Geometry of population Dynamics

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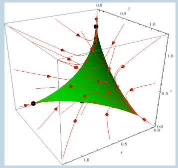
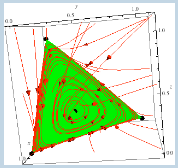
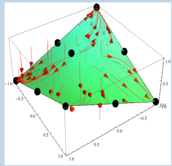


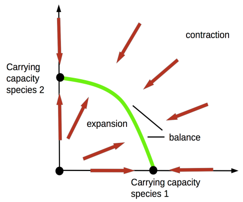




(For the mathematical details, please see preprints referenced)

A balancing principle:

In simple terms: If small populations grow and large populations contract there is a manifold of balance, e.g. carrying capacity, or *carrying simplex* – the green curve joining the carrying capacities:



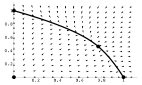
The carrying simplex is a compact attractor of all (nontrivial) population states. All asymptotic population densities lie on this invariant manifold (point, curve, surface…) – if one density is known, the other can be well-approximated after sufficient time.

To the right are 2 phase portraits for planar competitive Lotka-Volterra systems. The carrying simplex is the thick black line. For these systems the simplex is always convex or concave. Coexistence is stable if the simplex is convex (a) and unstable if it is concave (b).

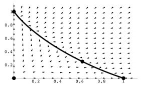
$$x' = px(1-x-ay)$$
$$y' = qy(1-y-bx)$$

x, y population densities

(a)

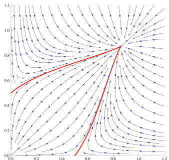


(b)

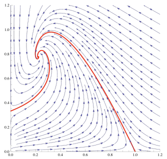


Such a balance also occurs for some cooperative or predator-prey models, although the smoothness of the attracting curve may be lost at an equilibrium:

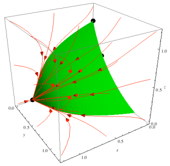
Wolin-Lawlor model (1984). Bounded cooperative dynamics with carrying simplex in red



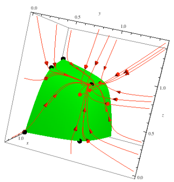
Lotka-Volterra with predator-Prey interactions (and alternative food source for predator)



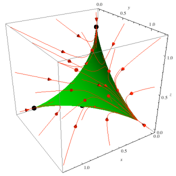
For 3 species competition the carrying simplex is attracting invariant surface: All population densities `eventually' evolve on this surface.



Saddlike-like



Convex

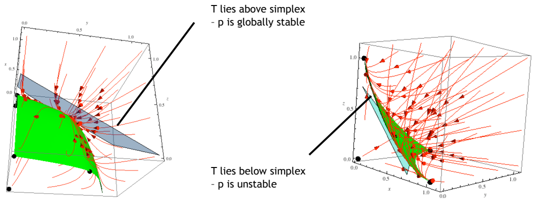


Concave

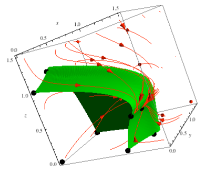
Stability from geometry

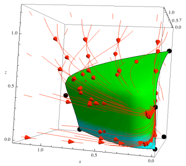
A simple test can determine the stability of coexistence states p: If the tangent plane

- T at p lies above the carrying simplex, p is *globally stable*.
- T lies below the carrying simplex, p is *unstable* (and solutions go to the boundary).



Some noncompetitve interactions also have a carrying simplex, and the geometry of these can be quite complex. The same stability results apply.





Additional Notes

1. There is another geometrical test - Split Lyapunov Method[b,d] - that can be used to determine stability.
2. Ultimately, stability is determined by the Guassian curvature[a,c] of the carrying simplex at the equilibrium.
3. Many carrying simplices are convex, concave or saddle-like[a,c].
4. Geometrical methods also confirm *local* stability for more general Kolmogorov systems[a].

References

a) Baigent, S. (2013) Geometry of carrying simplices of 3-species competitive Lotka-Volterra systems. Nonlinearity 26(4), 1001-1029.
b) Baigent, S and Hou, Z. (2012) On the global stability of fixed points for Lotka-Volterra systems. Differential Equations and Dynamical Systems. 20(1), 53-66.
c) Baigent, S. (2012) Convexity-preserving flows of totally competitive planar Lotka-Volterra equations and the geometry of the carrying simplex. Proceedings of the Edinburgh Mathematical Society Vol 55. No 1, 53-63.
d) Hou, Z. and Baigent, S. (2011) Fixed Point Global Attractors and Repellors in Competitive Lotka-Volterra Systems. Dynamical Systems Vol 26, No. 4. 367-390.

Quasilinkage Equilibrium models

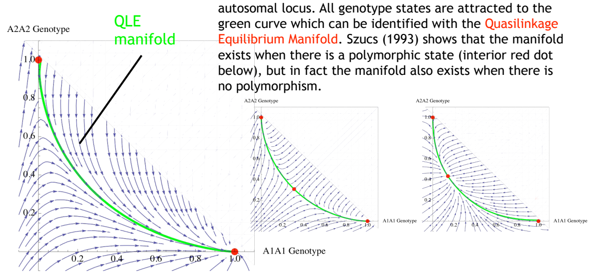
without the assumption of weak interactions

(nb: work in progress)

Selection at a diallelic autosomal locus

for a monoecious population genotypes A1A1, A1A2, A2A2.

The Nagylaki-Crow model for selection at an diallelic autosomal locus. All genotype states are attracted to the green curve which can be identified with the Quasilinkage Equilibrium Manifold. Szucs (1993) shows that the manifold exists when there is a polymorphic state (interior red dot below), but in fact the manifold also exists when there is no polymorphism.



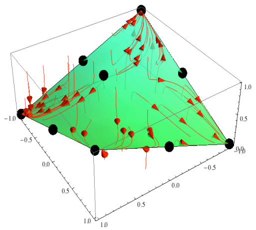
2-locus, diallelic system, selection and recombination

x_i frequencies of genotypes AB,aB, Ab, ab

$$x_i' = x_i(m_i - \overline{m}) + \varepsilon_i rD$$

$$m_i = (Ax)_i,\ \overline{m} = x^T Ax,\ D = x_1x_4 - x_2x_3,\ r \in (0,1],\ \varepsilon_i = -1,1,1,-1.$$

Using a coordinate change from genotype frequencies , the Quasilinkage Equilibrium (QLE) Manifold can (often) be shown to exist even when selection is not weak. Again the manifold exists due to a balance between two opposing processes - here selection and recombination.



(In the new coordinates the green QLE manifold is easily computed by solving a 1st order partial differential equation on [-1,1]^2.)