

Conditional Moments and Independence

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Consider two random variables X and Y . In initial probability and statistics courses, a discussion of various concepts of dissociation between X and Y is customary. These concepts typically involve independence and uncorrelatedness. An example is shown where $E(Y^n|X) = E(Y^n)$ and $E(X^n|Y) = E(X^n)$ for $n = 1, 2, \dots$ and yet X and Y are not stochastically independent. The bivariate distribution is constructed using a well-known example in which the distribution of a random variable is not uniquely determined by its sequence of moments. Other similar families of distributions with identical moments can be used to display such a pair of random variables. It is interesting to note in class that even such a degree of dissociation between the moments of X and Y does not imply stochastic independence.

KEY WORDS: Independence; Indeterminate distributions; Moments.

1. INTRODUCTION

In this article, I construct two random variables X and Y , such that $E(Y^n|X) = E(Y^n)$ and $E(X^n|Y) = E(X^n)$ for all positive integers n ; consequently, $\text{cov}(X^m, Y^n) = 0$ for all positive integers m and n . Yet X and Y are not stochastically independent. The point is that even such a strong degree of dissociation is not enough to ensure independence between two random variables. In constructing X and Y , I use a family of (distinct) distributions that are known to possess the same sequence of moments. I also point out that similar families can be used to construct bivariate distributions with similar properties.

Here is my example. Take two random variables X and Y distributed according to the following probability density function:

$$f(x, y) = \frac{1}{2\pi} \frac{1}{xy} \exp\left\{-\frac{[(\ln x)^2 + (\ln y)^2]}{2}\right\} \times [1 + \sin(2\pi x) \sin(2\pi y)], \quad (1)$$

for all positive x and y , and zero otherwise. This positive function integrates to 1 and is thus a valid probability density function. The marginal probability density functions for X and Y are

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log-normal distributions. From this expression, the conditional pdf of Y given X is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp\left[-\frac{(\ln y)^2}{2}\right] \times [1 + \sin(2\pi \ln x) \sin(2\pi \ln y)] \quad (2)$$

for all positive x and y , and the conditional pdf of X given Y is

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{(\ln x)^2}{2}\right] \times [1 + \sin(2\pi \ln y) \sin(2\pi \ln x)]$$

for all positive x and y . These expressions can be used to show that

$$E(Y^n|X) = E(Y^n) \text{ and } E(X^n|Y) = E(X^n)$$

for all positive integers n . To see this, check that

$$\begin{aligned} E(Y^n|X=x) &= \int_{\mathbf{R}_+} y^n f_{Y|X}(y|x) dy \\ &= \int_{\mathbf{R}_+} y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp(-(\ln y)^2/2) \\ &\quad \times [1 + \sin(2\pi \ln x) \sin(2\pi \ln y)] dy \\ &= \int_{\mathbf{R}_+} y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp[-(\ln y)^2/2] dy \\ &= E(Y^n), \end{aligned}$$

which does not depend on X . Here \mathbf{R}_+ denotes the positive half line. The above calculation uses the surprising fact that, for any positive integer, n

$$\begin{aligned} \int_0^\infty y^n \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp\left[-\frac{(\ln y)^2}{2}\right] \sin(2\pi \ln y) dy \\ = \frac{1}{\sqrt{2\pi}} e^{n^2/2} \int_{-\infty}^\infty e^{-s^2/2} \sin(2\pi s) ds = 0, \end{aligned}$$

where the first equality follows by a change of variables argument ($\ln y = s + n$) and the second holds since the integrand is an odd function. Similar computations deliver conditional moments independence of X given Y .

Yet X and Y are dependent because the conditional pdf of Y given X clearly depends on X , because X determines the weight on $\sin(2\pi \ln y)$ in the formula (2) for the conditional pdf. A similar conclusion follows from the analysis of the pdf of X given Y . In other words, in general

$$f_{Y|X}(y|x) \neq f_Y(y) \quad \text{and} \quad f_{X|Y}(x|y) \neq f_X(x).$$

To obtain this example, I used the following class of pdf's:

$$f(x; \lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp(-(\ln x)^2/2) [1 + \lambda \sin(2\pi \ln x)], & x > 0 \\ 0, & x \leq 0 \end{cases},$$

where $|\lambda| \leq 1$. When $\lambda = 0$ this corresponds to the pdf of a log-normal random variable. For different values of λ , these densities are easily seen to differ. This class of pdf's is nonetheless known to possess the same set of moments [see Billingsley (1995, Sect. 30), or Casella and Berger (2001, Example 2.3.10)].

