Contrasting measures of irreversibility in stochastic and deterministic dynamics

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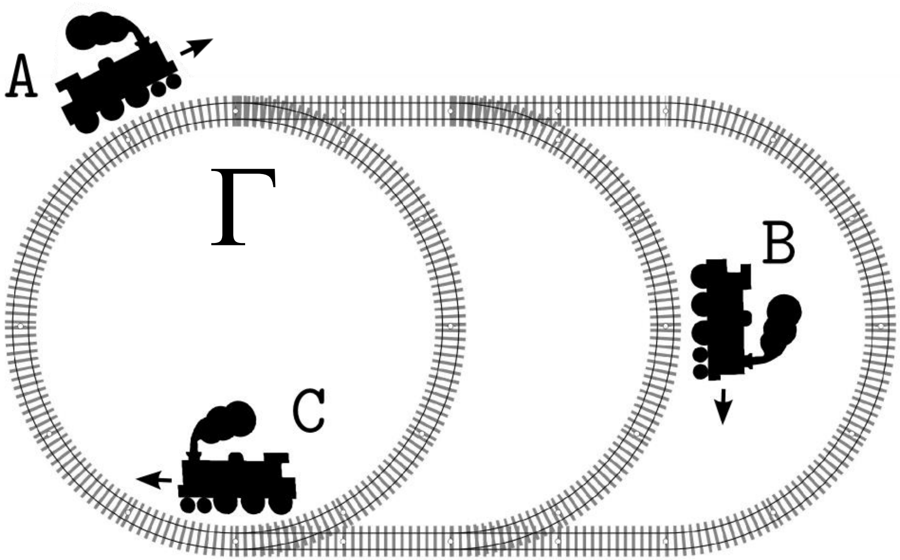
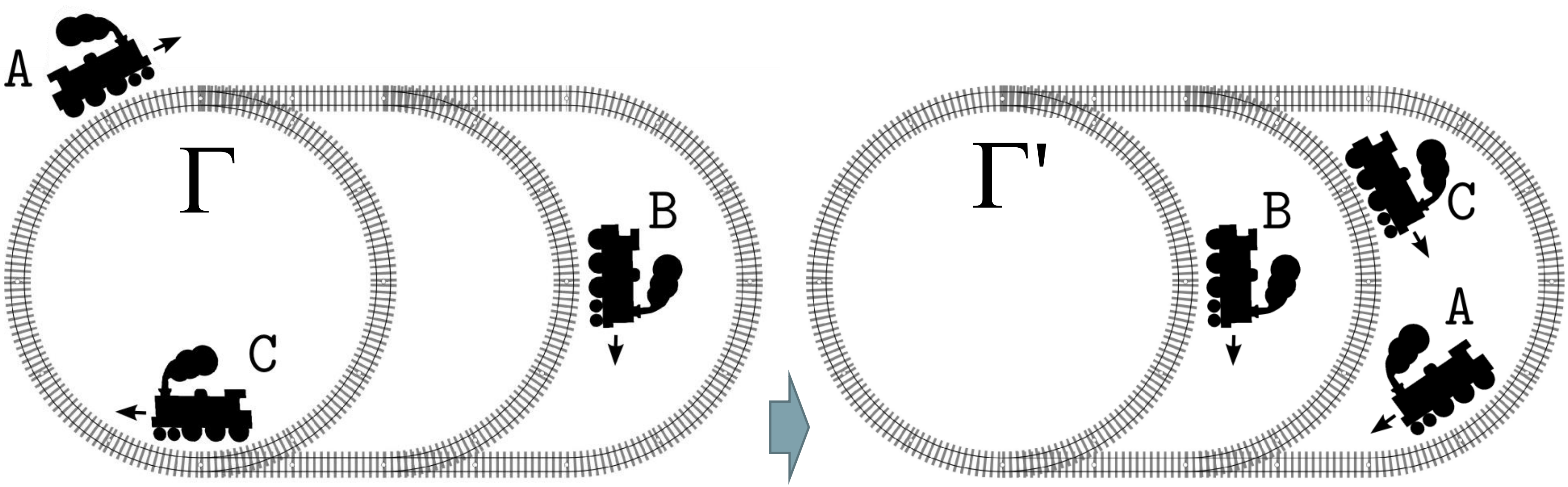
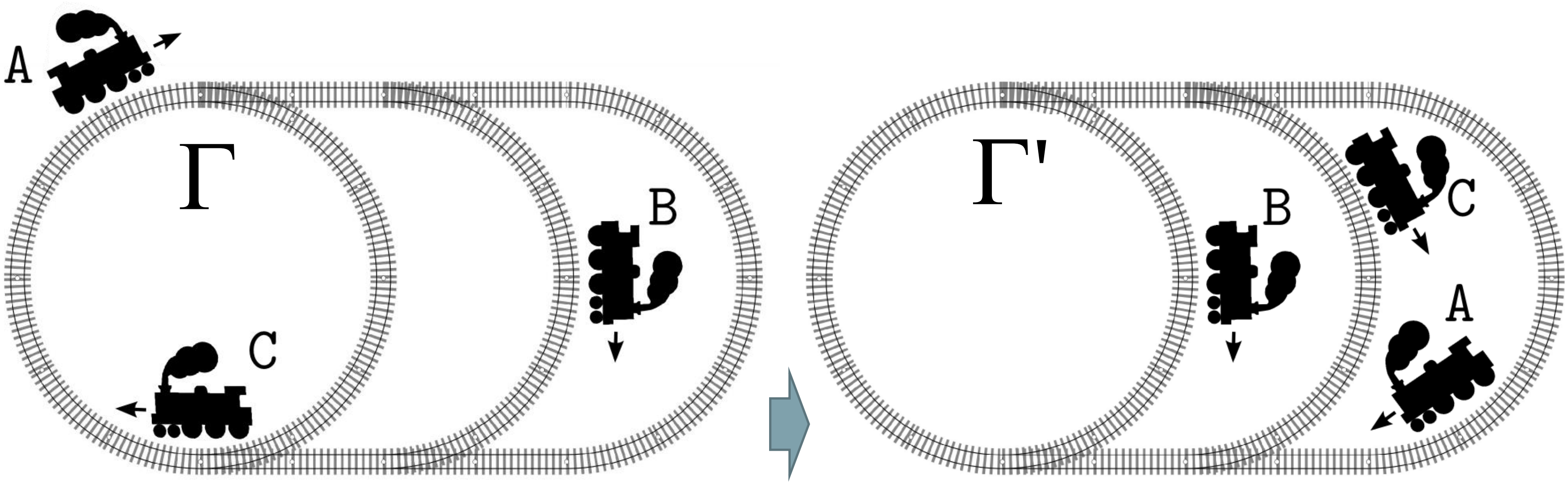
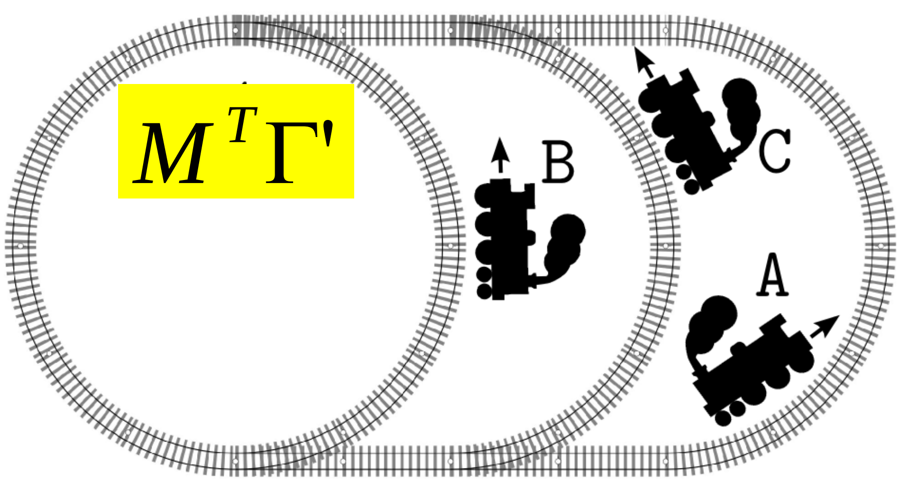
THE MESSAGE

