DEFRA Project: Improved Methods for National Spatial-Temporal Rainfall and Evaporation Modelling for BSM

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Mathematical expressions of generalised moments used in single-site rainfall models

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Contents

1 Introduction

A range of single-site rainfall models are being compared for assessment in Workpackage 1. Insofar as the selected fitting method is the generalised method of moments (Onof & Lekkas, 2003), mathematical expressions for each of these moments are required for input into the objective function.

Many such expressions are already available in the literature. However, as the result of a fairly high rate of typographical errors, they cannot be used without some form of checking, which may involve the full derivation of the formula.

Other expressions are not readily available. This is either because their derivations are unpublished, because the model in question has not yet been examined, or because the statistic in question has not yet been used.

This report seeks to bring all the relevant expressions together for use within the model comparison exercise. When useful or interesting, key elements of the derivations are given. To avoid unnecessary length, the report focusses upon models driven by a Bartlett-Lewis point process. The comparative exercise will in fact include one model driven by a Neyman-Scott point process, for which the relevant analytical expressions will be found in a later report.

2 Background: the Bartlett-Lewis Rectangular Pulse Model

Since single-site models driven by a Bartlett-Lewis point process are all modifications of the Barlett-Lewis Rectangular Pulse model (BLRPM), the key expressions for this model are given here. First, we present the notation used throughout the report, as well as that specific for this model.

2.1 Notation and model specification

2.1.1 Three levels of description

Single site models represent the continuous-time rainfall $Y(t)$. They are calibrated and validated by examining properties of one or both the following processes:

- discrete time aggregated process at time-scale h: $Y_i^{(h)} = \int_{(i-1)h}^{ih} Y(t)\,dt$
- continuous-time moving average process at time-scale h: $Y_{(h)}(t) = \frac{1}{h}\int_{t-h/2}^{t+h/2} Y(t)\,dt$

Data sets of observed data at time-scale h can be considered as samples of the aggregated process, but also as providing samples of the continuous-time moving average process.

2.1.2 Model description

The main Bartlett-Lewis point process is a cluster Poisson process characterised by the arrival of random clusters of points according to a Poisson process. In terms of the representation of rainfall, the clusters are *storms* and the points correspond to the arrivals of *cells*.

Within each cluster, points arrive throughout a period of storm activity which is a random variable.

To each cell arrival time is assigned a rainfall pulse of random duration and intensity.

2.1.3 Parameters

The following notation is used throughout:

- λ: Poisson cluster (storm) arrival rate
- β: Poisson point (cell) arrival rate activity

In the BLRPM, the storm and cell durations are standardly taken as exponentially distributed:

- γ: Exponential parameter of storm duration
- η: Exponential parameter of cell duration

The cell intensity distribution is characterised by three parameters:

- μ_x: Mean cell intensity
- μ_{x^2}: Mean of squares of cell intensities
- μ_{x^3}: Mean of cubes of cell intensities

Three distributions are considered for the intensity: the exponential, Gamma and general Pareto distributions. Details of the notation used and the main relevant properties of these distributions are given in appendix A.

In the distributions considered here, one or two parameters are sufficient to fully characterise the distribution. The BLRPM can therefore be characterised by the following set of parameters:

$$\{\lambda, \mu_x, \mu_{x^2}, \eta, \beta, \gamma\}$$

These parameters do not, however, all have direct physical meaning. It is therefore useful to re-parameterize the model in terms of a set of *mechanistic parameters*:

$$\{\lambda, \mu_x, \sigma_x, \delta_c, \mu_c, \delta_s\}$$

defined by:

Storm arrival rate	λ		(hr^{-1})
Mean cell intensity	μ_x		$(mm.hr^{-1})$
Standard deviation of cell intensity	σ_x	$= \sqrt{\mu_{x^2} - \mu_x^2}$	$(mm.hr^{-1})$
Mean cell duration	δ_c	$= 1/\eta$	(hr)
Mean number of cells per storm	μ_c	$= 1 + \beta/\gamma$	
Mean duration of storm activity	δ_s	$= 1/\gamma$	(hr)

For the sake of simplicity, the equations are given in terms of the original parameter set. They can easily be re-expressed in terms of the mechanistic parameters using the following relations:

$$\begin{aligned} \mu_{x^2} &= \sigma_x^2 + \mu_x^2 \\ \eta &= 1/\delta_c \\ \beta &= (\mu_c - 1)/\delta_s \\ \gamma &= 1/\delta_s \end{aligned}$$

2.1.4 Properties of the aggregated process

The properties of the process $Y_i^{(h)}$ which are considered for model calibration and validation are functions of the following:

- $M(h)$: mean of the rainfall depth (in mm)
- $V(h)$: variance of the rainfall depth (in mm^2)
- $C(k, h)$: autocovariance lag-k of the depth (in mm^2)
- $A(k, h)$: autocorrelation lag-k of the depth (in mm^2)
- $M^p(h)$: non-centered moment of order p ($p > 1$) of the intensity (in mm^p)
- $P_d(h)$: proportion of dry periods
- $M_d(h)$: mean duration of a dry period (in hours)
- $M_w(h)$: mean duration of a wet period (in hours)

where all the properties are for time-scale h hours.

2.1.5 Properties of the moving average process

Note that the properties below could also be expressed as properties of the aggregated process, since $\text{var}\left[Y_{(h)}(t)\right] = \text{var}\left[\frac{Y_i^{(h)}}{h}\right]$. The main property of interest is the variance of the moving average process. This variance can be related to the variance of the underlying continuous-time process $Y(t)$ by defining a variance reduction factor called the *variance function* and denoted $\omega(h)$:

$$\text{var}\left[Y_{(h)}(t)\right] = \omega(h)\text{var}[Y(t)]$$

The variance function can easily be calculated (VanMarcke, 1993) as:

$$\omega(h) = \frac{1}{h^2}\int_0^h \int_0^h \rho(t_1 - t_2)\, dt_1\, dt_2 = \frac{2}{h}\int_0^h \left(1 - \frac{\tau}{h}\right) \rho(\tau)\, d\tau \tag{1}$$

where $\rho(\tau)$ is the autocorrelation function at lag τ of process $Y(t)$.

Of particular interest is the behaviour of the variance function as the scale increases. If this is not a long memory process, we must have:

$$\lim_{h\to\infty} \omega(h) = 0$$

For many processes, the convergence to 0 is in $1/h$. Consequently, Vanmarcke (1993) defines the *scale of fluctuation* as:

$$\Theta = \lim_{h\to\infty} h\omega(h)$$

Since:

$$\omega(h) = \frac{1}{2h}\left[\int_0^h \rho(\tau)\, d\tau - \frac{1}{h}\int_0^h \tau\rho(\tau)\, d\tau\right]$$

and since the scale of fluctuation only exists if

$$\lim_{h\to\infty} \frac{1}{h}\int_0^h \tau\rho(\tau)\, d\tau = 0$$

we therefore obtain (Vanmarcke, 1993):

$$\Theta = \lim_{h\to\infty} h\omega(h) = 2\int_0^\infty \rho(\tau)\, d\tau$$

The scale of fluctuation is so called since, when h is large, $Y_{(h)}(t)$ has a variance which is approximately $\text{var}[Y(t)]\,\Theta/h$. $Y_{(h)}(t)$ is therefore equivalent to the mean of h/Θ independent observations from the continuous-time process.

To summarize, the following moving average process properties can be used in the calibration:

- $\omega(h)$: Variance function at time-scale h
- Θ: Scale of fluctuation

2.2 BLRPM Continuous-time properties

The two important properties for the calculation of the aggregated process properties are the mean and covariance function. From Rodriguez-Iturbe et al. (1987), we have:

$$E[Y(t)] = \lambda \mu_c \mu_x / \eta \tag{2}$$

where $\mu_c = 1 + \frac{\beta}{\gamma}$ is the mean number of cells per storm, and

$$c_Y(\tau) = \frac{\lambda \mu_c}{\eta} \left[\mu_{x^2} + \frac{\beta \gamma \mu_x^2}{\gamma^2 - \eta^2} \right] e^{-\eta \tau} - \frac{\lambda \mu_c}{\eta} \frac{\beta \eta \mu_x^2}{\gamma^2 - \eta^2} e^{-\gamma \tau} \tag{3}$$

2.3 BLRPM First- and second order aggregated depth moments

The first two moments of the marginal distribution of rainfall depths are obtained by integration:

$$\begin{aligned} M(h) &= E[Y_i^{(h)}] &&= E[Y(t)] && (4) \\ V(h) &= \text{var}[Y_i^{(h)}] &&= 2 \int_0^h (h - u) c_Y(u)\, du && (5) \\ C(k,h) &= \text{cov}[Y_i^{(h)}, Y_{i+k}^{(h)}] &&= \int_{-h}^{h} (h - |v|) c_Y(kh + v)\, dv && (6) \end{aligned}$$

The moments are given by (Rodriguez-Iturbe et al., 1987):

$$M(h) = \frac{\lambda h \mu_x \mu_c}{\eta} \tag{7}$$

$$V(h) = \frac{2\lambda \mu_c}{\eta} \left[\frac{(\mu_{x^2} + \beta \mu_x^2/\gamma) h}{\eta} + \frac{\mu_x^2 \beta \eta (1 - e^{-\gamma h})}{\gamma^2 (\gamma^2 - \eta^2)} - \left(\mu_{x^2} + \frac{\beta \gamma \mu_x^2}{\gamma^2 - \eta^2} \right) \frac{1 - e^{-\eta h}}{\eta^2} \right] \tag{8}$$

The autocovariance of lag-k is given by:

$$C(k,h) = \frac{\lambda \mu_c}{\eta} \left[\left(\mu_{x^2} + \frac{\beta \gamma \mu_x^2}{\gamma^2 - \eta^2} \right) \frac{(1 - e^{-\eta h})^2 e^{-\eta (k-1) h}}{\eta^2} - \frac{\mu_x^2 \beta \eta (1 - e^{-\gamma h})^2 e^{-\gamma (k-1) h}}{\gamma^2 (\gamma^2 - \eta^2)} \right] \tag{9}$$

2.4 BLRPM Wet-dry discrete-time properties

The expression for the proportion dry is (Rodriguez-Iturbe et al., 1987):

$$P_d(h) = \exp\{-\lambda(h + \mu_T) + \lambda G_P^*(0,0)(\gamma + \beta e^{-(\beta+\gamma)h})/(\beta + \gamma)\} \tag{10}$$

where μ_T is the mean storm duration, given by (Onof, 1992):

$$\mu_T = \frac{1}{\gamma} + \frac{\gamma}{\eta^2} \int_0^1 v^{-1}\, dv \int_0^1 t^{\frac{\gamma}{\eta} - 1} \left[1 - (1 - vt) e^{-\frac{\beta v (1-t)}{\eta}} \right] dt$$

and

$$G_P^*(z,s) = \eta^{-1} e^{-\frac{\beta(1-z)}{\eta}} \int_0^1 t^{\frac{\gamma+s}{\eta}-1} [1-(1-z)t] e^{\frac{\beta(1-z)t}{\eta}} \, dt$$

Since these expressions are not easy to compute, the following approximations can be used. They are valid if $\beta << \eta$ and $\gamma << \eta$ (i.e. if there is enough cell overlap and cell durations are much smaller than storm durations):

$$\mu_T \approx \frac{1}{\gamma}\left\{1 + \frac{\gamma(\beta+\gamma/2)}{\eta^2} - \frac{\gamma(5\gamma\beta+\beta^2+2\gamma^2)}{4\eta^3} + \frac{\gamma(4\beta^3+31\beta^2\gamma+99\beta\gamma^2+36\gamma^3)}{72\eta^4}\right\}$$

and

$$G_P^*(z,s) \approx \frac{1}{\gamma}\left\{1 - \frac{\beta+\gamma}{\eta} + \frac{3\beta\gamma+2\gamma^2+\beta^2}{2\eta^2}\right\}$$

as in Onof (1992).

