then from (20) we infer that H-type strictly prefers reporting H if his opponent plays L; since (BIC_H) binds, he strictly prefers reporting L when his opponent plays H. About L-type, he strictly prefers reporting L when his opponent plays H because $\theta_H u(q^*_L) - t^R_{LH} > \theta_H u(q^*_H) - t^R_{HH}$ implies $\theta_L u(q^*_L) - t^R_{LH} > \theta_L u(q^*_H) - t^R_{HH}$. Furthermore, he strictly prefers reporting L when his opponent plays L because (20) implies $\theta_L u(q^*_L) - t^R_{LL} > \theta_L u(q^*_H) - t^R_{HL}$ when $\alpha = 0$ or $\alpha > 0$ is small.

References

[1] Alger, I. (1999), "Buyer Strategies Limiting the Monopolist's Power: Multiple and Joint Purchases", *Rand Journal of Economics,* 30: 736-757.

[2] Ausubel, L. and P. Cramton (1999), "The Optimality of Being Efficient", Mimeo, University of Maryland.

[3] Baron D. P. and D. Besanko (1999), "Informational Alliances", *Review of Economic Studies,* 66: 743–768.

[4] Brusco, S. and G. Lopomo (2002), "Collusion via Signalling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without Complementarities", *Review of Economic Studies,* 69: 407-436.

[5] Caillaud, B. and P. Jehiel (1998), "Collusion in Auctions with Externalities", *Rand Journal of Economics,* 29: 680-702.

[6] Cramton, P.C. and T.R. Palfrey (1995), "Ratifiable Mechanisms: Learning from Disagreement", *Games and Economic Behavior*, 10: 255-283.

[7] Crémer J. and R.P. McLean (1985), "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent", *Econometrica*, 53: 345-361.

[8] Dequiedt, V. (2002), "Optimal Collusion in Optimal Auctions", mimeo, Universite de Tolouse.

[9] Graham, D.A. and R.C. Marshall (1987), "Collusive Bidder Behavior at Single-Object Second-Price and English Auctions", *Journal of Political Economy*, 95: 1217-1239.

[10] Heflebower, R. (1980) *Cooperatives and Mutuals in the Market System,* University of Wisconsin Press.

[11] Innes, R. and Sexton, R. (1993), "Customer Coalitions, Monopoly Price Discrimination and Generic Entry Deterrence", *European Economic Review*, 37: 1569-1597.

[12] Innes, R. and Sexton, R. (1994), "Strategic Buyers and Exclusionary Contracts", *American Economic Review*, 84: 566-584.

[13] Laffont, J.-J. and D. Martimort (1997), "Collusion under Asymmetric Information", *Econometrica*, 65: 875-911.

[14] Laffont, J.-J. and D. Martimort (2000), "Mechanism Design with Collusion and Correlation", *Econometrica*, 68: 309-342.

[15] Maskin, E. and J. Riley (1984), "Monopoly under Incomplete Information", *Rand Journal of Economics,* 15: 171-196.

[16] Mussa, M. and S. Rosen (1978), "Monopoly and Product Quality", *Journal of Economic Theory,* 18: 301-317.

[17] McAfee, R.P. and J. McMillan (1992), "Bidding Rings", *American Economic Review*, 82: 579-599.

[18] Rey, P. and J. Tirole (1986), "The Logic of Vertical restraints", *American Economic Review*, 76: 921-939.

[19] Rochet, J. C. and L. A. Stole (2002), "Nonlinear Pricing with Random Participation", *Review of Economic Studies*, 69: 277-311.

[20] Tirole, J. (1988), *The Theory of Industrial Organization*, MIT Press.

[21] Zheng, C. (2001), "Optimal Auctions with Resale", Mimeo, Northwestern University