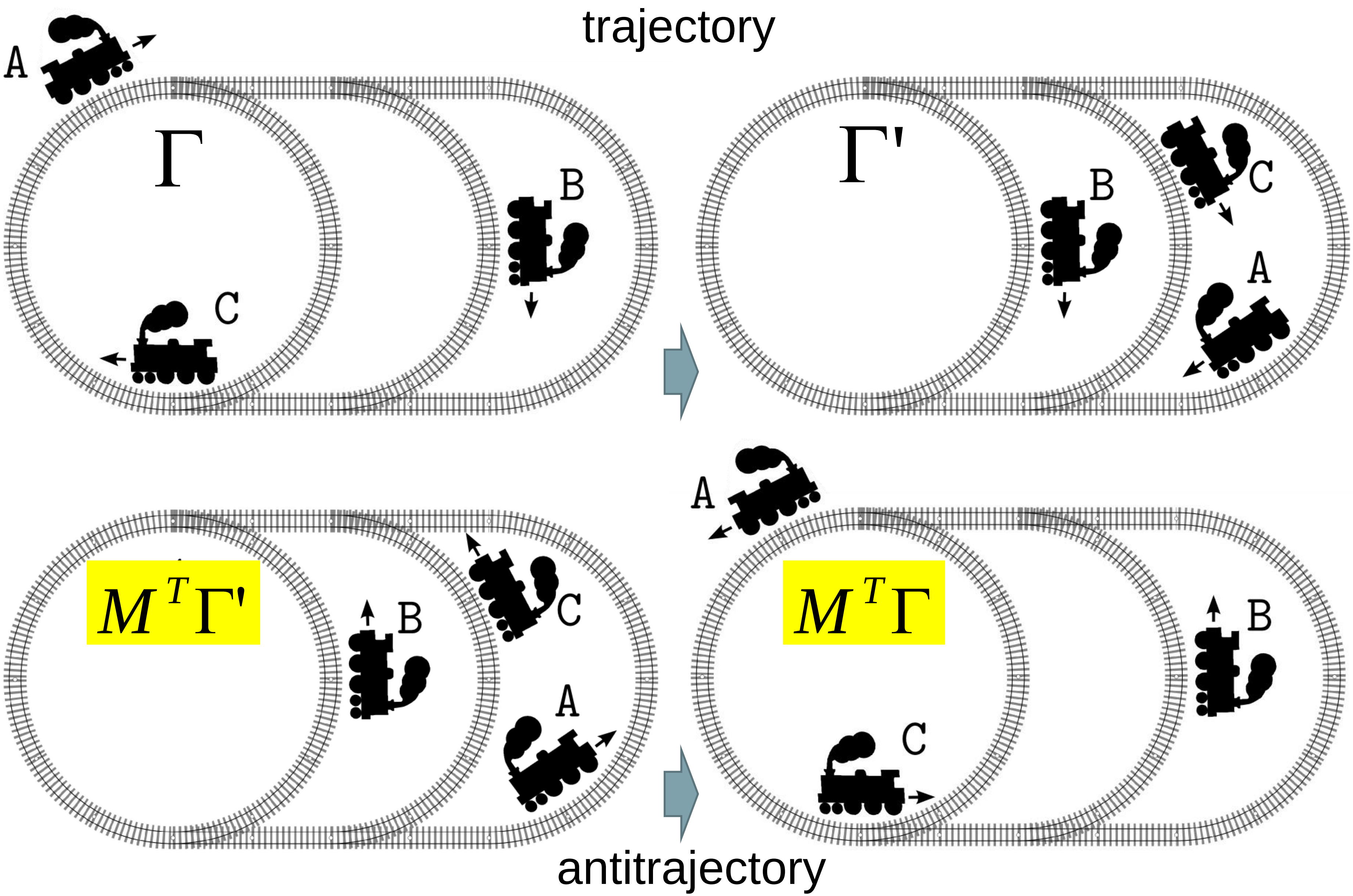
- How to measure **irreversibility**?
- For stochastic dynamics:
  - stochastic entropy production measures ***reversibility***
- For deterministic dynamics:
  - Evans-Searles dissipation function
  - But this assumes a velocity-symmetric pdf
- A modified dissipation function can be defined [Ford (2015)]
  - 'dissipation production' measures ***obversibility***
- Examples in classical and quantum dynamics


A
Γ
B
C


A
Γ
B
C
Γ'
B
C
A


A
Γ
B
C
Γ'
B
C
A

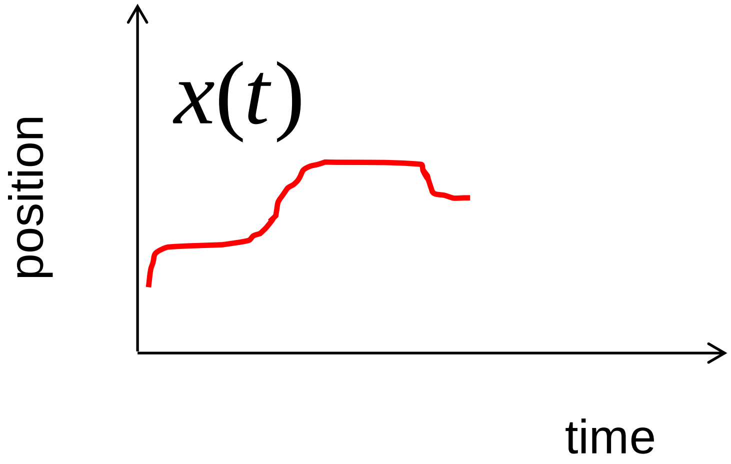

$M^T\Gamma'$
B
C
A

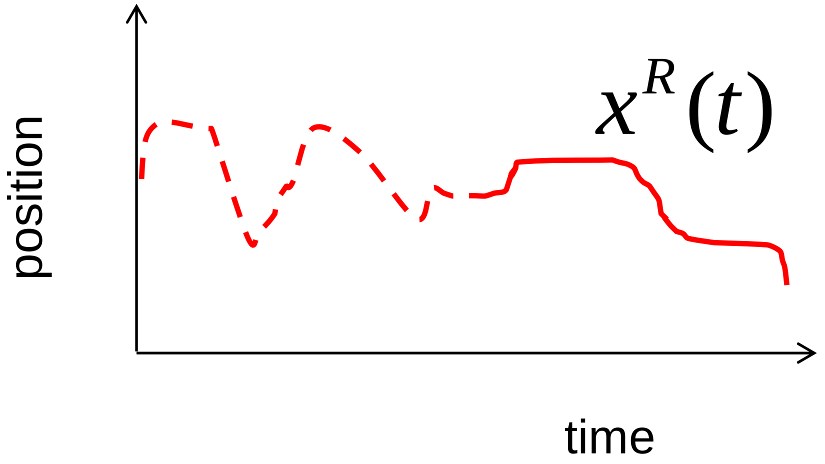

trajectory
A
Γ
B
C
Γ'
B
C
A
A
$M^T\Gamma'$
B
C
A
$M^T\Gamma$
B
C
antitrajectory

Definition of stochastic entropy production

$$\Delta s_{\mathrm{tot}}[x(t)] = \ln\left(\frac{\mathrm{prob}[\mathrm{trajectory}\ x(t), v(t)]}{\mathrm{prob}[\mathrm{antitrajectory}\ x^R(t), v^R(t)]}\right)$$

Sekimoto, Seifert, etc

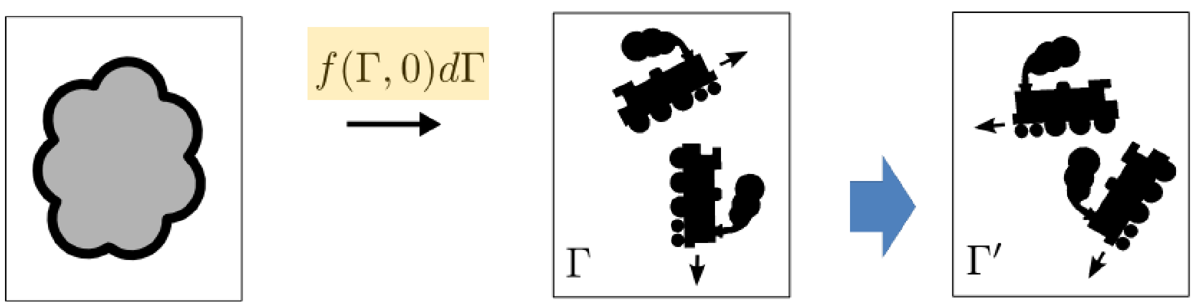




Reversibility measure

Transition probabilities

$$\Delta s_{\mathrm{tot}}[\Gamma \to \Gamma'] = \ln\left[\frac{f(\Gamma,0)T_t(\Gamma \to \Gamma')}{f(\Gamma',t)T_t(M^T\Gamma' \to M^T\Gamma)}\right]$$