However, if these requirements on small values of β and γ are not fulfilled, we can approximate these terms follows:

$$\mu_T \approx \eta^{-1}\left(1 + \phi\sum_{j=1}^{M} \frac{(-\kappa)^{j-1}(\kappa - j^2 - j)}{j(j+1)!} B(j+1,\phi) + \phi^{-1}\right)$$

and

$$G_P^*(0,0) \approx \eta^{1-} e^{-\kappa}\left(\sum_{j=0}^{M'} \frac{\kappa^j}{j!} B(j+\phi,2) + \frac{\delta_{M'}(\kappa)}{(M'+\phi+1)(M'+\phi+2)}\right)$$

with

$$\delta_{M'}(\kappa) = e^{\kappa} - \sum_{j=0}^{M'} \frac{\kappa^j}{j!} \tag{11}$$

where $\kappa = \beta/\eta$ and $\phi = \gamma/\eta$ and the values of M and M' are to be chosen large enough so as to reduce the error. For the LR model (discussed further), identical approximations are required. Upper bounds for the errors involved are estimated in appendix E, together with numerical investigations into their values for different values of κ and ϕ. Appendix E also presents the derivation of these approximations.

The mean duration dry is then a function of the proportion dry (Onof et al., 1994):

$$M_d(h) = \frac{P_d(h)}{P_d(h) - P_d(2h)} \tag{12}$$

Note we can easily derive another useful statistic, namely the mean number of events at time-scale h in a period of duration $n(h)$ time-intervals of h hours. Since the probability of the arrival of an event at time-scale h hours is given by:

$$p_e(h) = Pr\{\text{event start in } [(n-1)h, nh)\} = Pr\{Y_n^{(h)} > 0 | Y_{n-1}^{(h)} = 0\}\, Pr\{Y_{n-1}^{(h)} = 0\}$$

this yields:

$$p_e(h) = \left(1 - \frac{P_d(2h)}{P_d(h)}\right) P_d(h) = P_d(h) - P_d(2h)$$

If the mean storm duration is very small compared to the duration of the period, i.e. $\left(\frac{\mu_T}{h}+1\right)p_e(h) << n(h)$ (the condition is, for instance, met if the period under consideration is the month and the time-scale less than 6 hours), then the mean number of events is approximately given by:

$$n_e(h) \approx (P_d(h) - P_d(2h))\, n(h) \tag{13}$$

since, as the event duration goes to zero, the distribution of the number of events is approximately binomially distributed $B(n(h), P_d(h) - P_d(2h))$.

2.5 BLRPM Third-order aggregated depth moment

The third-order moment yields information about the asymmetry of a distribution. Because of the need to obtain a good fit for extreme values, it is useful to include this moment in the fitting process. The main steps of the derivation are as follows.

2.5.1 General form of the integral

As a first step, we need to relate this moment to moments of the underlying continuous-time process. In general, we can write that, if

$$Y_i^{(h)} = \int_{(i-1)h}^{ih} Y(u)\, du$$

then,

$$E\left[Y_i^{(h)}, Y_{i+j}^{(h)}, Y_{i+k}^{(h)}\right] = E\left[\int_{(i-1)h}^{ih}\int_{(i+j-1)h}^{(i+j)h}\int_{(i+k-1)h}^{(i+k)h} Y(u)Y(v)Y(w)\, du\, dv\, dw\right]$$

The change of variables:

$$x = u; y = v - u - jh; z = w - u - kh$$

yields the domain of integration shown in figure 1.

The sum of six integrals must then be computed:

$$E\left[Y_i^{(h)}, Y_{i+j}^{(h)}, Y_{i+k}^{(h)}\right] = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{14}$$

where:

$$\begin{aligned}
I_1 &= \int_{z=0}^{h}\int_{y=-h+z}^{0}\int_{x=(i-1)h-y}^{ih-z} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\, dx\, dy\, dz \\
I_2 &= \int_{z=0}^{h}\int_{y=0}^{z}\int_{x=(i-1)h}^{ih-z} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\, dx\, dy\, dz \\
I_3 &= \int_{z=0}^{h}\int_{y=z}^{h}\int_{x=(i-1)h}^{ih-y} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\, dx\, dy\, dz
\end{aligned}$$

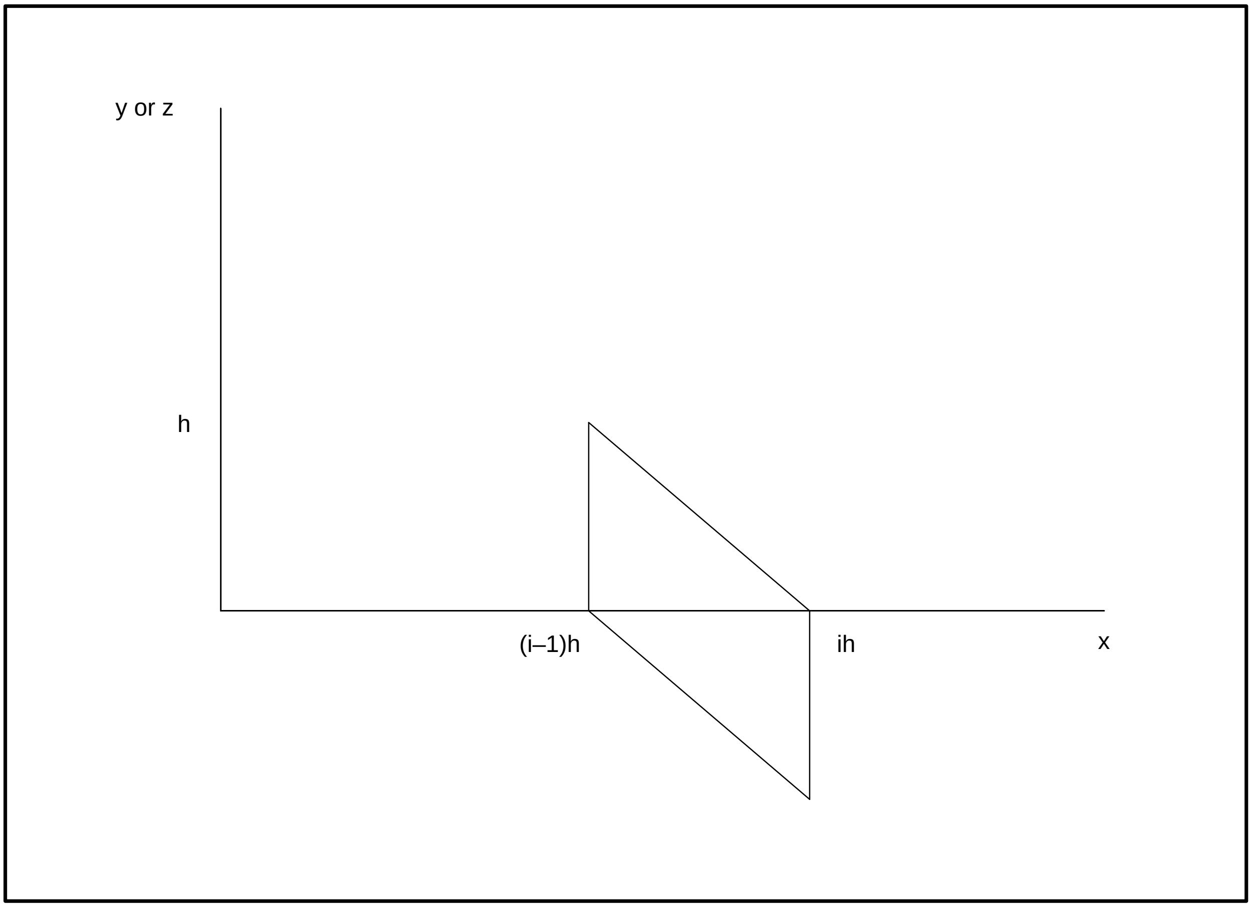

Figure 1: Domain of integration in the (x,y) or (x,z) planes

$$
\begin{aligned}
I_4 &= \int_{z=-h}^{0} \int_{y=-h}^{z} \int_{x=(i-1)h-y}^{ih} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\,dx\,dy\,dz \\
I_5 &= \int_{z=-h}^{0} \int_{y=z}^{0} \int_{x=(i-1)h-z}^{ih} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\,dx\,dy\,dz \\
I_6 &= \int_{z=-h}^{0} \int_{y=0}^{z+h} \int_{x=(i-1)h-z}^{ih-y} E[Y(x)Y((x+y+jh)Y((x+z+kh)]\,dx\,dy\,dz
\end{aligned}
$$

This corresponds to the subdivision of the domain of integration according to figure 2.

For the third-order moment, we have $j = k = 0$. The computation of the integrand, $E[Y(x)Y(x+y)Y(x+z)]$, requires that third-order properties of counts and of cell intensities be computed.

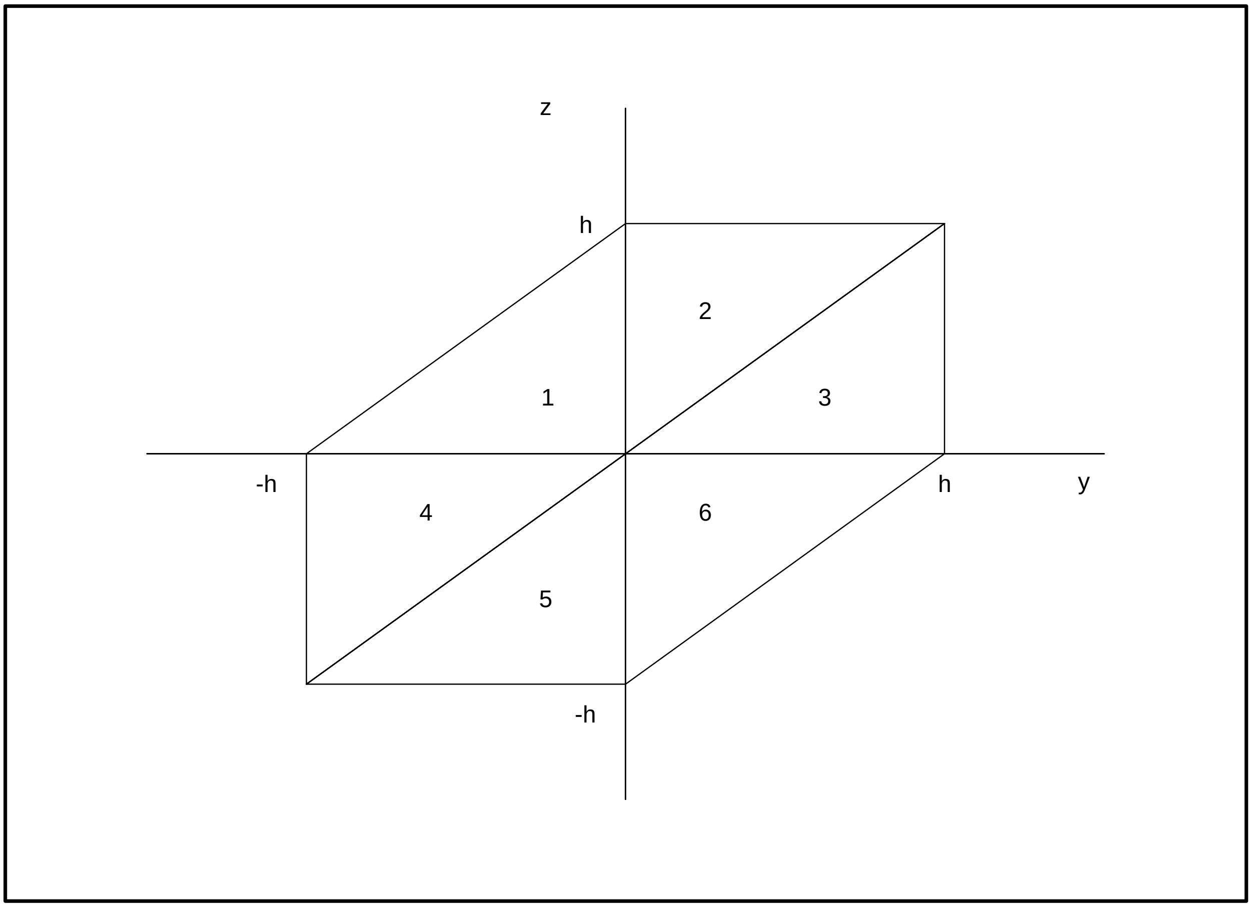

Figure 2: Domain of integration in the (y,z) plane

2.5.2 Third-order properties of counts

These are obtained for $x_1 < x_2 < x_3$ as follows:

$$\begin{aligned} E\left[\delta N(x_1)\delta N(x_2)\delta N(x_3)\right] &= (\lambda\mu_c)^3\, dx_1\, dx_2\, dx_3 \\ &\quad +\lambda\mu_c\beta^2 e^{-\gamma(x_3-x_1)}\, dx_1\, dx_2\, dx_3 \\ &\quad +(\lambda\mu_c)^2\beta\left[e^{-\gamma(x_3-x_2)}+e^{-\gamma(x_3-x_1)}+e^{-\gamma(x_2-x_1)}\right]\, dx_1\, dx_2\, dx_3 \end{aligned}$$

where the first term corresponds to 3 cells in different storms, the second to 3 cells in the same storm and the third to 1 cell in one storm and two in another. This yields:

$$\frac{E\left[\delta N(x_1)\delta N(x_2)\delta N(x_3)\right]}{dx_1\, dx_2\, dx_3} = \left\{(\lambda\mu_c)^3+\lambda\mu_c\beta^2 e^{-\gamma(x_3-x_1)}+(\lambda\mu_c)^2\beta\left[e^{-\gamma(x_3-x_2)}+e^{-\gamma(x_3-x_1)}+e^{-\gamma(x_2-x_1)}\right]\right\} \tag{15}$$

