Butterfly effect

The Butterfly Effect serves as a metaphor for what in technical language is called ‘sensitive dependence on initial conditions’ or ‘deterministic chaos’, the fact that small causes can have large effects.

As recounted by Gleick [3], in the early 60s Lorenz was doing computer experiments on a 12-dimensional weather model. One day he decided to run a particular time series for longer. In order to save time he restarted his code from data from a previous printout. After returning from a coffeee break he found that his weather had diverged sharply from that of his earlier run. After some checks he could only conclude that the difference was caused by the difference in initial conditions: he had typed in only the first three of the six decimal digits the computer worked with internally. Apparently, his assumption that the fourth digit would be unimportant was false.

Lorenz realised the importance of his observation: “If, then, there is any error whatever in observing the present state – and in any real system such errors seem inevitable – an acceptable prediction of an instantaneous state in the distant future may well be impossible.” [5]. Indeed, the error made by discarding the fourth and higher digits is so small that it can be imagined to represent the effect of the flap of the wings of a butterfly. In fact, Lorenz originally used the image of a seagull. The more lasting name seems to have come from his address at the annual meeting of the American Association for the Advancement of Science in Washington, 29 December 1972, which was entitled ‘Predictability: does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?’. The text of this talk was never published but is presented in its original form as an appendix in [6].

Sensitive dependence on initial conditions forces us to distinguish between determinism and predictability, two concepts often confused by scientists and popular writers alike. Determinism has to do with how Nature (or, less ambitiously, any system under consideration) behaves, while predictability has to do with what we, human beings, are able to observe, analyse and compute. We have determinism if we have a law, a formula, describing exactly, and fully, how the system behaves given its present state. To have predictability we need in addition to be able to measure the present state of the system with sufficient precision and to compute with the given formula (‘to solve the equations’) in a sufficiently accurate computational scheme.

Determinism is most famously expressed by Laplace [4]:

> An intelligence that, at a given instant, could comprehend all the forces by which nature is animated and the respective situation of the beings that make it up, if moreover it were vast enough to submit these data to analysis, would encompass in the same formula the movements of the greatest bodies of the universe and those of the lightest atoms. For such an intelligence nothing would be uncertain, and the future, like the past, would be open to its eyes.

Laplace’s dramatic statement is often erroneously interpreted as a belief in perfect predictability now rendered untenable by ‘Chaos Theory’. But he was describing determinism: given the state of the system (the universe) at some time, we have a formula (a set of differential equations) that gives in principle the state of the system at any later time. Nowhere will one find a claim about the computability, by us humans, of all the consequences of the laws of mechanics. Indeed, the quote appears in the introduction of a book on probability! Laplace is, in fact, assuming incomplete knowledge from the start and uses probabilities to make rational inferences. If it weren’t for quantum mechanics, Laplace’s statement would still stand, unaffected by deterministic chaos.

To illustrate the problems with computability consider the simple but important example of the (deterministic) Bernoulli shift map defined by

$$f : [0, 1] \to [0, 1] : \ x_{n+1} = 2x_n \pmod 1.$$

On numbers in binary representation this map has a particularly simple effect: shift the binary point one place to the right and discard the first digit. For example, if $x_0 = 0.10110$ (which corresponds to the decimal 0.6875), then $x_1 = 0.01100$ (decimal 0.375). Now, any rational starting number x_0 is represented by a repeating sequence of 0s and 1s and hence leads to a periodic orbit of f, while any irrational x_0 is represented by a nonrepeating sequence of 0s and 1s and hence leads to a nonperiodic orbit. This latter sequence would look as unpredictable as the sequence of heads and tails generated by flipping a coin, the quintessentially random process. Since there is an irrational number arbitrarily close to every rational number and vice versa, the map exhibits sensitive dependence on initial conditions. In practice, on a computer, numbers are always represented with finite precision; hence the computations become completely meaningless once, after a finite number of iterations, all significant digits have been removed. In the standard 32-bit (4-byte) floating point arithmetic with 23-bit mantissa, this will be after roughly 23 iterations.

The significance of the Bernoulli shift map is that dynamical systems theory tells us that its dynamics lies at the heart of the so-called *horseshoe dynamics* which in turn is commonly found in (the wide class of) systems with homoclinic (i.e., expanding and reinjecting) orbits [10] (this is the content of the Smale-Birkhoff homoclinic theorem). It means that in many situations all we can say about a system's dynamics is of a statistical nature. A quantitative measure of the sensitivity on initial conditions, and therefore a measure of the predictability 'horizon', is provided by the leading Lyapunov exponent.

The possibility of small causes having large effects (in a perfectly deterministic universe) was anticipated by many scientists before Lorenz, and even before the birth of dynamical systems theory, which is generally accepted to have its origins in Poincaré's work on differential equations towards the end of the 19th century. Maxwell [7] wrote: "There is a maxim which is often quoted, that 'The same causes will always produce the same effects.'" After discussing the meaning of this principle, he adds: "There is another maxim which must not be confounded with [this], which asserts that 'Like causes produce like effects.' This is only true when small variations in the initial circumstances produce only small variations in the final state of the system." He then gives the example of how a small displacement of railway points sends a train on different courses.

Others have often used the image of the weather:

Wiener [9]:

> It is quite conceivable that the general outlines of the weather give us a good, large picture of its course for hours or possibly even for days. However, I am profoundly skeptical of the unimportance of the unobserved part of the weather for longer periods. To assume that these factors which determine the infinitely complicated pattern of the winds and the temperature will not in the long run play their share in determining major features of weather, is to ignore the very real possibility of the self-amplification of small details in the weather map. A tornado is a highly local phenomenon, and apparent trifles of no great extent may determine its exact track. Even a hurricane is probably fairly local where it starts, and phenomena of no great importance there may change its ultimate track by hundreds of miles.

Poincaré [8]:

> Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance, so that many people think it quite natural to pray for rain or fine weather, though they would consider it ridiculous to ask for an eclipse by prayer? We see that great disturbances are generally produced in

> regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and that is the reason why it all seems due to the intervention of chance.

Even earlier, Franklin [2] had used an analogy surprisingly similar to Lorenz's:

> . . . an infinitesimal cause may produce a finite effect.
>
> Long range detailed weather prediction is therefore impossible, and the only detailed prediction which is possible is the inference of the ultimate trend and character of a storm from observations of its early stages; and the accuracy of this prediction is subject to the condition that the flight of a grasshopper in Montana may turn a storm aside from Philadelphia to New York!

Duhem [1] used Hadamard's theorem of 1898 on the complicated geodesic motion on surfaces of negative curvature to "expose fully the absolutely irremediable physical uselessness of certain mathematical deductions." If such incomputable behaviour is possible in mechanics, "the least complex of physical theories," Duhem goes on to ask rhetorically, "Should we not meet that ensnaring conclusion in a host of other, more complicated problems, if it were possible to analyse the solutions closely enough?"

Many had contemplated the possibility of sensitive dependence on initial conditions, but Lorenz was the first to see it actually happening quantitatively in the numbers spit out by his Royal McBee computing machine, and to be sufficiently intrigued by it to study it more closely in the delightfully simple system of equations now bearing his name [5]. Indeed, while most scientists, with Duhem, had looked to complicated systems for unpredictable behaviour, Lorenz found it in simple ones and thereby made it amenable to analysis.

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References

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