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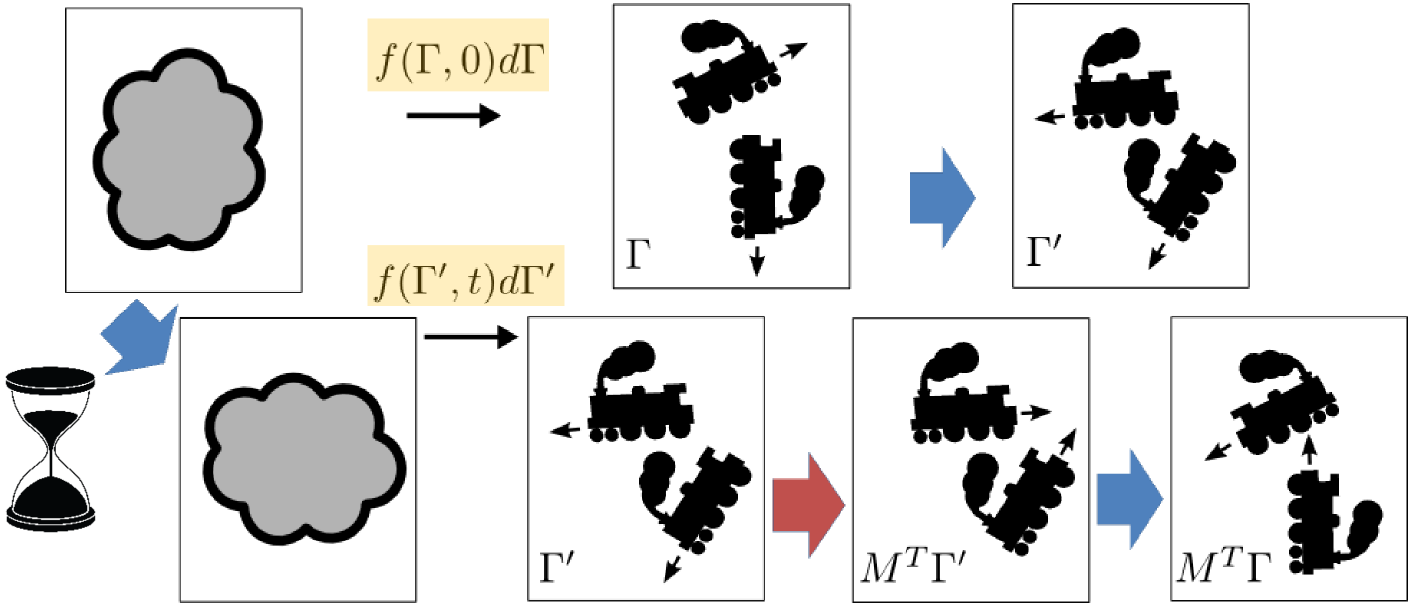
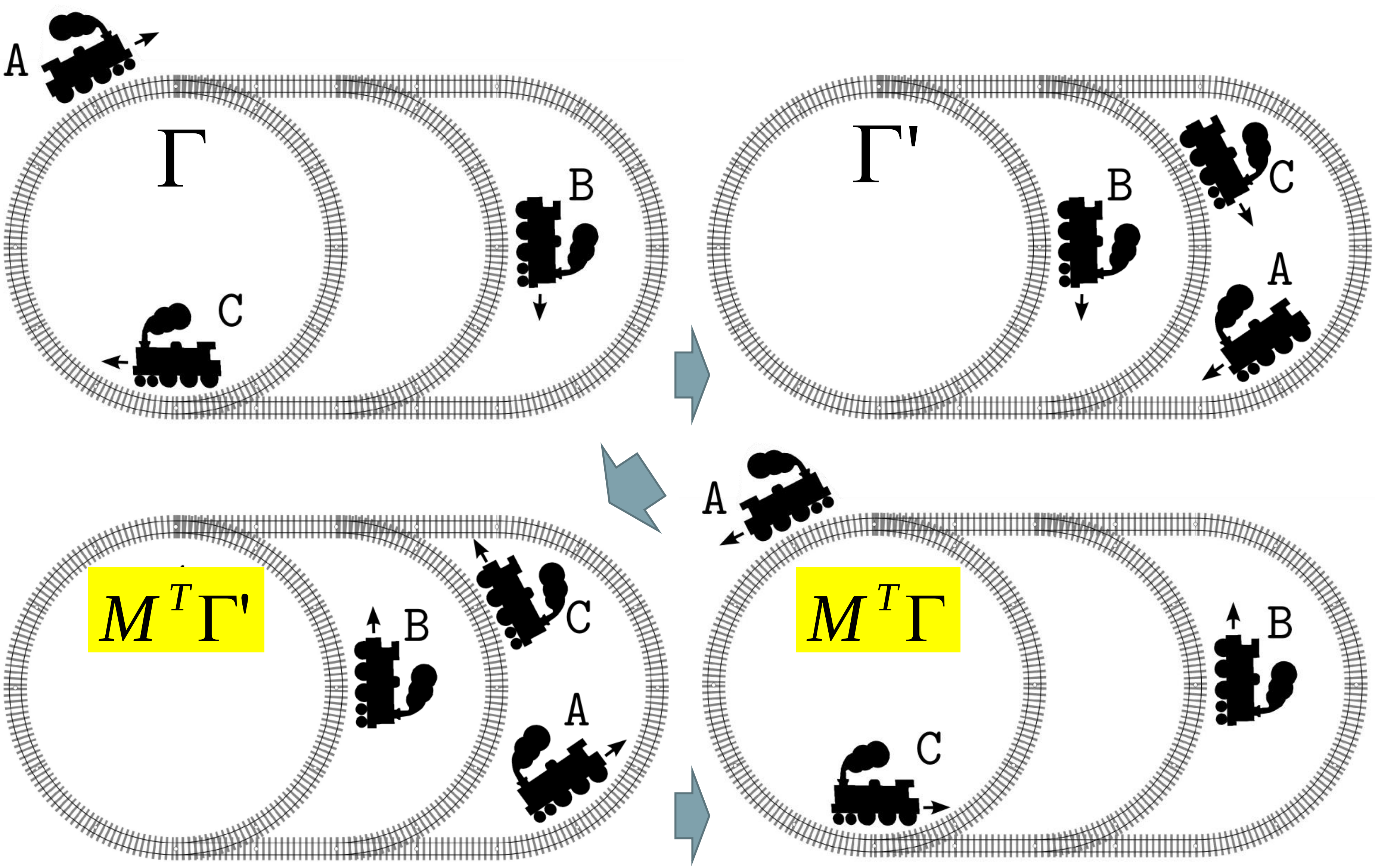


Deterministic dynamics

- Newton’s equations / Schrödinger equation
- Possibly with a deterministic thermal constraint
- How might we measure irreversibility?
  - stochastic entropy production is zero

In deterministic dynamics reversibility is automatic



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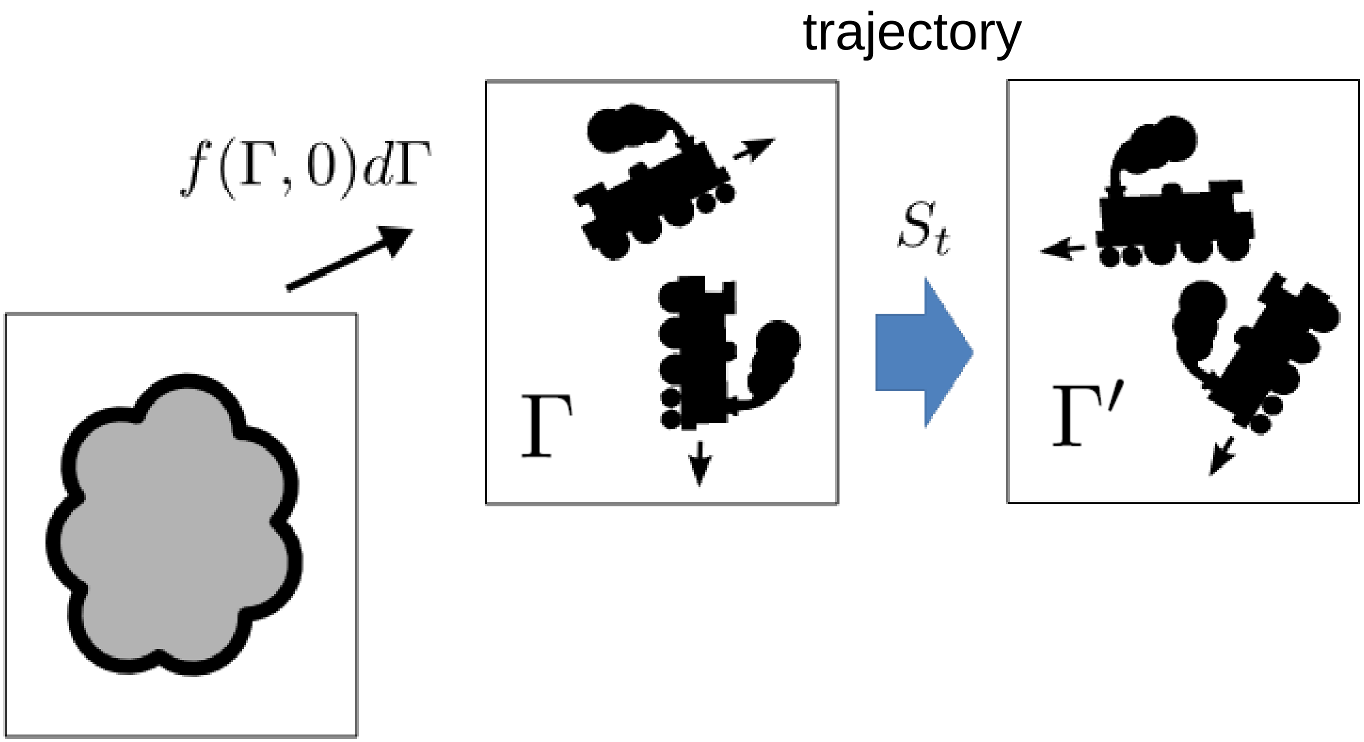
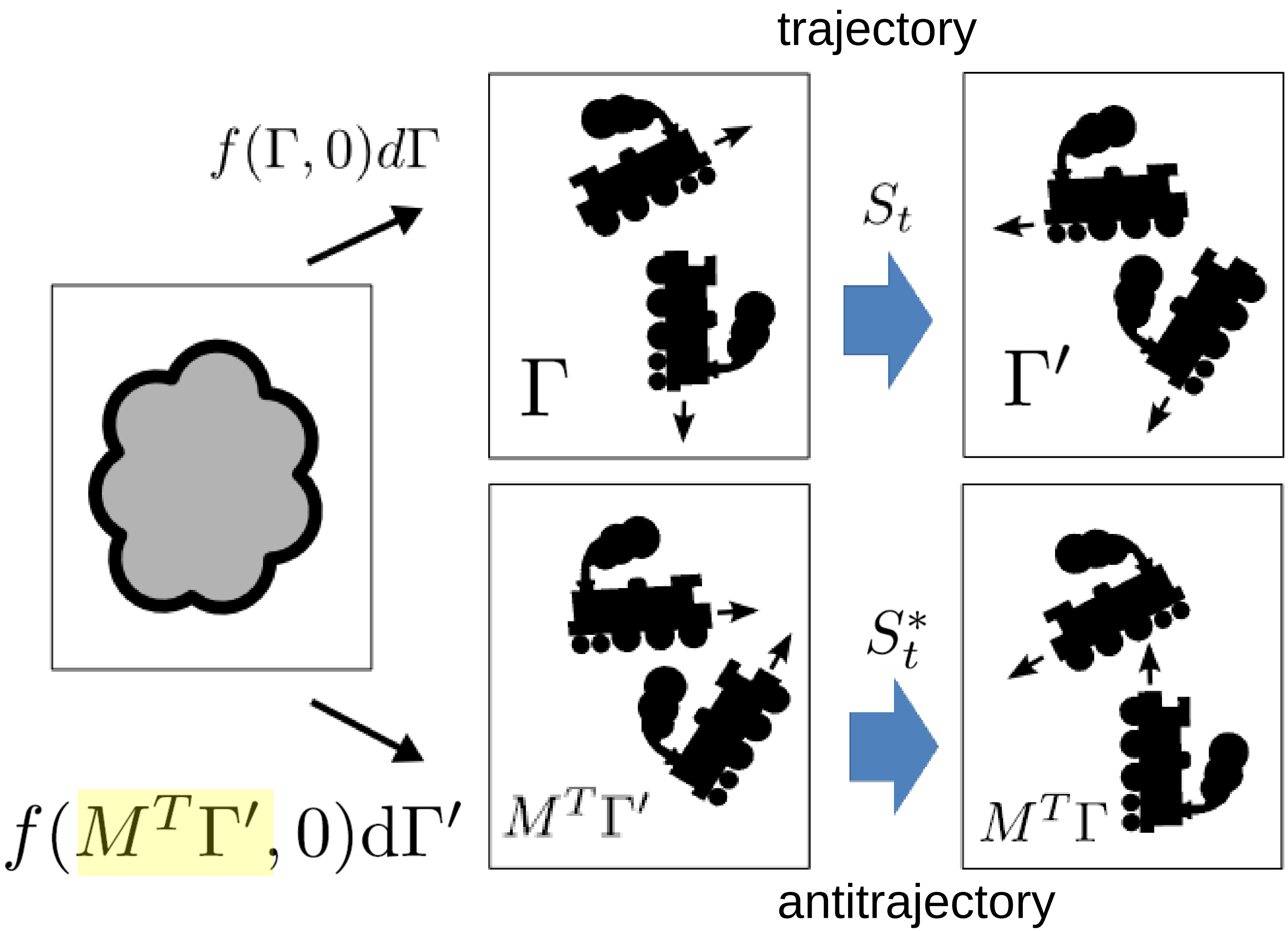
Sekimoto, Seifert, etc