2.5.3 Third-order properties of cell intensities

The following expression is required in the computations:

$$
\begin{aligned}
E\left[X_{t_1-u_1}(u_1)X_{t_1+\tau_1-u_2}(u_1)X_{t_1+\tau_1+\tau_2-u_3}(u_3)\right] \;=\; & \mu_{x^3}e^{-\eta(u_1+\tau_1+\tau_2)} \\
& +\mu_x m u_{x^2}e^{-\eta(u_1+u_2+\tau_2)} \\
& +\mu_x m u_{x^2}e^{-\eta(u_1+u_2+\tau_1+\tau_2)} \\
& +\mu_x m u_{x^2}e^{-\eta(u_1+u_3+\tau_1)} \\
& +\mu_x^3 e^{-\eta(u_1+u_2+u_3)}
\end{aligned}
\tag{16}
$$

where the terms correspond to the following respective 5 cases:

1.	$t_1-u_1=t_1+\tau_1-u_2$	$t_1-u_1=t_1+\tau_1+\tau_2-u_3$	
2.	$u_3=u_2+\tau_2$	$u_2\neq u_1+\tau_1$	
3.	$u_3=u_1+\tau_1+\tau_2$	$u_2\neq u_1+\tau_1$	
4.	$u_2=u_1+\tau_1$	$u_3\neq u_2+\tau_2$	
5.	$u_2=u_1+\tau_1$	$u_3\neq u_2+\tau_2$	$u_3\neq u_1+\tau_1+\tau_2$

2.5.4 Third-order moment of continuous-time process

The main term to compute is the expected value of a product of three rainfall depths of the continuous-time process. This is:

ch

$$
\begin{aligned}
& E\left[Y(x)Y(x+y)Y(x+z)\right] = \\
& \quad E[\int_0^\infty X_{x-u}(u)\,dN(x-u)\;\int_0^\infty X_{x+y-v}(v)\,dN(x+y-v) \\
& \quad\quad \int_0^\infty X_{x+z-w}(w)\,dN(x+z-w)]
\end{aligned}
\tag{17}
$$

This is evaluated as the following sum:

$$
\begin{aligned}
& E\left[Y(x)Y(x+y)Y(x+z)\right] = \\
& \quad \int_{u=0}^\infty\int_{v=0,v\neq u+y}^\infty\int_{w=0,w\neq u+z,w\neq v+z-y}^\infty E[X_{x-u}(u)]E[X_{x+y-v}(v)]E[X_{x+z-w}(w)]\,E[dN(x-u)dN(x+y-v)dN(x+z-w)] \\
& \quad +\int_{u=0}^\infty\int_{v=0,v\neq u+y}^\infty E[X_{x-u}(u)]E[X_{x+y-v}(v)X_{x+y-v}(v+z-y)]\,E[dN(x-u)dN(x+y-v)] \\
& \quad +\int_{u=0}^\infty\int_{v=0,v\neq u+y}^\infty E[X_{x-u}(u)X_{x-u}(u+z)]E[X_{x+y-v}(v)]\,E[dN(x-u)dN(x+y-v)] \\
& \quad +\int_{u=0}^\infty\int_{w=0,v\neq u+z}^\infty E[X_{x-u}(u)X_{x-u}(u+y)]E[X_{x+z-w}(w)]\,E[dN(x-u)dN(x+z-w)] \\
& \quad +\int_{u=0}^\infty E[X_{x-u}(u)X_{x-u}(u+y)X_{x-u}(u+z)]\,E[dN(x-u)]
\end{aligned}
$$

The computation of this integral and the final form of its analytical expression are detailled in appendix B.

2.5.5 Third-order moment of discrete-time process

The six integrals in 14 can be rewritten as:

$$
\begin{aligned}
I_1 &= \int_{z=0}^{h} \int_{y=-h+z}^{0} \int_{x=(i-1)h-y}^{ih-z} E[Y(x+y)Y((x+y)-y)Y((x+y)-y+z)]\,dx\,dy\,dz \\
I_2 &= \int_{z=0}^{h} \int_{y=0}^{z} \int_{x=(i-1)h}^{ih-z} E[Y(x)Y(x+y)Y(x+z)]\,dx\,dy\,dz \\
I_3 &= \int_{z=0}^{h} \int_{y=z}^{h} \int_{x=(i-1)h}^{ih-y} E[Y(x)Y(x+z)Y(x+y)]\,dx\,dy\,dz \\
I_4 &= \int_{z=-h}^{0} \int_{y=-h}^{z} \int_{x=(i-1)h-y}^{ih} E[Y(x+y)Y((x+y)+(z-y))Y((x+y)-y)]\,dx\,dy\,dz \\
I_5 &= \int_{z=-h}^{0} \int_{y=z}^{0} \int_{x=(i-1)h-z}^{ih} E[Y(x+z)Y((x+z)+(y-z))Y((x+z)-z)]\,dx\,dy\,dz \\
I_6 &= \int_{z=-h}^{0} \int_{y=0}^{z+h} \int_{x=(i-1)h-z}^{ih-y} E[Y(x+z)Y((x+z)-z)Y((x+z)+(y-z))]\,dx\,dy\,dz
\end{aligned}
$$

where the integrands have been written so as to contain products of $Y(r)Y(r+s)Y(r+t)$ with $0 \le s \le t$. This involves the following transformations:

For I_1:	$r = x + y$	$s = -y$	$t = z - y$
For I_2:	$r = x$	$s = y$	$t = z$
For I_3:	$r = x$	$s = z$	$t = y$
For I_4:	$r = x + y$	$s = z - y$	$t = -y$
For I_5:	$r = x + z$	$s = y - z$	$t = z$
For I_6:	$r = x + z$	$s = -z$	$t = y - z$

By introducing this change of variables, we find that all 6 integrals are identical to I so that:

$$
M^3(h) = 6I = 6 \int_{s=0}^{h} \int_{t=s}^{h} \int_{r=(i-1)h}^{ih-t} E[Y(r)Y(r+s)Y(r+t)]\,dr\,dt\,ds \tag{18}
$$

The final expression for $M^3(h) = E[(Y_i^{(h)})^3]$ is given in appendix C.

2.6 BLRPM moving average properties

Since, for the continuous-time process $Y(t)$, we have:

$$
\text{var}[Y(t)] = \frac{\lambda \mu_c}{\eta} \left[\mu_{x^2} + \frac{\beta}{\gamma + \eta} \mu_x^2 \right]
$$

we have, for the variance function:

$$\omega(h) = \frac{2\left[\frac{(\mu_{x^2}+\beta\mu_x^2/\gamma)}{h\eta} + \frac{\mu_x^2\beta\eta(1-e^{-\gamma h})}{h^2\gamma^2(\gamma^2-\eta^2)} - \left(\mu_{x^2} + \frac{\beta\gamma\mu_x^2}{\gamma^2-\eta^2}\right)\frac{1-e^{-\eta h}}{h^2\eta^2}\right]}{\mu_{x^2} + \frac{\beta}{\gamma+\eta}\mu_x^2} \tag{19}$$

The scale of fluctuation is therefore given by:

$$\Theta = \frac{2}{\eta}\,\frac{\mu_{x^2} + \frac{\beta}{\gamma}\mu_x^2}{\mu_{x^2} + \frac{\beta}{\gamma+\eta}\mu_x^2} \tag{20}$$

Note that if $\gamma \to \infty$, we find the scale of fluctuation of the simpler Poisson Rectangular Pulse Model (Rodriguez-Iturbe et al., 1987), namely $\frac{2}{\eta}$.

3 The Dependent Depth-Duration Model

3.1 Model specification

One way in which the above model can be altered so as to improve its wet-dry properties is by introducing a dependence between cell intensity and cell duration distributions. This option, the DD model, has been examined by Kakou (1997).

The model is characterised by the same parameters $\lambda, \beta, \gamma, \eta$ for the storm and cell arrival rates, storm activity and cell duration. But the cell intensities X are now specified through the distribution of X conditional upon the cell duration L, i.e. $X|L$.

A first way of specifying the dependence is by choosing:

$$E[X|L=l] = fe^{-cl}$$

Kakou (1997) assumed an exponential distribution. More generally, we shall consider a second-order moment specified as:

$$E[X^2|L=l] = ge^{-dl}$$

Note that this entails the following first- and second-order unconditional moments:

$$\begin{aligned} E[X] &= \frac{f\eta}{c+\eta} \\ E[X^2] &= \frac{g\eta}{d+\eta} \end{aligned}$$

We shall refer to this as the DD1 model. Since the temporal structure is identical to that of the BLRPM, we need only examine its depth properties.

3.1.1 Parameters

The proposed DD1 model has 8 parameters:

$$\{\lambda, c, d, f, g, \eta, \beta, \gamma\}$$

Note that it is likely we may wish to simplify this and assume a relation between c and d for instance (in the exponential case considered by Kakou (1997) , $d = 2c$ and $g = 2f^2$).