My example uses the above class of pdf's to generate the conditional pdf's for two variables X and Y . Since I use λ as the function of the conditioning random variable and since the moments of this class of distributions do not depend on this parameter, the conditional moments are, in the end, equal to the unconditional ones. Finally, these two conditional distributions are consistent with a well-defined bivariate joint distribution (i.e., they are compatible). In fact, they can be obtained from the joint pdf displayed in (1).

The above construction makes use of a well-known class of distinct distributions possessing the same sequence of moments. Other such families of distributions exist and can similarly be used to generate bivariate distributions with the same properties. Examples were given by Berg (1988) and Stieltjes (1894/1895). These families are typically given by

$$f(x) = h(x)[1 + \lambda g(x)], \quad x \in \mathcal{S},$$

where $h(x)$ is a pdf and $g(x)$ is a periodic function. A condition $|\lambda| < c$ is typically imposed to ensure positivity of $f(x)$. A bivariate distribution $f(x, y)$ can then be obtained from the following conditional distributions

$$f_{Y|X}(y|x) = h(y)[1 + \lambda' g(x)g(y)],$$

and

$$f_{X|Y}(x|y) = h(x)[1 + \lambda' g(x)g(y)],$$

where a bound on $|\lambda'|$ may be necessary to guarantee that the joint distribution is positive and $(x, y) \in \mathcal{S}^2$. That these conditional distributions are compatible with a consistent bivariate distribution can be obtained from Theorem 2.2.1 in Arnold et al. (1992). As a matter of fact, the conditional distributions above are compatible with the joint distribution

$$f_{X,Y}(x, y) = h(x)h(y)[1 + \lambda' g(x)g(y)]$$

with support \mathcal{S}^2 . Furthermore, they demonstrate that X and Y are not stochastically independent and the conditional moments can be computed and shown to be equal to the unconditional ones.

2. A SIMPLE APPLICATION

I now present a stylized illustration for the points discussed in this note. I start with the observation that since stock prices are a function of the cash flow generated by companies, one would expect that dissociation of earnings across firms would be accompanied by dissociation of stock prices across firms. Pindyck and Rotemberg (1993) noted that while firm earnings across sufficiently distinct sectors tend to be uncorrelated, stock prices for the same firms are not. The following very simple illustration shows that such a phenomenon can occur when profits are associated in the subtle manner depicted by the family of multivariate distributions introduced previously in this article. In fact, a stronger version of earnings uncorrelatedness, uncorrelatedness for all positive powers of earnings, is still consistent with correlated stock prices in this very simple environment. For a comprehensive analysis of this issue, see Pindyck and Rotemberg (1993) or Barberis et al. (2005).

Consider an economic environment with two firms, A and B . A traditional model for firm profits in period $t + 1$, P_{t+1}^A and P_{t+1}^B , given profits in the period $t + 1$ postulates that

$$\frac{P_{t+1}^A}{P_t^A} = R_P^A \epsilon_{t+1}^A \quad \frac{P_{t+1}^B}{P_t^B} = R_P^B \epsilon_{t+1}^B,$$

where $\log(\epsilon_{t+1}^A)$ and $\log(\epsilon_{t+1}^B)$ are standard normal random variables and R_P^A and R_P^B are the per-period returns for the two firms. This follows if one models P_t as a geometric Brownian motion as in McDonald and Siegel (1985). In this simple financial environment, stock prices in period $t + 1$, S_{t+1}^A and S_{t+1}^B , equal the expected discounted sum of utilities provided by earnings in each period conditional on profits at that period. Assuming a logarithmic utility function this delivers

$$S_{t+1}^A = E \left[\sum_{i=1}^{\infty} \beta^{i-1} \log(P_{t+i}^A) | P_{t+1}^A \right],$$

where β is the discount factor. After some calculation, this gives

$$S_{t+1}^A = \frac{1}{1-\beta} \log P_{t+1}^A + \frac{\beta}{(1-\beta)^2} \log R_P^A$$

and an analogous expression holds for S_{t+1}^B . If we assume that the joint probability density function of ϵ_{t+1}^A and ϵ_{t+1}^B is given by (1), the logarithms of ϵ_{t+1}^A and ϵ_{t+1}^B are standard normals and firm earnings are uncorrelated. As a matter of fact, all positive powers of P_{t+1}^A and P_{t+1}^B are uncorrelated! Yet,

$$\text{cov}(S_{t+1}^A, S_{t+1}^B) \neq 0$$

since the logarithms of ϵ_{t+1}^A and ϵ_{t+1}^B are correlated. Therefore, even if all positive powers of profits are uncorrelated, those may still be dependent in ways that would lead to correlated stock prices. As the example indicates, independence can be described in terms of covariance, but nonlinear test functions are

needed. More specifically, X and Y are independent if and only if $\text{cov}(f(X), g(Y)) = 0$ for all bounded continuous functions f and g .

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