The Evans-Searles test



The Evans-Searles test



The Evans-Searles dissipation function

time-integrated
dissipation function

Probability of selecting
configuration Γ at t=0

$$\Omega_t(\Gamma) = \int_0^t \Omega(S_s\Gamma)ds = \ln\left[\frac{f(\Gamma,0)d\Gamma}{f(M^T\Gamma',0)d\Gamma'}\right]$$

$$\boxed{\langle\Omega_t\rangle \geq 0}$$

Probability of selecting, at t=0, the *velocity inverted evolved configuration* $M^T\Gamma'$

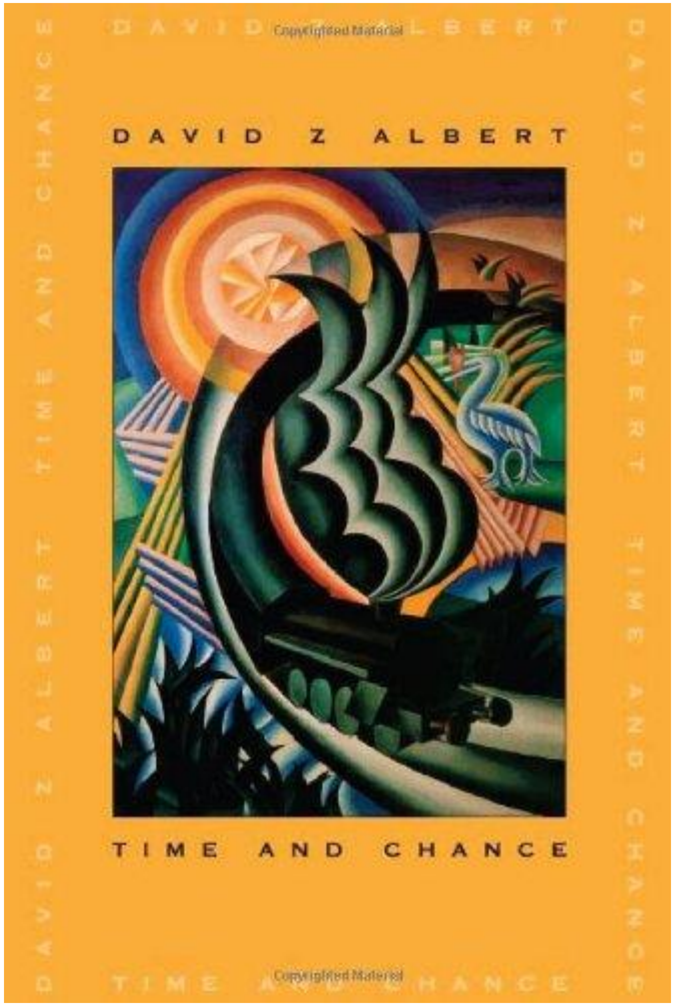
Important assumption: initial pdf is *velocity symmetric.*

Ensures that $\Omega_{t=0}(\Gamma) = 0$

The Past Hypothesis

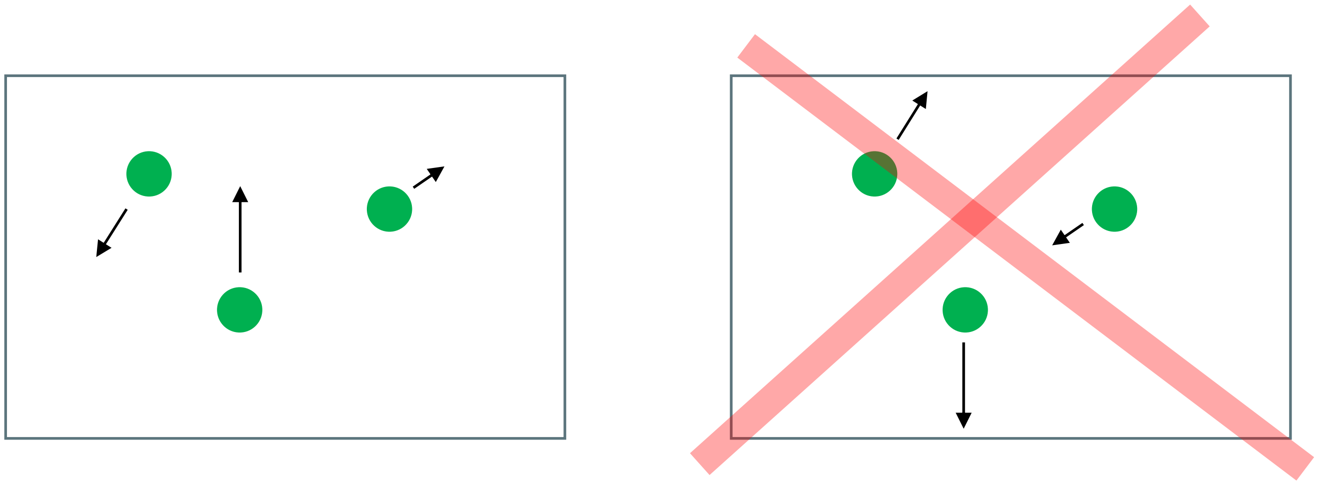
- The possible states of the universe in the past differ from those in the future