The *mechanistic parameters* for this model are:

$$\{\lambda, \mu_x, \sigma_x, \mu_{x|0}, \sigma_{x|0}, \delta_c, \mu_c, \delta_s\}$$

defined by:

Storm arrival rate	λ		(hr^{-1})
Mean cell intensity	μ_x	$= \frac{f\eta}{c+\eta}$	$(mm.hr^{-1})$
Std. deviation of cell intensity	σ_x	$= \frac{g\eta}{d+\eta} - \left[\frac{f\eta}{c+\eta}\right]^2$	$(mm.hr^{-1})$
Conditional mean cell intensity limit for 0 cell duration	$\mu_{x\|0}$	$= f$	$(mm.hr^{-1})$
Conditional std. deviation of cell intensity limit for 0 cell duration	$\sigma_{x\|0}$	$= \sqrt{g - f^2}$	$(mm.hr^{-1})$
Mean cell duration	δ_c	$= 1/\eta$	(hr)
Mean number of cells per storm	μ_c	$= 1 + \beta/\gamma$	
Mean duration of storm activity	δ_s	$= 1/\gamma$	(hr)

As before, the equations are given in terms of the original parameter set which can easily be re-expressed in terms of the mechanistic parameters using:

$$\begin{aligned}
c &= \frac{1}{\delta_c}\left(\frac{\mu_{x|0}}{\mu_x} - 1\right) \\
d &= \frac{1}{\delta_c}\left(\frac{\sigma_{x|0}^2 + \mu_{x|0}^2}{\sigma_x^2 + \mu_x^2} - 1\right) \\
f &= \mu_{x|0} \\
g &= \sigma_{x|0}^2 + \mu_{x|0}^2 \\
\eta &= 1/\delta_c \\
\beta &= (\mu_c - 1)/\delta_s \\
\gamma &= 1/\delta_s
\end{aligned}$$

3.2 Continuous-time depth properties

As in Kakou (1997), we find that for the DD model, the mean depth is:

$$E[Y(t)] = \lambda\mu_c E[XL] \tag{21}$$

For DD1, this yields:

$$E[Y(t)] = \lambda\mu_c \frac{\eta f}{(\eta + c)^2} \tag{22}$$

The other important property is the covariance of lag τ, which we derive as:

$$\begin{aligned}
c_Y(\tau) \;=\; & \lambda\mu_c \int_\tau^\infty (l-\tau)\, E[X^2|l]\, f_L(l)\, dl + \frac{\lambda\beta\mu_c}{\gamma^2} \\
& \{2\gamma D(0,\tau,l+\tau,0) + 2\gamma B(0,l+\tau,\infty,0) - 2\gamma\tau A(0,\tau,l+\tau,0) \\
& -e^{-\gamma\tau}A(0,0,\infty,0) + e^{\gamma\tau}A(0,\tau,\infty,\gamma) - e^{\gamma\tau}A(-\gamma,l+\tau,\infty,\gamma) \\
& +e^{-\gamma\tau}A(\gamma,0,\infty,0) - e^{-\gamma\tau}A(\gamma,0,l+\tau,-\gamma) + e^{-\gamma\tau}A(0,0,\tau,-\gamma)\}
\end{aligned} \tag{23}$$

where:

$$\begin{aligned}
A(\theta,a,b,\xi) &= \int_0^\infty dl \int_a^b dl'\, E[X|l]\, f_L(l)\, E[X'|l']\, f_L(l')\, e^{-\theta l} e^{-\xi l'} \\
B(\theta,a,b,\xi) &= \int_0^\infty dl \int_a^b dl'\, E[X|l]\, f_L(l)\, E[X'|l']\, f_L(l')\, l\, e^{-\theta l} e^{-\xi l'} \\
D(\theta,a,b,\xi) &= \int_0^\infty dl \int_a^b dl'\, E[X|l]\, f_L(l)\, E[X'|l']\, f_L(l')\, l'\, e^{-\theta l} e^{-\xi l'}
\end{aligned}$$

For DD1, this becomes:

$$c_Y(\tau) = \frac{\lambda\mu_c g\eta e^{-(d+\eta)\tau}}{(d+\eta)^2} + \frac{\lambda\mu_c f^2\beta\eta^2 \left[(c+\eta)e^{-\gamma\tau} - \gamma e^{-(c+\eta)\tau}\right]}{(c+\eta)^3\left[-\gamma^2 + (c+\eta)^2\right]} \tag{24}$$

3.3 Discrete-time depth properties

3.3.1 First- and second-order moments

The following relations (Rodriguez-Iturbe et al., 1987) are used to obtain first- and second-order properties of the aggregated process:

$$\begin{aligned}
M(h) &= hE[Y(t)] \\
V(h) &= 2\int_0^h (h-\tau)\, c_Y(\tau)\, d\tau \\
C(k,h) &= \int_{-h}^{+h} (h-|\tau|)\, c_Y(kh+\tau)\, d\tau
\end{aligned}$$

For model DD1, we trivially find for the mean depth at time-scale h:

$$M(h) = h\lambda\mu_c \frac{\eta f}{(c+\eta)^2} \tag{25}$$

The variance is derived as:

$$\begin{aligned} V(h) &= \frac{2\lambda\mu_c g\eta}{(d+\eta)^4}\left(e^{-(d+\eta)h} - 1 + (d+\eta)h\right) \\ &+ \frac{2\lambda\mu_c f^2\beta\eta^2}{(c+\eta)^3\left[(c+\eta)^2-\gamma^2\right]}\left[\frac{c+\eta}{\gamma^2}(e^{-\gamma h} + \gamma h - 1) - \frac{\gamma}{(c+\eta)^2}\left(e^{-(c+\eta)h} + (c+\eta)h - 1\right)\right] \end{aligned} \tag{26}$$

and the covariance as:

$$\begin{aligned} C(k,h) &= \frac{\lambda\mu_c g\eta}{(d+\eta)^4}e^{-(d+\eta)(k-1)h}\left(1-e^{-(d+\eta)h}\right)^2 \\ &+ \frac{\lambda\mu_c f^2\beta\eta^2}{(c+\eta)^3\left[(c+\eta)^2-\gamma^2\right]}\left[\frac{c+\eta}{\gamma^2}(1-e^{-\gamma h})^2 e^{-\gamma(k-1)h}\right. \\ &\left.- \frac{\gamma}{(c+\eta)^2}\left(1-e^{-(c+\eta)h}\right)^2 e^{-(c+\eta)(k-1)h}\right] \end{aligned} \tag{27}$$

3.3.2 Wet-Dry properties

These properties are the same as for the BLRPM model, so that equations (10) and (12) can be used, as well as the approximations (11), where, as above, $\kappa = \beta/\eta$ and $\phi = \gamma/\eta$.

3.3.3 Third-order moment

As with the BLRPM model, the calculation of $M^3(h)$ first requires the evaluation of

$$E[Y(x)Y(x+y)Y(x+z)] \text{ with } y > 0\, z > y.$$

This integral involves more extensive calculations than for the BLRPM. It is the sum of 14 terms which are analytically derivable, but are not presented here because they are too cumbersome. To illustrate this, the first term is shown in appendix D. The expressions for all these terms are available in Maple.

The computation of the third-order moment of the discrete-time process involves a triple integral of the sum of these 14 terms, as in the following expression in equation

$$M^3(h) = 6I = 6\int_{y=0}^{h}\int_{z=y}^{h}\int_{r=(i-1)h}^{ih-z} E[Y(x)Y(x+y)Y(x+z)]\,dx\,dz\,dy \tag{28}$$

which, since (as a result of stationarity) the integrand is not a function of x, reduces to:

$$M^3(h) = 6\int_{y=0}^{h}\int_{z=y}^{h} E[Y(x)Y(x+y)Y(x+z)](h-z)\,dx\,dz\,dy \tag{29}$$

These integrals can be computed analytically for the DD1 model, and the results are available in Maple.

3.4 Moving average properties

Since, for the continuous-time process $Y(t)$, we have, for the DD1 model:

$$\mathrm{var}[Y(t)] = \frac{\lambda\mu_c g\eta}{(d+\eta)^2} + \frac{\lambda\mu_c f^2\beta\eta^2}{(c+\eta)^3(c+\eta+\gamma)}$$

we have, for the variance function:

$$\begin{aligned}\omega(h) &= \frac{\frac{2\lambda\mu_c g\eta}{h^2(d+\eta)^4}\left(e^{-(d+\eta)h} - 1 + (d+\eta)h\right)}{\frac{\lambda\mu_c g\eta}{(d+\eta)^2} + \frac{\lambda\mu_c f^2\beta\eta^2}{(c+\eta)^3(c+\eta+\gamma)}} \\ &\quad + \frac{\frac{2\lambda\mu_c f^2\beta\eta^2}{h^2(c+\eta)^3[(c+\eta)^2-\gamma^2]}\left[\frac{c+\eta}{\gamma^2}(e^{-\gamma h} + \gamma h - 1) - \frac{\gamma}{(c+\eta)^2}\left(e^{-(c+\eta)h} + (c+\eta)h - 1\right)\right]}{\frac{\lambda\mu_c g\eta}{(d+\eta)^2} + \frac{\lambda\mu_c f^2\beta\eta^2}{(c+\eta)^3(c+\eta+\gamma)}}\end{aligned} \tag{30}$$

The scale of fluctuation is therefore given by:

$$\Theta = 2\,\frac{\frac{g}{(d+\eta)^3} + \frac{f^2\eta\beta}{(c+\eta)^4\gamma}}{\frac{g}{(d+\eta)^2} + \frac{f^2\eta\beta}{(c+\eta)^3(c+\eta+\gamma)}} \tag{31}$$

4 The N-Cell Model

4.1 Model specification

Since empirical observations confirm that rainfall produced by convective and frontal mechanisms have different features, and that many climates tend to experience both types, the model can be transformed to generate n types of cells. These are characterised by:

- n random variables for the intensity distributions $\{X_i, i = 1, .., n\}$, with means $\{\mu_{x_i}, i = 1, .., n\}$ and mean square intensities $\{\mu_{x_i^2}, i = 1, .., n\}$
- n duration distributions, with exponential parameters $\{\eta_i, i = 1, .., n\}$
- n probabilities $\{\psi_i, i = 1, .., n\}$ for each cell type

This is a model defined by $4n + 3$ parameters. Because of the constraint $\sum_{i=1}^{n} \psi_i = 1$, there are in effect $4n + 2$ parameters. In practice, it is likely that $n = 2$ will be used.

The parameters are:

$$\{\lambda, \mu_{x_1}, ..., \mu_{x_n}, \mu_{x_1^2}, ..., \mu_{x_n^2}, \eta_1, ...\eta_n, \psi_1, ..., \psi_n, \beta, \gamma\}$$

For this model, the following *mechanistic parameters* can be used:

$$\{\lambda, \mu_{x_1}, ...\mu_{x_n}, \sigma_{x_1}, ..., \sigma_{x_n}, \delta_{c_1}, ..., \delta_{c_n}, \psi_1, ..., \psi_n, \mu_c, \delta_s\}$$

defined by:

Storm arrival rate	λ		(hr^{-1})
Mean cell intensities	μ_{x_i}	for $i = 1, ..., n$	$(mm.hr^{-1})$
Standard deviations of cell intensities	σ_{x_i}	$= \sqrt{\mu_{x_i^2} - \mu_{x_i}^2}$ for $i = 1, ..., n$	$(mm.hr^{-1})$
Mean cell durations	δ_{c_i}	$= 1/\eta_i$ for $i = 1, ..., n$	(hr)
Proportion of each rainfall type	ψ_i	for $i = 1, ..., n$	
Mean number of cells per storm	μ_c	$= 1 + \beta/\gamma$	
Mean duration of storm activity	δ_s	$= 1/\gamma$	(hr)

The original parameters, in terms of which the equations are written, are the following functions of these *mechanistic parameters*:

$$\begin{aligned}
\mu_{x_i^2} &= \sigma_{x_i}^2 + \mu_{x_i}^2 \\
\eta_i &= 1/\delta_{c_i} \\
\beta &= (\mu_c - 1)/\delta_s \\
\gamma &= 1/\delta_s
\end{aligned}$$

4.2 Continuous-time depth properties

We derive the following expressions for the mean and covariance of the continuous-time process:

$$E[Y(t)] = \lambda \mu_c \sum_{i=1}^{n} \frac{\psi_i \mu_{x_i}}{\eta_i} \tag{32}$$

$$\begin{aligned}
c_Y(\tau) = {} & \lambda \mu_c \sum_{i=1}^{n} \frac{\psi_i \mu_{x_i^2} e^{-\eta_i \tau}}{\eta_i} \\
& + \lambda \mu_c \beta e^{-\gamma \tau} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\psi_i \psi_j \mu_{x_i} \mu_{x_j}}{(\eta_i - \gamma)(\eta_j + \gamma)} \\
& + 2\lambda \mu_c \beta \gamma \sum_{j=1}^{n} \frac{\psi_j \mu_{x_j} e^{-\eta_j \tau}}{\gamma^2 - \eta_j^2} \sum_{i=1}^{n} \frac{\psi_i \mu_{x_i}}{\eta_i + \eta_j}
\end{aligned} \tag{33}$$