- Implies asymmetry in *velocity* statistics



A constraint on configurations

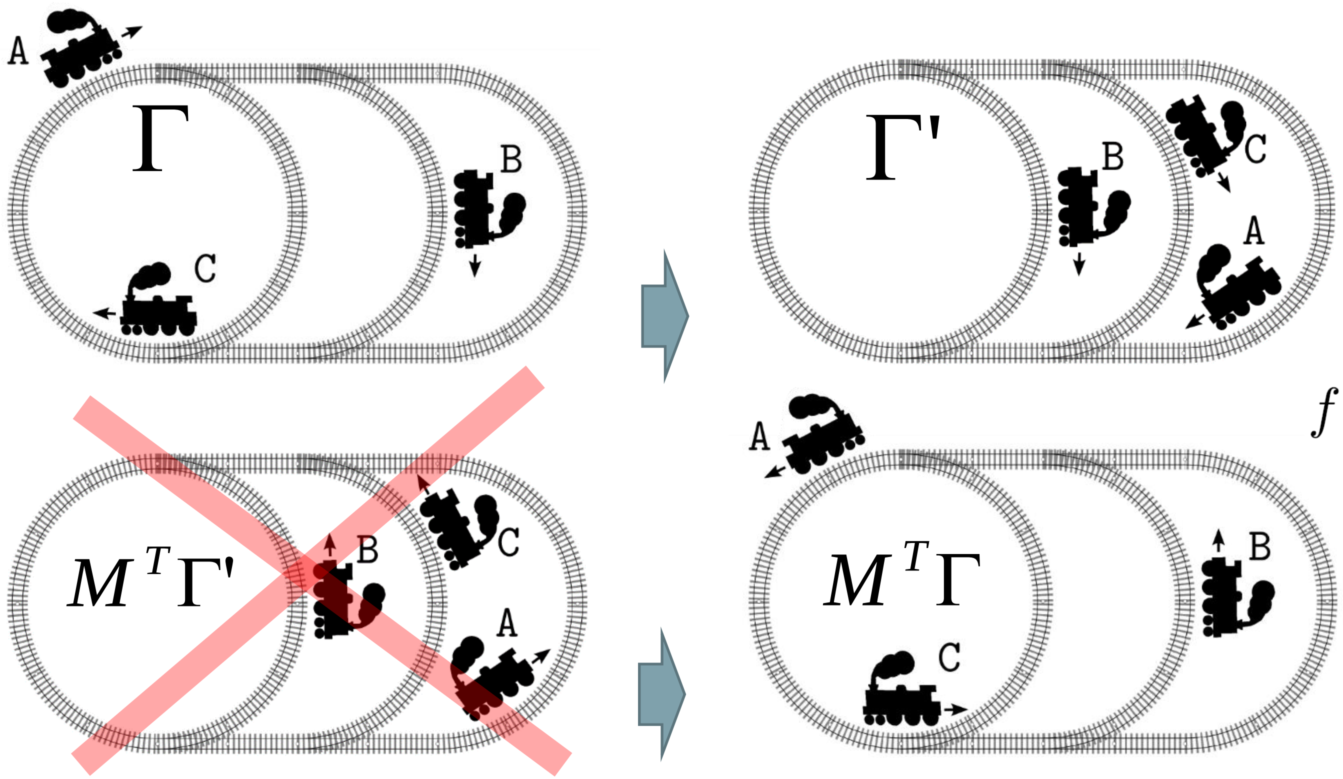
- Configuration and its velocity inverse cannot be equally possible (except in thermal equilibrium)



- If so future would be like the past

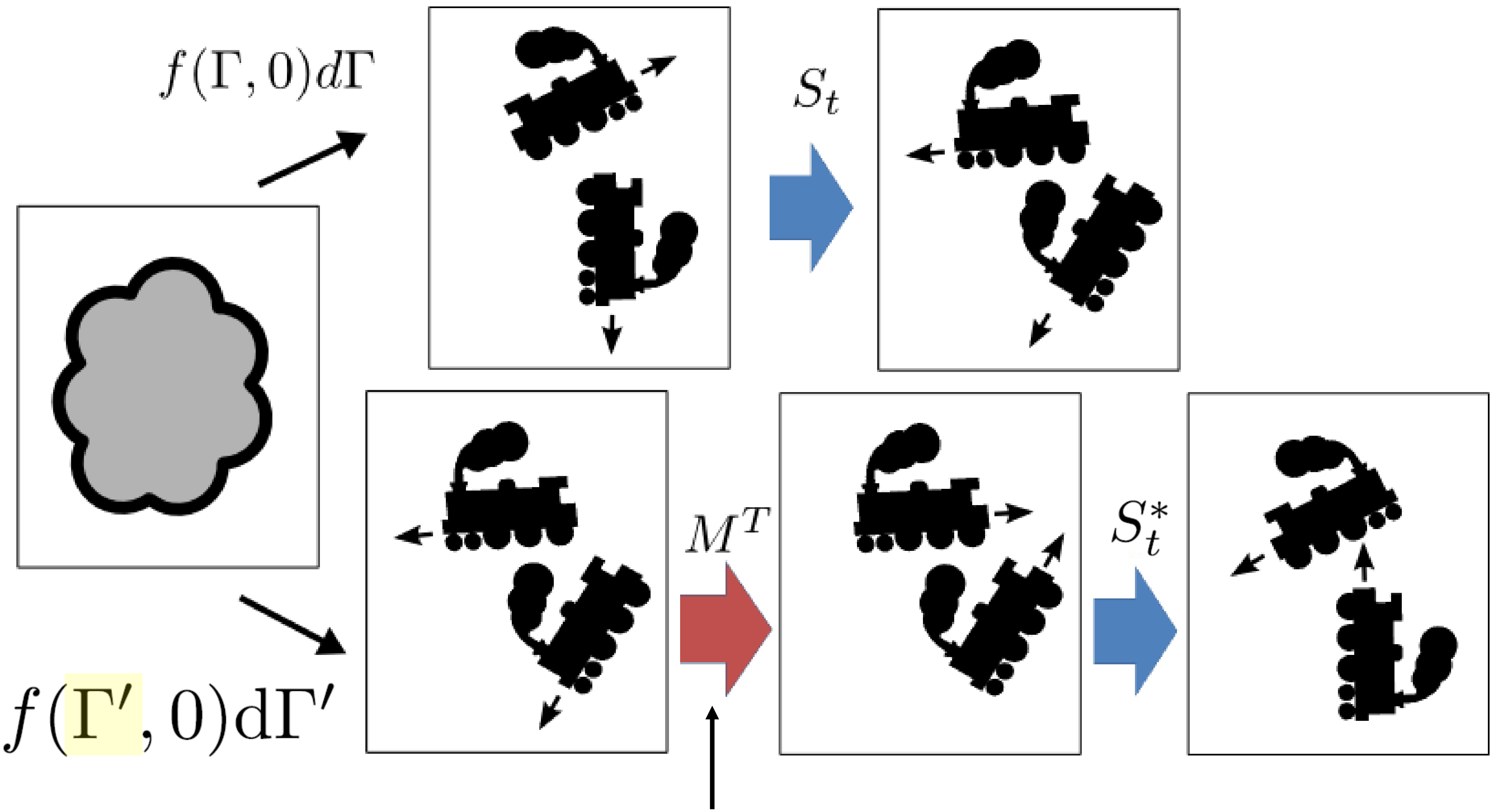
An illustration of the problem

- Suppose the initial ensemble contains trains that can only go forwards?
  - a form of Past Hypothesis.



$$f(M^T\Gamma', 0)d\Gamma' = 0$$

Need a *modified* Evans-Searles test



Select configuration, **then** invert

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The modified dissipation function

Probability of selecting configuration Γ at $t=0$

$$\omega_t(\Gamma) = \int_0^t \omega(S_s\Gamma)ds = \ln\left[\frac{f(\Gamma,0)d\Gamma}{f(\Gamma',0)d\Gamma'}\right]$$

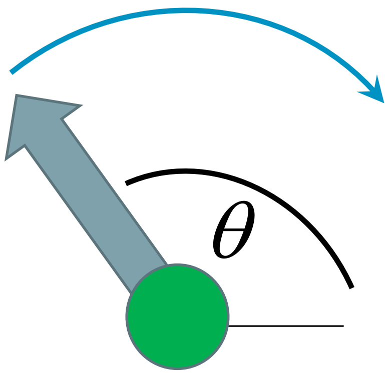
$$\boxed{\langle\omega_t\rangle \geq 0}$$

Dissipation production

Probability of selecting, at $t=0$, the ~~velocity inverted~~ evolved configuration Γ'