This can be re-expressed as:

$$c_Y(\tau) = \sum_{i=1}^{n} C_i e^{-\eta_i \tau} + D e^{-\gamma \tau} \tag{34}$$

using the following notation:

$$
\begin{aligned}
C_i &= \lambda\mu_c \left[\frac{\psi_i \mu_{x_i^2}}{\eta_i} + 2\frac{\beta\gamma\psi_i\mu_{x_i}}{\gamma^2-\eta_i^2} \sum_{j=1}^{n} \frac{\psi_i \mu_{x_i}}{\eta_j+\eta_i} \right] \\
D &= \lambda\mu_c\beta \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\psi_i\psi_j\mu_{x_i}\mu_{x_j}}{(\eta_i-\gamma)(\eta_j+\gamma)}
\end{aligned}
$$

4.3 Discrete-time properties

4.3.1 First- and second-order depth moments

The following first- and second-order properties are obtained by integration:

$$M(h) = h\lambda\mu_c \sum_{i=1}^{n} \frac{\psi_i\mu_{x_i}}{\eta_i} \tag{35}$$

$$V(h) = \sum_{i}^{n} \frac{2C_i}{\eta_i^2}(h\eta_i + e^{-\eta_i h} - 1) + \frac{2D}{\gamma^2}(\gamma h + e^{-\gamma h} - 1) \tag{36}$$

$$C(k,h) = \sum_{i}^{n} \frac{C_i e^{-\eta_i(k-1)h}}{\eta_i^2}(1-e^{-\eta_i h})^2 + \frac{De^{-\gamma(k-1)h}}{\gamma^2}(1-e^{-\gamma h})^2 \tag{37}$$

4.3.2 Wet-dry properties

The proportion of dry periods can easily be derived on the basis of the derivation of the same property for BLRPM in Rodriguez-Iturbe et al. (1987). The two terms which depend upon the cell duration parameter η in the exact expression of $P_d(h)$ (see equation (10))are the mean duration of a storm μ_T and the term $G_P^*(0,0)$. Both these terms are functions of the probabilities $p_r(t)$ and $q_r(t)$ defined as:

$$
\begin{aligned}
p_r(t) &= \Pr\{\text{Storm live and r cells active at time t}\} \\
q_r(t) &= \Pr\{\text{Storm terminated and r cells active at time t}\}
\end{aligned}
$$

which satisfy the following differential equations:

$$
\begin{aligned}
dp_r(t)/dt &= -(\beta+\gamma+r\sum_{i=1}^{n}\psi_i\eta_i)\,p_r(t) + (r+1)(\sum_{i=1}^{n}\psi_i\eta_i)\,p_{r+1}(t) + \beta\,p_{r-1}(t) \\
dq_r(t)/dt &= -(r\sum_{i=1}^{n}\psi_i\eta_i)\,q_r(t) + \gamma\,p_r(t) + (r+1)(\sum_{i=1}^{n}\psi_i\eta_i)\,q_{r+1}(t)
\end{aligned}
$$

so that, if we define $\eta = \sum_{i=1}^{n}\psi_i\eta_i$, we have the same differential system as for the RBLPM (see Rodriguez-Iturbe et al., 1987, section 4.2).

We therefore have:

$$P_d(h) = \exp\{-\lambda(h+\mu_T) + \lambda G_P^*(0,0)(\gamma + \beta e^{-(\beta+\gamma)h})/(\beta+\gamma)\} \tag{38}$$

with:

$$\mu_T = \frac{1}{\gamma} + \frac{\gamma}{\eta^2}\int_0^1 v^{-1}\,dv \int_0^1 t^{\frac{\gamma}{\eta}-1}\left[1-(1-vt)e^{-\frac{\beta v(1-t)}{\eta}}\right]dt$$

and

$$G_P^*(z,s) = \eta^{-1}e^{-\frac{\beta(1-z)}{\eta}}\int_0^1 t^{\frac{\gamma+s}{\eta}-1}\left[1-(1-z)t\right]e^{\frac{\beta(1-z)t}{\eta}}\,dt$$

Since these expressions are not easy to compute, the following approximations can be used. They are valid if $\beta << \eta$ and $\gamma << \eta$ (i.e. if there is enough cell overlap and cell durations are much smaller than storm durations):

$$\mu_T \approx \frac{1}{\gamma}\left\{1 + \frac{\gamma(\beta+\gamma/2)}{\eta^2} - \frac{\gamma(5\gamma\beta+\beta^2+2\gamma^2)}{4\eta^3} + \frac{\gamma(4\beta^3+31\beta^2\gamma+99\beta\gamma^2+36\gamma^3)}{72\eta^4}\right\}$$

and

$$G_P^*(z,s) \approx \frac{1}{\gamma}\left\{1 - \frac{\beta+\gamma}{\eta} + \frac{3\beta\gamma+2\gamma^2+\beta^2}{2\eta^2}\right\}$$

as in Onof (1992).

As with previous models, we also require approximations when these conditions upon β and γ are not fulfilled. Defining, as previously, $\kappa = \beta/\eta$ and $\phi = \gamma/\eta$, μ_T and $G_P^*(0,0)$ can be approximated as in equation (11).

4.3.3 Third-order moment

The evaluation of this moment involves very lengthy analytical developments. It will therefore not be computed for the purpose of this project.

4.4 Moving-average properties

The variance function is easily obtained as:

$$\omega(h) = \frac{\sum_i^n \frac{2C_i}{\eta_i^2 h^2}(h\eta_i + e^{-\eta_i h} - 1) + \frac{2D}{\gamma^2 h^2}(\gamma h + e^{-\gamma h} - 1)}{\sum_i^n C_i + D} \tag{39}$$

The scale of fluctuation is therefore given by:

$$\Theta = 2\,\frac{\sum_i^n C_i/\eta_i + D/\gamma}{\sum_i^n C_i + D} \tag{40}$$

5 The Linear Random Parameter Model

5.1 Model specification

An important modification of the original BLRPM was proposed by Rodriguez-Iturbe et al. (1988). The observation that this model does not provide a satisfactory reproduction of the proportion of dry periods suggested introducing a greater diversity of the internal wet-dry structure of storms.

A first way in which this could be done would be to consider introducing a range of m types of storms, such that each storm is characterised by one of η_i, β_i and γ_i, with $i = 1, ..., M$, each type appearing with probability ϵ_1. To preserve the overall structure of storms, β_1 and γ_i would be chosen proportional to η_i according to:

$$\begin{aligned} \beta_i &= \kappa\eta_i \\ \gamma_i &= \phi\eta_i \end{aligned}$$

so that the model parameters would be:

$$\{\lambda, \mu_x, \mu_{x^2}, \eta_1, ...\eta_m, \epsilon_1, ..., \epsilon_m, \kappa, \phi\}$$

thus yielding a $2m + 4$ parameter model (since one ϵ_i can be calculated from the knowledge of the others to satisfy the condition that these parameters add up to 1).

The observation that the above approach amounts to randomising parameter η by assigning it a discrete distribution characterised by the m probabilities $\epsilon_i, /, i = 1, ..., m$, suggests the second approach which is adopted here. This consists in using a continuous distribution to randomise parameter η. A flexible candidate is the Gamma distribution. Thus, η is now sampled for each storm from the distribution $\Gamma(\alpha, \nu)$, while β and γ are proportional according to the relations:

$$\begin{aligned} \beta &= \kappa\eta \\ \gamma &= \phi\eta \end{aligned}$$

As a consequence, we have a 7 parameter model characterised by the following set of parameters:

$$\{\lambda, \mu_x, \mu_{x^2}, \alpha, \nu, \kappa, \phi\}$$

The following *mechanistic parameters* can be used:

$$\{\lambda, \mu_x, \sigma_x, \delta_c, \sigma_c, \mu_c, \delta_s\}$$

defined by:

Storm arrival rate	λ		(hr^{-1})
Mean cell intensity	μ_x		$(mm.hr^{-1})$
Standard deviation of cell intensity	σ_x	$=\sqrt{\mu_{x^2}-\mu_x^2}$	$(mm.hr^{-1})$
Mean cell duration	δ_c	$=\frac{\nu}{\alpha-1}$	(hr)
Inter-Storm standard deviation of cell duration	ϵ_c	$=\frac{\nu}{\sqrt{(\alpha-1)^2(\alpha-2)}}$	(hr)
Mean n^o of cells/storm	μ_c	$=1+\frac{\kappa}{\phi}$	
Mean duration of storm activity	δ_s	$=\frac{\nu}{(\alpha-1)\phi}$	(hr)

The original parameters, which are used in the equations below, are expressed in terms of the mechanistic parameters as follows:

$$\begin{aligned}
\mu_{x^2} &= \sigma_x^2+\mu_x^2 \\
\alpha &= 2+\frac{\delta_c^2}{\epsilon_c^2} \\
\nu &= \delta_c\left(1+\frac{\delta_c^2}{\epsilon_c^2}\right) \\
\kappa &= \frac{\delta_c}{\delta_s}(\mu_c-1) \\
\phi &= \frac{\delta_c}{\delta_s}
\end{aligned}$$

5.2 Continuous-time properties

The expressions below are obtained by derivation as for the BLRPM (see Rodriguez-Iturbe et al., 1988). Note that they can also be obtained by integrating the equivalent expressions for the BLRPM over the parameter η.

$$E[Y(t)] = \lambda\mu_c\mu_x\frac{\nu}{\alpha-1} \tag{41}$$

where $\mu_c = 1+\frac{\kappa}{\phi}$ is the mean number of cells per storm, and

$$c_Y(\tau) = \frac{\lambda\mu_c\nu}{\alpha-1}\left[\left\{\mu_{x^2}+\frac{\kappa\phi}{\phi^2-1}\mu_x^2\right\}\left(\frac{\nu}{\nu+\tau}\right)^{\alpha-1} - \frac{\kappa\mu_x^2}{\phi^2-1}\left(\frac{\nu}{\nu+\phi\tau}\right)^{\alpha-1}\right] \tag{42}$$

Rodriguez-Iturbe et al. (1988) note that for $1<\alpha<2$, the integral

$$\int_0^\infty c_Y(\tau)\,d\tau$$

diverges, indicating asymptotic self-similarity over that range.

5.3 Discrete-time depth properties

5.3.1 First- and second-order moments

The mean depth is (Onof, 1992):

$$M(h) = \lambda h \mu_c \mu_x \frac{\nu}{\alpha - 1} \tag{43}$$

and the variance and co-variance are (Rodriguez-Iturbe et al., 1988):

$$V(h) = 2A_1 \left\{(\alpha-3)h\nu^{2-\alpha} - \nu^{3-\alpha} + (\nu+h)^{3-\alpha}\right\} - 2A_2 \left\{\phi(\alpha-3)h\nu^{2-\alpha} - \nu^{3-\alpha} + (\nu+\phi h)^{3-\alpha}\right\} \tag{44}$$

and

$$\begin{aligned} C(k,h) &= A_1 \left\{[\nu+(k+1)h]^{3-\alpha} - 2[\nu+kh]^{3-\alpha} + [\nu+(k-1)h]^{3-\alpha}\right\} \\ &\quad - A_2 \left\{[\nu+(k+1)\phi h]^{3-\alpha} - 2[\nu+k\phi h]^{3-\alpha} + [\nu+(k-1)\phi h]^{3-\alpha}\right\} \end{aligned} \tag{45}$$

where:

$$\begin{aligned} A_1 &= \frac{\lambda \mu_c \nu^\alpha}{(\alpha-1)(\alpha-2)(\alpha-3)} \left\{\mu_{x^2} + \frac{\kappa\phi\mu_x^2}{\phi^2-1}\right\} \\ A_2 &= \frac{\lambda\mu_c\kappa\mu_x^2\nu^\alpha}{\phi^2(\phi^2-1)(\alpha-1)(\alpha-2)(\alpha-3)} \end{aligned}$$

5.3.2 Wet-dry properties

As shown by Rodriguez-Iturbe et al. (1988), the proportion dry for the Linear Randomised model is obtained by starting with the expression for this property in the BLRPM and taking expectations over the term which is exponentiated. This leads to the following for the proportion of dry periods (Onof, 1992):

$$P_d(h) = \exp\left[-\lambda(h+\mu_T) + \frac{\lambda\nu e^{-\kappa}}{\alpha-1} \times \frac{\phi + \kappa\left(\frac{\nu}{\nu+(\kappa+\phi)h}\right)^{\alpha-1}}{\phi+\kappa} \int_0^1 dt\, t^{\phi-1}(1-t)e^{\kappa t}\right] \tag{46}$$