Initial pdf can be velocity *asymmetric*: no conflict with Past Hypothesis

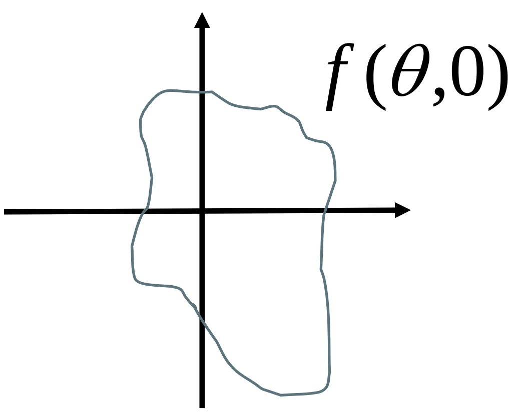
Example: alignment of particle direction of motion with a field



$$\frac{d\theta}{dt} = -\sin\theta$$

pdf over directions

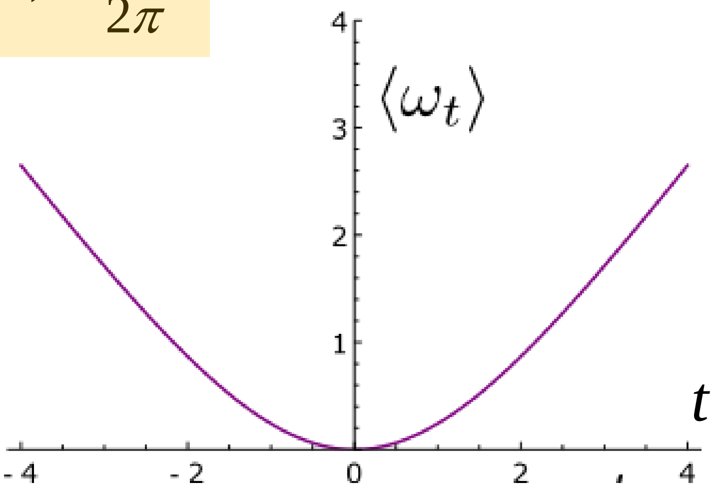
$$f(\theta,0) \rightarrow f(\theta,t)$$

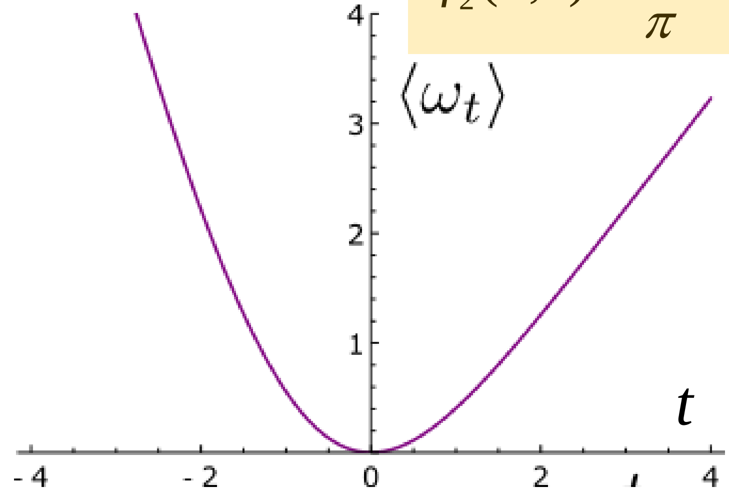


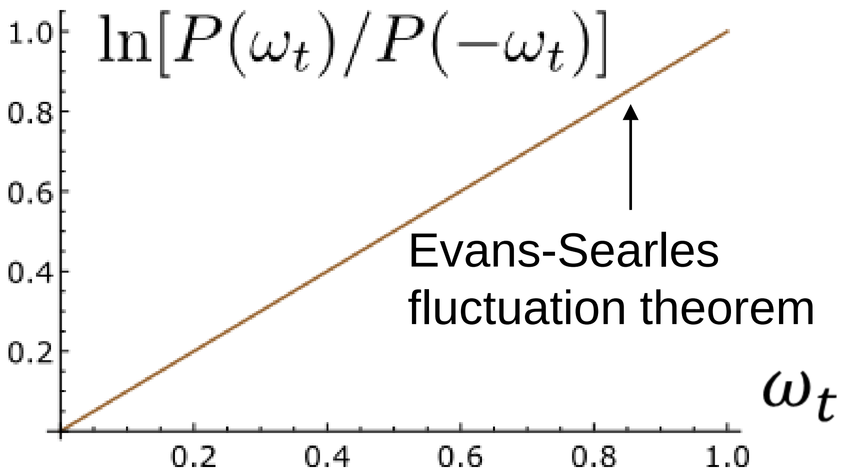
Initial velocity symmetry and asymmetry

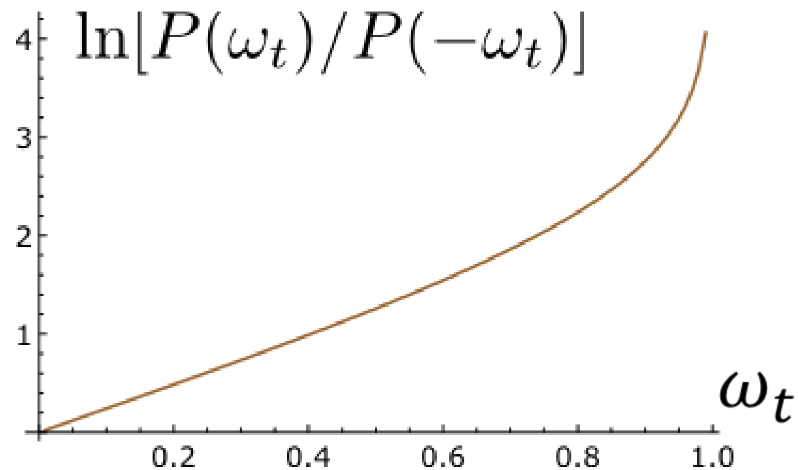
$$f_1(\theta,0) = \frac{1}{2\pi}$$

$$f_2(\theta,0) = \frac{1}{\pi}\cos^2\theta$$

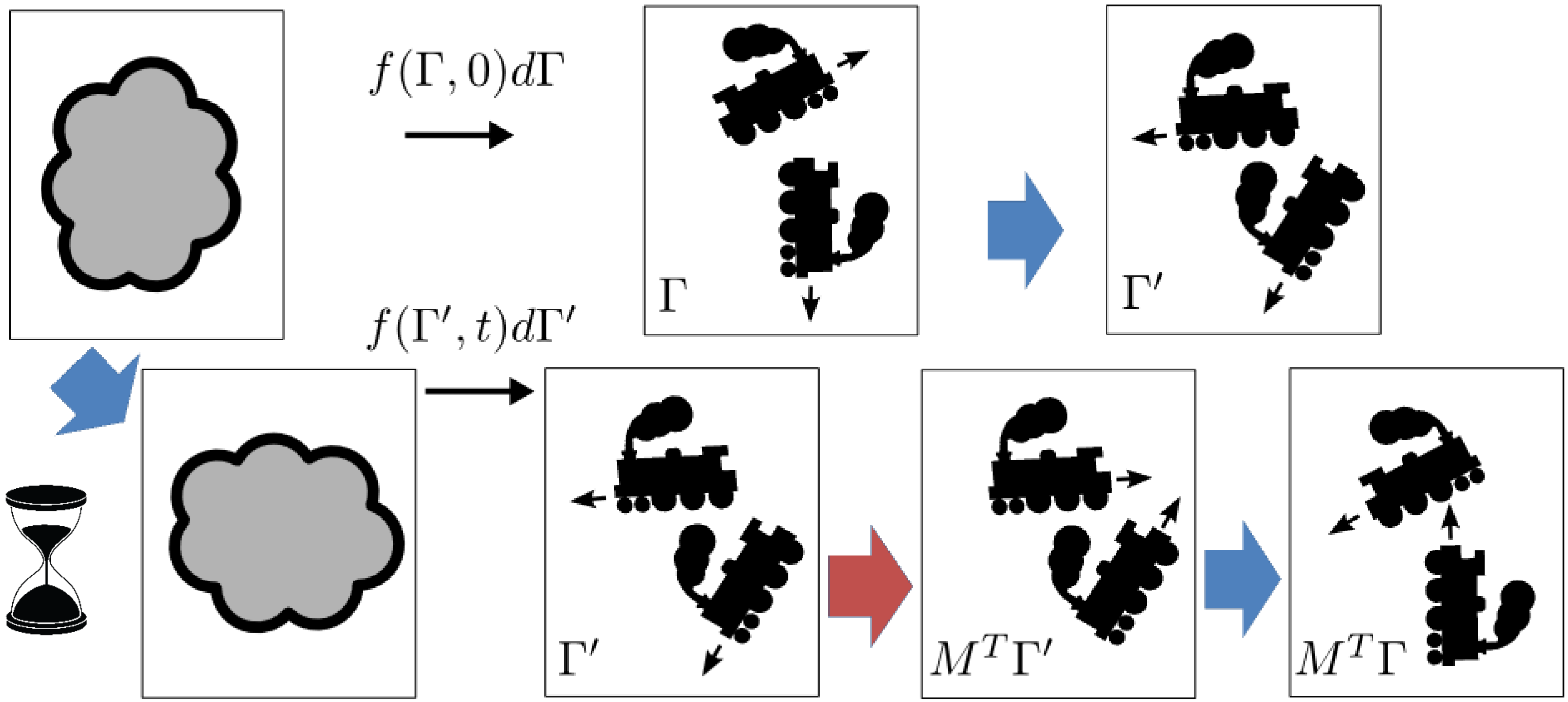






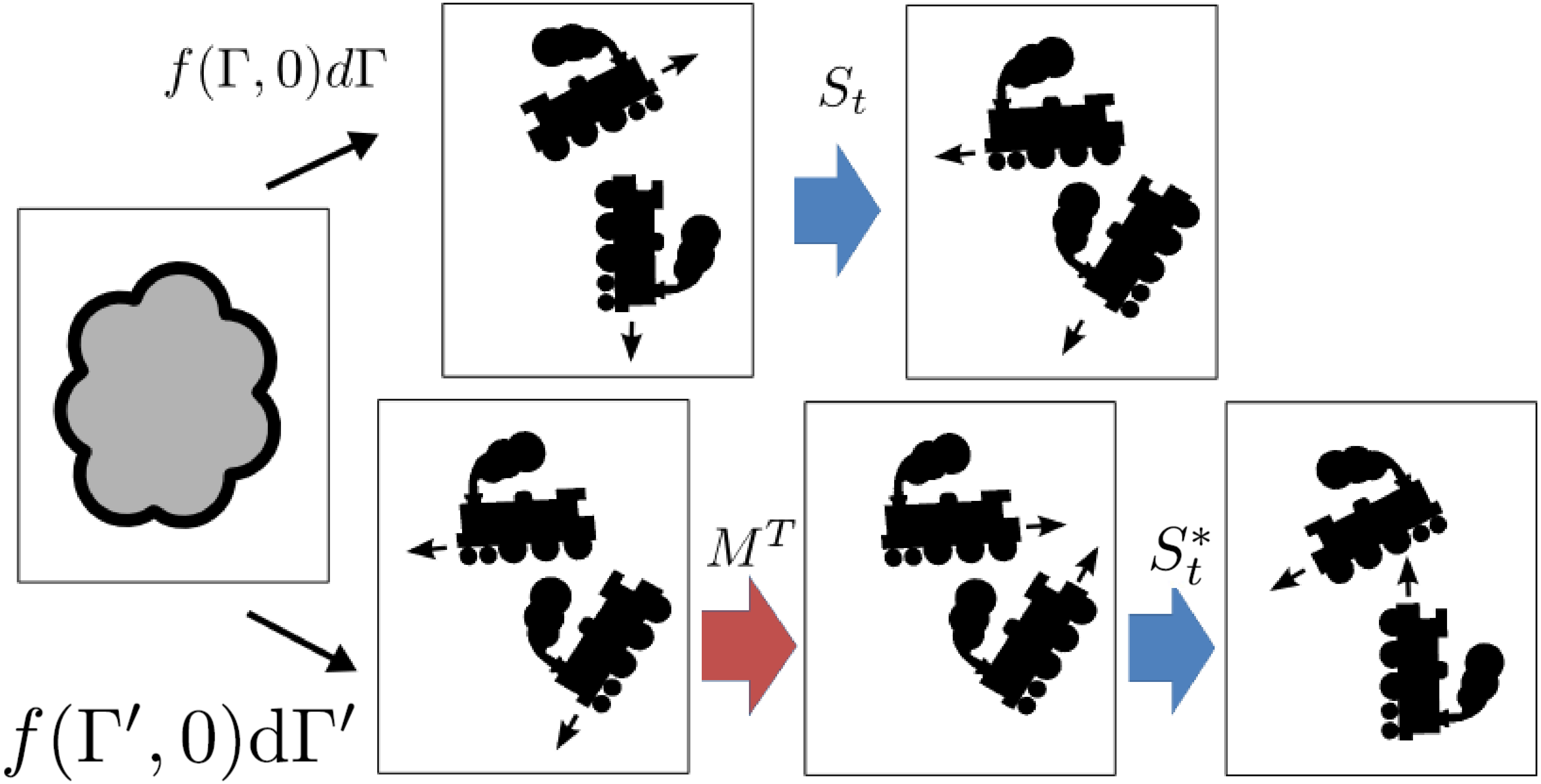


Reversibility tested by Δs_{tot}



How likely is an antitrajectory of previous behaviour?

Obversibility tested by ω_t



How likely is an antitrajectory from the start?

The obverse and reverse trajectories

- Obverse trajectory runs *concurrent* with the trajectory

- Reverse trajectory runs *subsequent* to the trajectory

Mathematical contrast (and similarity)

Transition probability from Γ to Γ'

$$\Delta s_{\mathrm{tot}}[\Gamma \to \Gamma'] = \ln \left[\frac{f(\Gamma,0)T_t(\Gamma \to \Gamma')}{f(\Gamma',t)T_t(M^T\Gamma' \to M^T\Gamma)} \right]$$

$$\omega_t(\Gamma) = \ln \left[\frac{f(\Gamma,0)d\Gamma}{f(\Gamma',0)d\Gamma'} \right]$$

Γ deterministically mapped to Γ'

Quantum example

with Claudia Clarke

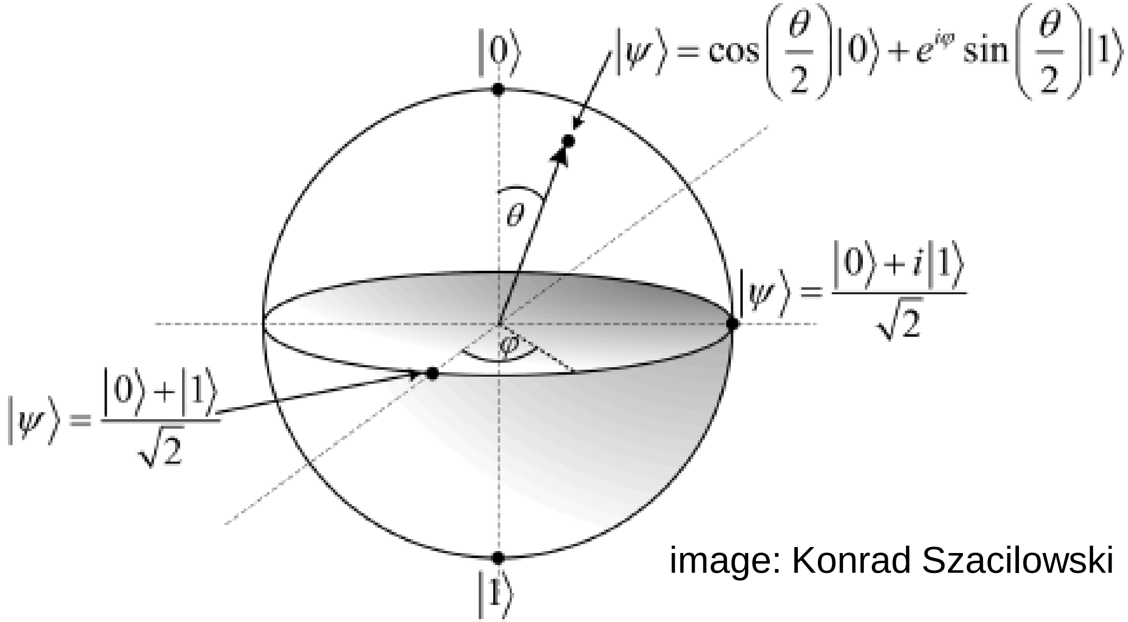


image: Konrad Szacilowski

- Two level system (qubit)

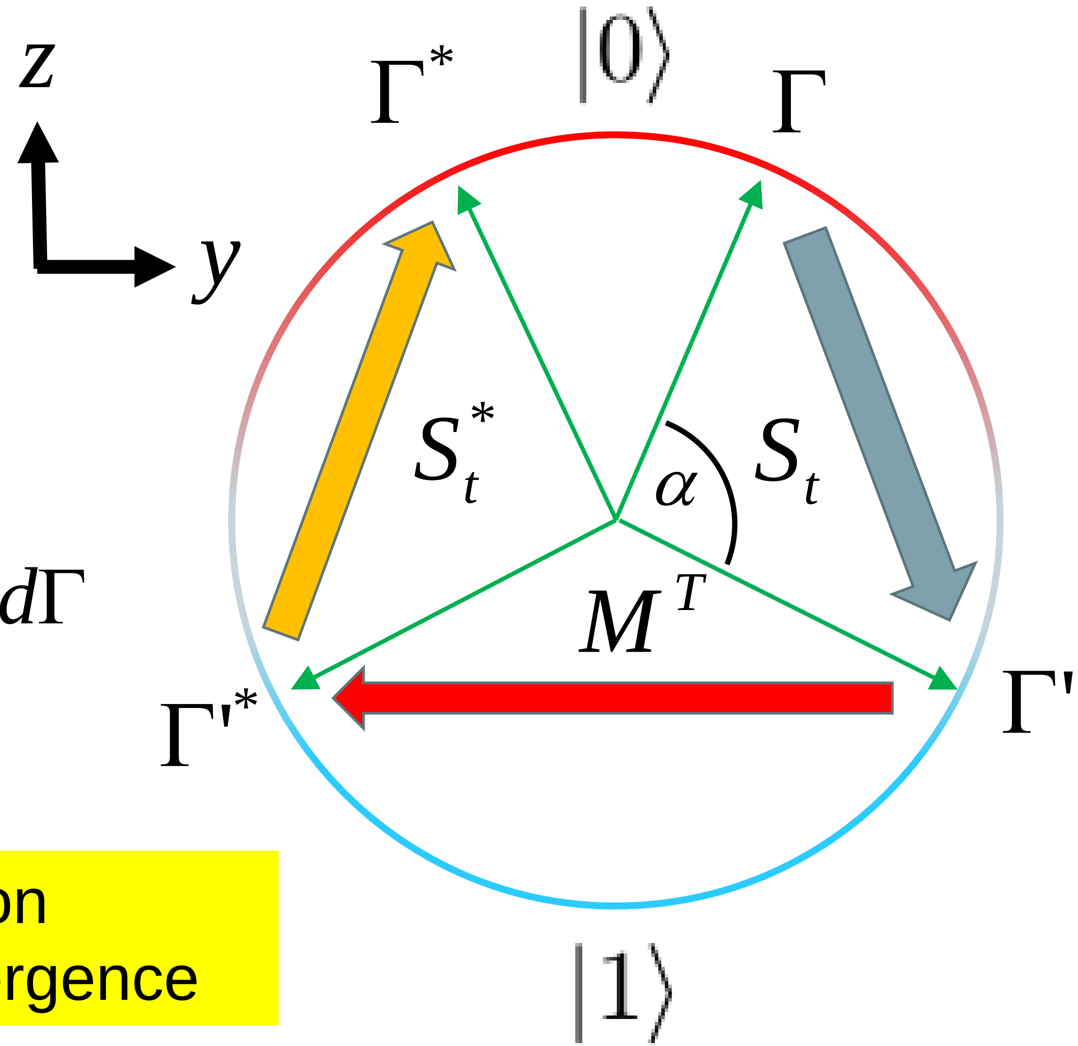
$$|\psi\rangle = |\theta(t), \phi(t)\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