If $\kappa << 1$ and $\phi << 1$, which means that there is enough cell overlap and cell durations are much smaller than storm durations, then the following approximation can be used (Onof, 1992):

$$\begin{aligned} P_d(h) \approx & \\ & \exp\left\{-\lambda h - \frac{\lambda\nu}{\phi(\alpha-1)}\left[1 + \phi\left(\kappa + \frac{\phi}{2}\right) - \frac{1}{4}\phi\left(5\phi\kappa + \kappa^2 + 2\phi^2\right) + \frac{1}{72}\phi\left(4\kappa^3 + 31\kappa^2\phi + 99\kappa\phi^2 + 36\phi^3\right)\right]\right. \\ & \left. + \frac{\lambda\nu}{\phi(\alpha-1)}\left(1 - \kappa - \phi + \frac{3}{2}\kappa\phi + \phi^2 + \frac{1}{2}\kappa^2\right)\left[\frac{\phi}{\phi+\kappa} + \frac{\kappa}{\phi+\kappa}\left(\frac{\nu}{\nu+(\kappa+\phi)h}\right)^{\alpha-1}\right]\right\} \end{aligned} \tag{47}$$

However, if these requirements on small values of κ and ϕ are not fulfilled, we can approximate the proportion dry as follows:

$$P_d(h) \approx \exp\left[-\lambda(h+\hat{\mu_{TM}}) + \frac{\lambda\nu e^{-\kappa}}{\alpha-1} \times \frac{\phi+\kappa\left(\frac{\nu}{\nu+(\kappa+\phi)h}\right)^{\alpha-1}}{\phi+\kappa}\hat{I}_{M'}\right] \tag{48}$$

where

$$\hat{\mu_{TM}} = \frac{\nu}{\alpha-1}\left(1+\phi\sum_{j=1}^{M}\frac{(-\kappa)^{j-1}(\kappa-j^2-j)}{j(j+1)!}B(j+1,\phi)+\phi^{-1}\right)$$

and

$$\hat{I}_{M'} = \sum_{j=0}^{M'}\frac{\kappa^j}{j!}B(j+\phi,2)+\frac{\delta_{M'}(\kappa)}{(M'+\phi+1)(M'+\phi+2)}$$

with

$$\delta_{M'}(\kappa) = e^{\kappa}-\sum_{j=0}^{M'}\frac{\kappa^j}{j!}$$

where M and M' are to be chosen large enough for a good approximation. Upper bounds for the errors involved and numerical investigations into their values are presented, together with the derivations for the approximations are found in Chandler (2003) and Onof (2003). These two notes are reproduced in appendix E.

5.3.3 Third-order moment

The third-order moment is best obtained by integration of the corresponding expression for the BLRPM multiplied by the density function of the gamma distribution $\Gamma(\alpha,\nu)$. The resulting expression can however not be integrated in a closed form. As a consequence, a numerical integration is required.

A note about this numerical integration is useful. For the integral:

$$E\left[\left(Y_i^{(h)}\right)^3\right] = \int_0^\infty f(\eta)\,d\eta \tag{49}$$

has the particularity that $f(\eta) \propto \frac{1}{\eta^4}$ in the neighbourhood of 0. $f(\eta)$ is therefore not integrable at 0. However, practically, values of η close to 0 are not physically representative (and their probability is very small). This would correspond to a storm with very long cells only, which is hardly appropriate for the representation of fine-scale rainfall. It is therefore realistic to neglect small values of η. Calculations with a lower bound of 10^{-7} were found to give results in line with the simulations. The upper bound does not have to be chosen as particularly large since the integrand $f(\eta)$ decreases very quickly. Thus we can approximate the integral as follows:

$$E\left[\left(Y_i^{(h)}\right)^3\right] = \int_{10^{-7}}^{100} f(\eta)\,d\eta \tag{50}$$

and calculate it using Simpson's rule.

5.4 Moving-average properties

The variance function is easily obtained as:

$$\omega(h) = \frac{2A_1\left\{(\alpha-3)h\nu^{2-\alpha} - \nu^{3-\alpha} + (\nu+h)^{3-\alpha}\right\} - 2A_2\left\{\phi(\alpha-3)h\nu^{2-\alpha} - \nu^{3-\alpha} + (\nu+\phi h)^{3-\alpha}\right\}}{\frac{h^2(\alpha-2)(\alpha-3)}{\nu^{\alpha-1}}(A_1 - \phi^2 A_2)} \tag{51}$$

The scale of fluctuation will then depend upon the value of α. If $\alpha < 2$, the scale is infinite, in line with the observation of asymptotic self-similarity. Else, if $\alpha > 2$ we find:

$$\Theta = \frac{2\nu}{\alpha-2} \times \frac{A_1 - A_2\phi}{A_1 - A_2\phi^2} \tag{52}$$

6 The Quadratic Random Parameter Model

6.1 Model specification

The observation that the LR model is liable to underestimate extreme rainfall depths suggests an alteration of the mechanism by which cells are produced in a storm. Rather than have the random cell arrival rate β depend linearly upon the cell duration parameter η, this dependence could include a power function. So as to facilitate the computations, let us assume a quadratic dependence. We therefore assume that:

$$\beta = \kappa_1\eta + \kappa_2\eta^2 \tag{53}$$

The model therefore has 8 parameters:

$$\{\lambda, \mu_x, \mu_{x^2}, \alpha, \nu, \kappa_1, \kappa_2, \phi\}$$

The following *mechanistic parameters* can be used:

$$\{\lambda, \mu_x, \sigma_x, \delta_c, \epsilon_c, \mu_c, \sigma_c, \delta_s\}$$

defined by:

Storm arrival rate	λ		(hr^{-1})
Mean cell intensity	μ_x		$(mm.hr^{-1})$
Std. deviation of cell intensity	σ_x	$= \sqrt{\mu_{x^2} - \mu_x^2}$	$(mm.hr^{-1})$
Mean cell duration	δ_c	$= \frac{\nu}{\alpha-1}$	(hr)
I.S. std. deviation of cell duration	ϵ_c	$= \frac{\nu}{\sqrt{(\alpha-1)^2(\alpha-2)}}$	(hr)
Mean n^o of cells/storm	μ_c	$= 1 + \frac{\kappa_1}{\phi} + \frac{\kappa_2\alpha}{\phi\nu}$	
Std. deviation of n^o of cells/storm	σ_c	$= \sqrt{\frac{2}{\phi^2}\left[(\kappa_1^2 + \kappa_1\phi) + (2\kappa_1\kappa_2 + \kappa_2\phi)\frac{\alpha}{\nu} + \kappa_2^2\frac{\alpha(\alpha+1)}{\nu^2}]\right]}$	
Mean duration of storm activity	δ_s	$= \frac{\nu}{(\alpha-1)\phi}$	(hr)

where 'I.S.' stands for 'Inter-Storm' and the calculation of the standard deviation of the number of cells per storm is shown in appendix F.

The equations below are given in terms of the original parameters. These can be re-expressed in terms of the mechanistic parameters using:

$$\begin{aligned}
\mu_{x^2} &= \sigma_x^2 + \mu_x^2 \\
\alpha &= 2 + \frac{\delta_c^2}{\epsilon_c^2} \\
\nu &= \delta_c\left(1 + \frac{\delta_c^2}{\epsilon_c^2}\right) \\
\kappa_1 &= \phi\left(\mu_c - 1 - \sqrt{\frac{(1-\mu_c)^3(\alpha+1)}{\alpha} + \frac{\sigma_c^2\alpha}{2}}\right) \\
\kappa_2 &= \frac{\nu\phi}{\alpha}\sqrt{\frac{(1-\mu_c)^3(\alpha+1)}{\alpha} + \frac{\sigma_c^2\alpha}{2}} \\
\phi &= \frac{\delta_c}{\delta_s}
\end{aligned}$$

6.2 Continuous-time properties

The expressions below are obtained by integrating the equivalent expressions for the BLRPM over the parameter η. The same integrals of functions of η are used which were computed for the LR model. This yields:

$$E[Y(t)] = \lambda\mu_x\left[\frac{\nu}{\alpha-1}\left(1 + \frac{\kappa_1}{\phi}\right) + \frac{\kappa_2}{\phi}\right] \tag{54}$$

and

$$
\begin{aligned}
c_Y(\tau) &= \frac{\lambda\left(1+\frac{\kappa_1}{\phi}\right)\nu}{\alpha-1}\left[\mu_{x^2}+\frac{\kappa_1\phi\mu_x^2}{\phi^2-1}\right]\left(\frac{\nu}{\nu+\tau}\right)^{\alpha-1} \\
&+\lambda\left[\frac{\left(1+\frac{\kappa_1}{\phi}\right)\kappa_2\phi\mu_x^2}{\phi^2-1}+\frac{\kappa_2}{\phi}\left(\mu_{x^2}+\frac{\kappa_1\phi\mu_x^2}{\phi^2-1}\right)\right]\left(\frac{\nu}{\nu+\tau}\right)^{\alpha} \\
&+\frac{\lambda\kappa_2^2\mu_x^2\alpha}{(\phi_2^2-1)\nu}\left(\frac{\nu}{\nu+\tau}\right)^{\alpha+1}-\frac{\lambda\mu_x^2}{\phi^2-1}\left[\left(1+\frac{\kappa_1}{\phi}\right)\left(\frac{\kappa_1\nu}{\alpha-1}+\kappa_2\right)+\frac{\kappa_2}{\phi}\left(\kappa_1+\kappa_2\frac{\alpha}{\nu}\right)\right]e^{-\gamma\tau}
\end{aligned}
\tag{55}
$$

We note that for $-1<\alpha<2$, the integral

$$\int_0^\infty c_Y(\tau)\,d\tau$$

diverges, indicating asymptotic self-similarity when $\alpha<2$.

6.3 Discrete-time properties

These properties can be obtained by using the general relations presented in equations (4) applied to the equations (54) and (55). But since the integration over the values of η can be performed last, they can also be obtained by directly integrating the properties of the BLRPM.

6.3.1 First- and second-order moments of depths

For these properties, the derivation proceeds by integrating the expressions in equations (7), (8) and (9). This yields:

$$M(h)=\lambda h\mu_x\left[\frac{\nu}{\alpha-1}\left(1+\frac{\kappa_1}{\phi}\right)+\frac{\kappa_2}{\phi}\right] \tag{56}$$

and

$$
\begin{aligned}
V(h) &= \frac{2\lambda h\kappa_2^2\mu_x^2}{\phi^2}+\frac{2\lambda\nu\kappa_2}{(\alpha-1)\phi}\left[h\mu_{x^2}+h\mu_x^2\left(1+\frac{2\kappa_1}{\phi}\right)+\frac{\mu_x^2\kappa_2}{\phi^2(\phi^2-1)}\xi(1,\phi)-\frac{\kappa_2\mu_x^2\phi}{\phi^2-1}\xi(1,1)\right] \\
&+\frac{2\lambda\nu^2}{(\alpha-1)(\alpha-2)}\left[\left(1+\frac{\kappa_1}{\phi}\right)h\left(\mu_{x^2}+\mu_x^2\frac{\kappa_1}{\phi}\right)+\frac{\kappa_2\mu_x^2}{\phi^2(\phi^2-1)}\left(1+\frac{2\kappa_1}{\phi}\right)\xi(2,\phi)\right. \\
&\left.-\kappa_2\xi(2,1)\left(\frac{\mu_{x^2}}{\phi}+\frac{\mu_x^2\phi}{\phi^2-1}\left(1+\frac{2\kappa_1}{\phi}\right)\right)\right] \\
&+\frac{2\lambda\nu^3\left(1+\frac{\kappa_1}{\phi}\right)}{(\alpha-1)(\alpha-2)(\alpha-3)}\left[\frac{\mu_x^2\kappa_1}{\phi^2(\phi^2-1)}\xi(3,\phi)-\left(\mu_{x^2}+\frac{\mu_x^2\phi}{\phi^2-1}\kappa_1\right)\xi(3,1)\right]
\end{aligned}
\tag{57}
$$

where:

$$\xi(k,l) = 1 - \left(\frac{\nu}{\nu + lh}\right)^{\alpha-k}$$

For the covariance, let us define:

$$\begin{aligned}
A_1(x) &= \frac{\nu}{\alpha-1}\left[\left(\frac{\nu}{\nu+x(k-1)h}\right)^{\alpha-1} - 2\left(\frac{\nu}{\nu+xkh}\right)^{\alpha-1} + \left(\frac{\nu}{\nu+x(k+1)h}\right)^{\alpha-1}\right] \\
A_2(x) &= \frac{\nu^2}{(\alpha-1)(\alpha-2)}\left[\left(\frac{\nu}{\nu+x(k-1)h}\right)^{\alpha-2} - 2\left(\frac{\nu}{\nu+xkh}\right)^{\alpha-2} + \left(\frac{\nu}{\nu+x(k+1)h}\right)^{\alpha-2}\right] \\
A_3(x) &= \frac{\nu^3}{(\alpha-1)(\alpha-2)(\alpha-3)}\left[\left(\frac{\nu}{\nu+x(k-1)h}\right)^{\alpha-3} - 2\left(\frac{\nu}{\nu+xkh}\right)^{\alpha-3} + \left(\frac{\nu}{\nu+x(k+1)h}\right)^{\alpha-3}\right]
\end{aligned}$$

In terms of these expressions, the covariance is derived as:

$$\begin{aligned}
C(k,h) &= \lambda\left(1+\frac{\kappa_1}{\phi}\right)\left[\mu_{x^2} + \frac{\kappa_1\phi\mu_x^2}{\phi^2-1}\right] A_3(1) \\
&\quad + \lambda\kappa_2\left[\frac{\mu_{x^2}}{\phi} + \frac{\phi\left(1+\frac{2\kappa_1}{\phi}\right)}{\phi^2-1}\mu_x^2\right] A_2(1) + \frac{\lambda\kappa_2^2\mu_x^2}{\phi^2-1}A_1(1) \\
&\quad - \frac{\lambda\mu_x^2}{\phi^2(\phi^2-1)}\left[\left(1+\frac{\kappa_1}{\phi}\right)\kappa_1 A_3(\phi) + \left(1+2\frac{\kappa_1}{\phi}\right)\kappa_2 A_2(\phi) + \frac{\kappa_2^2}{\phi}A_1(\phi)\right] \qquad (58)
\end{aligned}$$

6.3.2 Wet-dry properties

For the proportion dry, a closed expression cannot be obtained since the following integrals need to be evaluated:

$$\begin{aligned}
I_1(t) &= \int_0^\infty \frac{\exp\left[-(\kappa_1+\kappa_2 u)(1-t) - \nu u\right] u^{\alpha-2}\nu^\alpha}{(\kappa_1+\kappa_2 u+\phi)\Gamma(\alpha)}\, du \\
I_2(t) &= \int_0^\infty \frac{\exp\left[-(\kappa_1+\kappa_2 u)(1-t) - (\kappa_1 u+\kappa_2 u^2+\phi u)h - \nu u\right] u^{\alpha-2}\nu^\alpha}{(\kappa_1+\kappa_2 u+\phi)\Gamma(\alpha)}\, du \\
I_3(t) &= \int_0^\infty \frac{\exp\left[-(\kappa_1+\kappa_2 u)(1-t) - (\kappa_1 u+\kappa_2 u^2+\phi u)h - \nu u\right] u^{\alpha-1}\nu^\alpha}{(\kappa_1+\kappa_2 u+\phi)\Gamma(\alpha)}\, du
\end{aligned}$$

In terms of these integrals, and with the following expression for the mean storm duration:

$$\mu_T = \phi\int_0^1 du \int_0^1 dt\, u^{-1}t^{\phi-1}\frac{\nu}{\alpha-1}\left[1-(1-ut)e^{-\kappa_1 u(1-t)}\left(\frac{\nu}{\nu+\kappa_2 u(1-t)}\right)^{\alpha-1}\right] + \frac{\phi\nu}{\alpha-1}$$

the proportion dry is:

$$P_d(h) = \exp\left[-\lambda(h+\mu_T) + \int_0^1 dt\, t^{\phi-1}(1-t)\lambda\left(\phi I_1(t) + \kappa_1 I_2(t) + \kappa_2 I_3(t)\right)\right] \qquad (59)$$

6.3.3 Third-order moment of depths

For the third-order moment of the Quadratic Randomised model, a similar numerical computation is required as in the case of the Linear Randomised model.In other words, the expression for the BLRPM multiplied by the density function of the gamma distribution $\Gamma(\alpha, \nu)$. The final expression is then integrated numerically using Simpson's rule. Because of the non-convergence of the integral in the neighbourhood of 0, the domain of integration is not $[0, \infty)$, but can, in practice, for instance be taken as $[10^{-7}, 100]$.

6.4 Moving-average properties

The variance function is given by:

$$\omega(h) = \frac{V(h)}{c_Y(0)h^2} \tag{60}$$

where the numerator and denominator are given in equations (57) and (55).

This leads to the following expression for the scale of fluctuation. If $\alpha < 2$, it is infinite, which reflects the asymptotic self-similarity. If $\alpha > 2$ we find::

$$\Theta = 2\frac{\frac{\lambda\kappa_2^2\mu_x^2}{\phi^2} + \frac{\lambda\nu\kappa_2}{(\alpha-1)\phi}\left[\mu_{x^2} + \mu_x^2\left(1 + \frac{2\kappa_1}{\phi}\right)\right] + \frac{\lambda\nu^2}{(\alpha-1)(\alpha-2)}\left[\left(1 + \frac{\kappa_1}{\phi}\right)\left(\mu_{x^2} + \mu_x^2\frac{\kappa_1}{\phi}\right)\right]}{\lambda\mu_x^2\left(1 + \frac{\kappa_1}{\phi}\right)\frac{\kappa_1\nu + \kappa_2(\alpha-1)}{(\alpha-1)(\phi+1)} + \frac{\lambda\kappa_2\mu_x^2}{\phi(\phi+1)}\left(\frac{\kappa_2\alpha}{\nu} + \kappa_1\right) + \lambda\mu_{x^2}\left[\frac{\nu}{\alpha-1}\left(1 + \frac{\kappa_1}{\phi}\right) + \frac{\kappa_2}{\phi}\right]} \tag{61}$$

A Appendix: Cell intensity distributions

This appendix includes density and cumulative distribution functions for the three distributions considered for the cell intensity. Also included are the third-order moments.

A.1 Exponential distribution

The exponential distribution is a one-parameter distribution. The density function is defined by:

$$f_X(x) = \frac{1}{\mu_x} e^{-x/\mu_x} \text{ for } x > 0 \tag{62}$$

and the cumulative distribution function (cdf) by:

$$F_X(x) = 1 - e^{-x/\mu_x} \text{ for } x > 0 \tag{63}$$

The following relations hold for the exponential distribution:

$$\mu_{x^2} = 2\mu_x^2 \text{ and } \mu_{x^3} = 3\mu_x^3 \tag{64}$$

A.2 Gamma distribution

The Gamma distribution is a two-parameter distribution denoted $\Gamma(\psi, \sigma)$, where ψ is the shape parameter and σ the scale parameter. The density function is given by:

$$f_X(x) = \frac{\sigma^\psi x^{\psi-1} e^{-\sigma x}}{\Gamma(\psi)} \text{ for } x > 0 \tag{65}$$

and the cdf is not available in a closed form:

$$F_X(x) = \int_0^x \frac{\sigma^\psi x^{\psi-1} e^{-\sigma t}}{\Gamma(\psi)} dt \text{ for } x > 0 \tag{66}$$

where:

$$\psi = \frac{\mu_x^2}{\mu_{x^2} - \mu_x^2} \text{ and } \sigma = \frac{\mu_x}{\mu_{x^2} - \mu_x^2} \tag{67}$$

The following relation holds for the Gamma distribution:

$$\mu_{x^3} = \frac{(2\mu_{x^2} - \mu_x^2)\mu_{x^2}}{\mu_x} \tag{68}$$

A.3 General Pareto distribution

The Pareto distribution is also a two parameter distribution, denoted $P(\psi, \sigma)$, where again ψ is the shape parameter and σ the scale parameter. The density function is given by:

$$f_X(x) = \frac{\psi \sigma^\psi}{x^{\psi+1}} \text{ for } x > \sigma \tag{69}$$

and the cdf by:

$$F_X(x) = 1 - (\frac{\sigma}{x})^{\psi} \text{ for } x > \sigma \tag{70}$$

where:

$$\psi = 1 + \sqrt{\frac{\mu_{x^2}}{\mu_{x^2} - \mu_x^2}} \text{ and } \sigma = \frac{\mu_x(\psi - 1)}{\psi} \tag{71}$$

The third-order moment is given by:

$$\mu_{x^3} = \frac{(\psi - 1)(\psi - 2)}{\psi(\psi - 3)} \mu_{x^2} \mu_x \tag{72}$$

Note that, in the above, the moment of order p is only defined for $p < \alpha$.

B Appendix: 3rd-order continuous-time BLRPM properties

The computation of the integrals described in the text require that the following expected values of differential products be computed:

$$\begin{aligned}&E\left[dN(x-u)\,dN(x+y-v)\,dN(x+z-w)\right]=\\&\quad\left\{(\lambda\mu_c)^3+\lambda\mu_c\beta^2e^{-\gamma(M-m)}\right.\\&\quad\left.+(\lambda\mu_c)^2\beta\left[e^{-\gamma|z-w-y+v|}+e^{-\gamma|z-w+u|}+e^{-\gamma|y-v+u|}\right]\right\}du\,dv\,dw\end{aligned}$$

and

$$\begin{aligned}E\left[dN(x-u)\,dN(x+y-v)\right] &= cov\left[dN(x-u)\,dN(x+y-v)\right]+E\left[dN(x-u)\right]\,E\left[dN(x+y-v)\right]\\&= \lambda\mu_c\left\{\delta(y-v+u)+h(y-v+u)-\lambda\mu_c\right\}\,du\,dv+(\lambda\mu_c)^2\,du\,dv\\&= \lambda\mu_c h(y-v+u)\,du\,dv\\&= \lambda\mu_c\left(\lambda\mu_c+\beta e^{-\gamma|y-v+u|}\right)\,du\,dv\end{aligned}$$

where: $m=\min(x-u,x+y-v,x+z-w)$ and $M=\max(x-u,x+y-v,x+z-w)$.

The computation of the integral then involves examining $M-m$ which is:

$$M-m=\max(y-v+u,z-w+u,z-w-y+v,-y+v-u,-z+w-u,-z+w+y-v)$$

for the different intervals of integration.