- Evolution on the Bloch sphere

Velocity inversion = complex conjugation

$$\omega_t(\Gamma) = \ln\left(\frac{f(\Gamma)}{f(\Gamma')}\right)$$

$$\langle \omega_t \rangle = \int f(\Gamma) \ln\left(\frac{f(\Gamma)}{f(\Gamma')}\right) d\Gamma$$

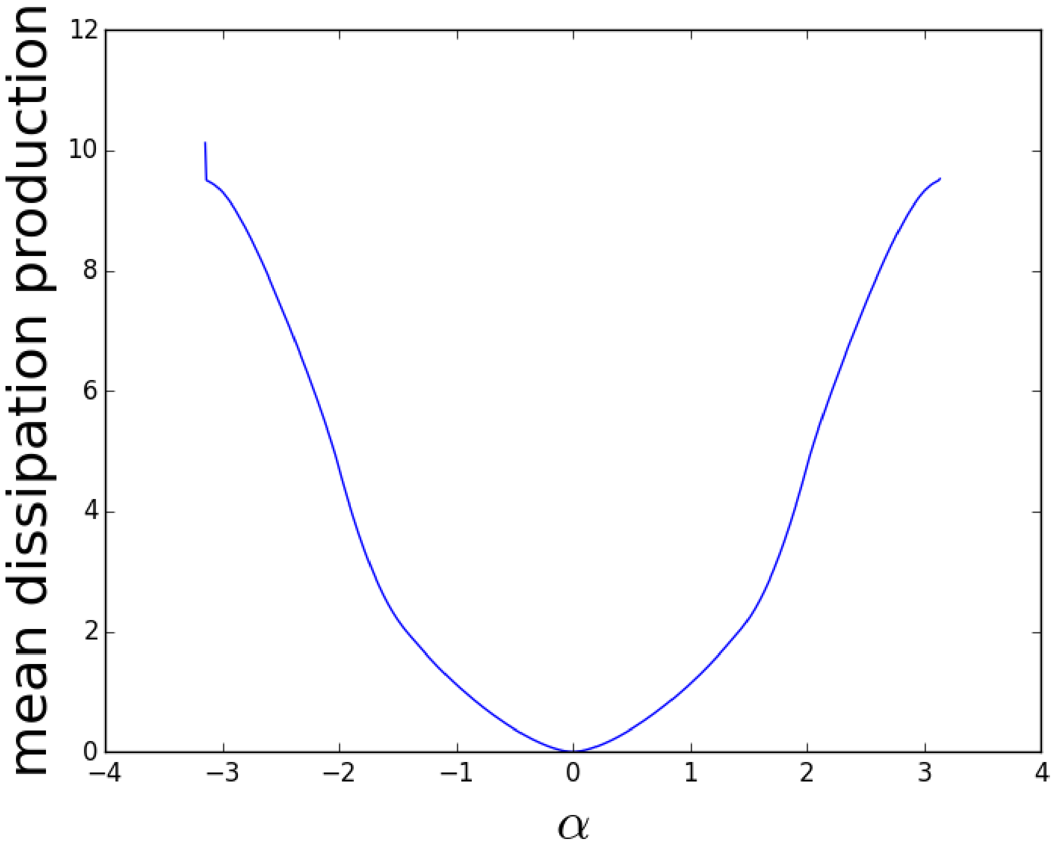


Mean dissipation production
= a Kullback-Leibler divergence

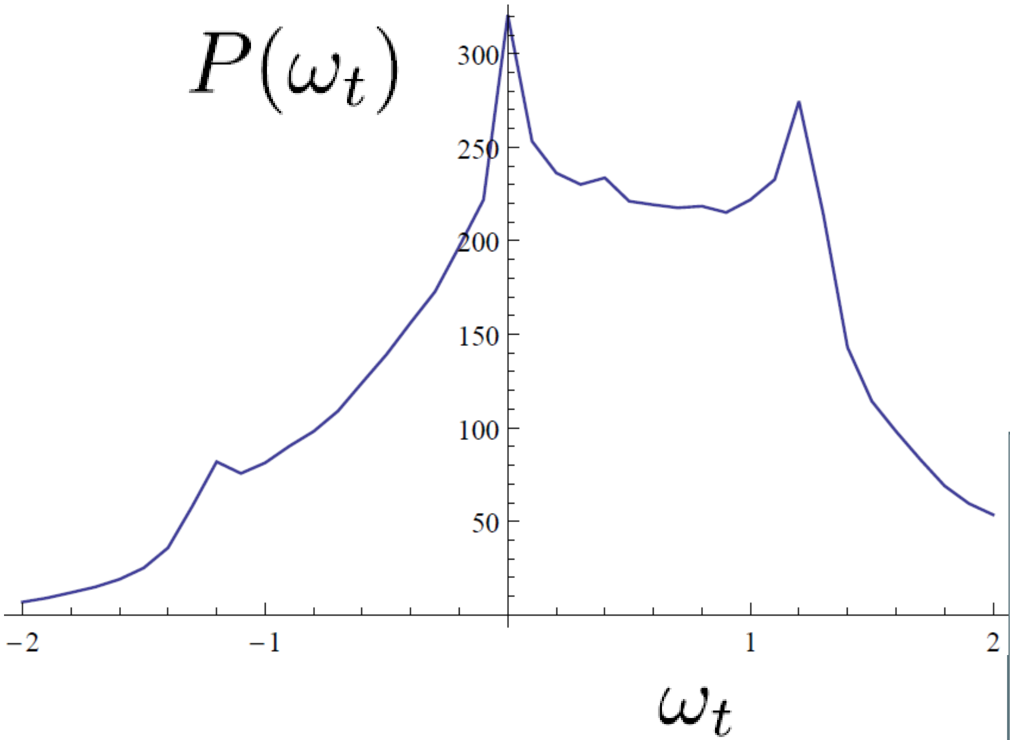
Symmetric pdf

$$f(\theta,\phi) = \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} = f(\theta,-\phi)$$

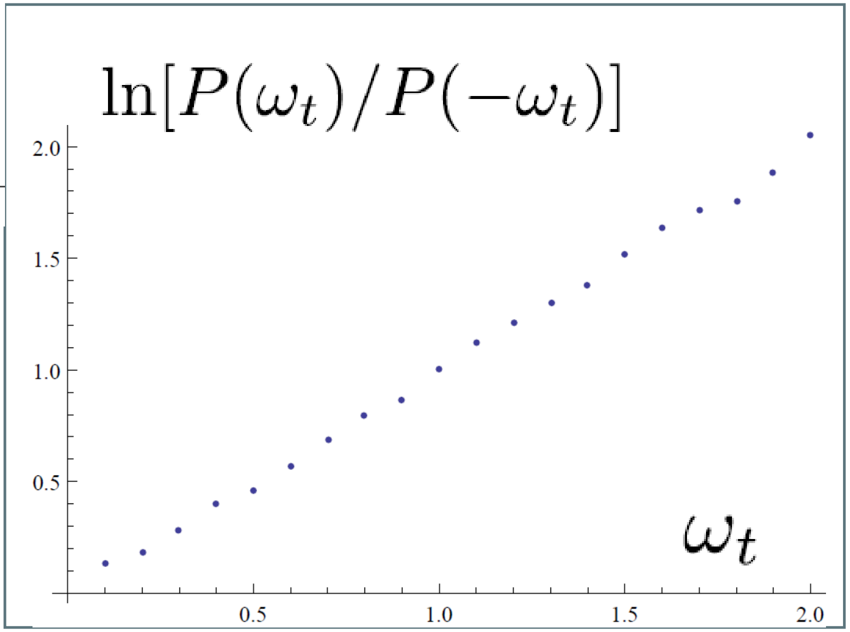
$$\langle \omega_t \rangle$$



pdf of dissipation production



Process: π/2 rotation about *x* axis



What have we learnt (so far)

- Dissipation production ω_t
  - Modified version of time-integrated Evans-Searles dissipation function Ω_t
  - Suitable for velocity asymmetric pdfs implied by Past Hypothesis
- Average increases with elapsed time
  - reminiscent of thermodynamic entropy (but different)
  - similar to stochastic entropy production (but different)
  - evolution can be asymmetric in time

THE MESSAGE

- How to measure **irreversibility**?
- For stochastic dynamics:
  - stochastic entropy production measures ***reversibility***
- For deterministic dynamics:
  - ‘dissipation production’ measures ***obversibility***

- **Thanks for listening!**

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