We find the following:

$$\begin{array}{rcl}\text{v}\in[0,y+u] & \text{and } w\in[0,z-y+v] & \Rightarrow M-m=z-w+u\\ \text{v}\in[0,y+u] & \text{and } w\in[z-y+v,z+u] & \Rightarrow M-m=y-v+u\\ \text{v}\in[0,y+u] & \text{and } w\in[z+u,\infty) & \Rightarrow M-m=-z+w+y-v\\ \text{v}\in[y+u,\infty] & \text{and } w\in[0,z+u] & \Rightarrow M-m=z-w-y+v\\ \text{v}\in[y+u,\infty] & \text{and } w\in[z+u,z-y+v] & \Rightarrow M-m=-y+v-u\\ \text{v}\in[y+u,\infty] & \text{and } w\in[z-y+v,\infty) & \Rightarrow M-m=-z+w-u\end{array}$$

Lengthy but standard computations of integrals of exponential functions then lead to the following expression for $y\le 0$ and $z\le y$:

$$\begin{aligned}E[Y(x)Y(x+y)Y(x+z)] &= {\mu_x}^3\lambda\,\mu_c\left(\frac{{\mu_c}^2\lambda^2}{\eta^3}+2\,\frac{e^{-\eta\,z}e^{-\eta\,y}\beta^2\gamma^2}{4\,\eta^5-5\,\eta^3\gamma^2+\eta\,\gamma^4}-\frac{e^{-\gamma\,z}e^{-\eta\,y}\beta^2\gamma}{\eta\,(\eta-\gamma)\,(\eta+\gamma)\,(2\,\eta+\gamma)}\right.\\&+\frac{e^{-\eta\,y}\beta\,\gamma\,\lambda\,\mu_c}{-\eta^4+\eta^2\gamma^2}+\frac{e^{-\eta\,z}e^{\eta\,y}\beta\,\gamma\,\lambda\,\mu_c}{-\eta^4+\eta^2\gamma^2}+\frac{e^{-\eta\,z}e^{(\eta-\gamma)y}\beta^2\gamma}{-2\,\eta^4+\eta^3\gamma+2\,\eta^2\gamma^2-\eta\,\gamma^3}\\&-\frac{e^{-\eta\,z}e^{-\gamma\,y}\beta^2\gamma}{\eta\,(\eta-\gamma)\,(\eta+\gamma)\,(2\,\eta+\gamma)}-\frac{e^{-\gamma\,z}e^{\gamma\,y}\lambda\,\mu_c\,\beta}{(-\eta+\gamma)\,(\eta+\gamma)\,\eta}\end{aligned}$$

$$
\begin{aligned}
&+\frac{e^{-\gamma\,z}\beta\,\left(\beta+\lambda\,\mu_c\right)}{\eta^3-\eta\,\gamma^2}+\frac{e^{-\eta\,z}\beta\,\gamma\,\lambda\,\mu_c}{-\eta^4+\eta^2\gamma^2}-\frac{e^{-\gamma\,y}\lambda\,\mu_c\,\beta}{\left(-\eta+\gamma\right)\left(\eta+\gamma\right)\eta}\Bigg)\\
&+\mu_x\,\mu_{x^2}\,\lambda\,\mu_c\,\Bigg(-2\,\frac{e^{-\eta\,z}e^{-\eta\,y}\beta\,\gamma}{\eta\,\left(\eta-\gamma\right)\left(\eta+\gamma\right)}+\frac{e^{-\gamma\,z}e^{-\eta\,y}\beta}{\left(\eta-\gamma\right)\left(\eta+\gamma\right)}+\frac{e^{-\eta\,y}\lambda\,\mu_c}{\eta^2}\\
&+\frac{e^{-\eta\,z}e^{\eta\,y}\lambda\,\mu_c}{\eta^2}+\frac{e^{-\eta\,z}e^{(\eta-\gamma)y}\beta}{\left(\eta-\gamma\right)\left(\eta+\gamma\right)}+\frac{e^{-\eta\,z}e^{-\gamma\,y}\beta}{\left(\eta-\gamma\right)\left(\eta+\gamma\right)}\\
&+\frac{e^{-\eta\,z}\left(\beta\,\eta\,\gamma+\left(\gamma^2-\eta^2\right)\lambda\,\mu_c\right)}{-\eta^4+\eta^2\gamma^2}\Bigg)+\frac{\mu_{x^3}\,\lambda\,\mu_c\,e^{-\eta\,z}}{\eta}
\end{aligned}
\tag{73}
$$

C Appendix: 3rd-order moment of BLRPM aggregated process

This is the final expression of the third-order moment of rainfall depths at a time-scale h hours. If we define the following:

$$\begin{aligned}
T1 = {} & 12\,\gamma^7\mu_x{}^3\beta^2e^{h(\eta+\gamma)}-48\,\mu_x{}^3e^{2\,h\eta}\eta^7\beta^2+72\,\gamma^7e^{h(2\,\eta+\gamma)}\mu_{x3}\,\eta^2+48\,\gamma\,\mu_x\,\mu_{x2}\,e^{h(2\,\eta+\gamma)}\beta\,\eta^7+24\,\gamma\,h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\eta^7\mu_c- \\
& 36\,\mu_x\,\mu_{x2}\,\gamma^7h^2e^{h(2\,\eta+\gamma)}\lambda\,\mu_c\,\eta^3-24\,\gamma\,h\mu_x{}^3e^{h(2\,\eta+\gamma)}\eta^7\beta^2+24\,\mu_x\,\mu_{x2}\,\gamma^4he^{h(\eta+\gamma)}\beta\,\eta^5+24\,\mu_x\,\mu_{x2}\,\gamma^2e^{h(2\,\eta+\gamma)}\beta\,\eta^6- \\
& 36\,\mu_x\,\mu_{x2}\,\gamma^3e^{h(2\,\eta+\gamma)}\beta\,\eta^5-6\,\gamma^8h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\mu_c-30\,\gamma^3h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\eta^5\mu_c-72\,\mu_x\,\mu_{x2}\,\gamma^6he^{h(2\,\eta+\gamma)}\beta\,\eta^3+ \\
& 6\,\gamma^5h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^3-54\,\mu_x\,\mu_{x2}\,\gamma^5he^{h(2\,\eta+\gamma)}\lambda\,\mu_c\,\eta^4-84\,\gamma^2\mu_x{}^3e^{h(2\,\eta+\gamma)}\eta^5\beta^2+30\,\gamma^6h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^2- \\
& 36\,\gamma^5h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta^2\eta^3+24\,\mu_x\,\mu_{x2}\,\gamma^3he^{h(2\,\eta+\gamma)}\lambda\,\eta^6\mu_c+54\,\gamma^3h\mu_x{}^3e^{h(2\,\eta+\gamma)}\eta^5\beta^2+36\,\mu_x\,\mu_{x2}\,\gamma^7he^{h(2\,\eta+\gamma)}\lambda\,\mu_c\,\eta^2+ \\
& 6\,\mu_x\,\mu_{x2}\,\gamma^5e^{h(2\,\eta+\gamma)}\beta\,\eta^3+6\,\gamma^7h\mu_x{}^3e^{h(2\,\eta+\gamma)}\eta\,\beta^2-24\,\mu_x\,\mu_{x2}\,\gamma^2e^{h(\eta+\gamma)}\beta\,\eta^6+117\,\mu_x\,\mu_{x2}\,\gamma^6e^{h(2\,\eta+\gamma)}\beta\,\eta^2- \\
& 18\,\gamma^4\mu_x{}^3e^{h(\eta+\gamma)}\beta^2\eta^3-30\,\gamma^6h\mu_x{}^3e^{h(\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^2+54\,\gamma^5e^{h(2\,\eta+\gamma)}h\mu_{x3}\,\eta^5+39\,\gamma^5\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta^2\eta^2- \\
& 36\,\gamma^7e^{h(2\,\eta+\gamma)}h\mu_{x3}\,\eta^3-24\,\gamma^3e^{h(2\,\eta+\gamma)}h\mu_{x3}\,\eta^7-12\,\gamma^9e^{h(2\,\eta+\gamma)}\mu_{x3}+6\,\eta\,\gamma^9h\mu_{x3}\,e^{h(\eta+\gamma)}
\end{aligned}$$

$$\begin{aligned}
T2 = {} & -24\,\gamma^4h\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^4+6\,\mu_x\,\mu_{x2}\,\gamma^4e^{2\,h\eta}\beta\,\eta^4-30\,\mu_x\,\mu_{x2}\,\gamma^6he^{h(\eta+\gamma)}\beta\,\eta^3-48\,\mu_x\,\mu_{x2}\,\gamma^2he^{h(2\,\eta+\gamma)}\beta\,\eta^7- \\
& 48\,\gamma\,\mu_x\,\mu_{x2}\,e^{2\,h\eta}\beta\,\eta^7-24\,\gamma\,h\mu_x{}^3e^{2\,h\eta}\eta^7\beta^2+30\,\gamma^3h\mu_x{}^3e^{2\,h\eta}\beta\,\lambda\,\eta^5\mu_c+54\,\gamma^4\mu_x{}^3h^2e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\eta^5\mu_c+ \\
& 6\,\gamma^5\mu_x{}^3e^{2\,h\eta}\beta^2\eta^2+6\,\mu_x\,\mu_{x2}\,\gamma^8he^{h(\eta+\gamma)}\beta\,\eta-36\,\mu_x\,\mu_{x2}\,\gamma^7he^{h(\eta+\gamma)}\lambda\,\mu_c\,\eta^2-138\,\mu_x\,\mu_{x2}\,\gamma^4e^{h(2\,\eta+\gamma)}\beta\,\eta^4+ \\
& 6\,\mu_x\,\mu_{x2}\,\gamma^9he^{h(\eta+\gamma)}\lambda\,\mu_c+48\,\mu_x{}^3e^{h(2\,\eta+\gamma)}\eta^7\beta^2+30\,\gamma^3h\mu_x{}^3e^{2\,h\eta}\eta^5\beta^2+54\,\mu_x\,\mu_{x2}\,\gamma^5h^2e^{h(2\,\eta+\gamma)}\lambda\,\eta^5\mu_c- \\
& 24\,\mu_x\,\mu_{x2}\,\gamma^2e^{2\,h\eta}\beta\,\eta^6+9\,\gamma^5\mu_x{}^3h^3e^{h(2\,\eta+\gamma)}\lambda^2\eta^5\mu_c{}^2+36\,\mu_x\,\mu_{x2}\,\gamma^3e^{2\,h\eta}\beta\,\eta^5+24\,\mu_x\,\mu_{x2}\,\gamma^3e^{h(\eta+\gamma)}\beta\,\eta^5+ \\
& 6\,\mu_x\,\mu_{x2}\,\gamma^9h^2e^{h(2\,\eta+\gamma)}\lambda\,\eta\,\mu_c+24\,\gamma^4h\mu_x{}^3e^{h(\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^4-24\,\mu_x\,\mu_{x2}\,\gamma^3he^{h(\eta+\gamma)}\lambda\,\eta^6\mu_c-132\,\mu_x\,\mu_{x2}\,\gamma^6e^{h(\eta+\gamma)}\beta\,\eta^2- \\
& 6\,\mu_x\,\mu_{x2}\,\gamma^5e^{2\,h\eta}\beta\,\eta^3-6\,\gamma^5h\mu_x{}^3e^{2\,h\eta}\beta\,\lambda\,\mu_c\,\eta^3+54\,\mu_x\,\mu_{x2}\,\gamma^5he^{h(\eta+\gamma)}\lambda\,\mu_c\,\eta^4-24\,\gamma\,h\mu_x{}^3e^{2\,h\eta}\beta\,\lambda\,\eta^7\mu_c+ \\
& 150\,\mu_x\,\mu_{x2}\,\gamma^4e^{h(\eta+\gamma)}\beta\,\eta^4-42\,\gamma^5\mu_x{}^3e^{h(\eta+\gamma)}\beta^2\eta^2-6\,\gamma^7\mu_x{}^3h^3e^{h(2\,\eta+\gamma)}\lambda^2\mu_c{}^2\eta^3+\gamma^9\mu_x{}^3h^3e^{h(2\,\eta+\gamma)}\lambda^2\mu_c{}^2\eta+ \\
& 6\,\gamma^8\mu_x{}^3h^2e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\eta\,\mu_c-6\,\gamma^5h\mu_x{}^3e^{2\,h\eta}\beta^2\eta^3+12\,\mu_x\,\mu_{x2}\,\gamma^8he^{h(2\,\eta+\gamma)}\beta\,\eta-6\,\mu_x\,\mu_{x2}\,\gamma^9he^{h(2\,\eta+\gamma)}\lambda\,\mu_c- \\
& 6\,\mu_x\,\mu_{x2}\,\gamma^5e^{h(\eta+\gamma)}\beta\,\eta^3-24\,\eta^5\mu_{x2}\,\mu_x\,\beta\,\gamma^3e^{h\eta}-12\,\eta^4\mu_{x2}\,\mu_x\,\gamma^4\beta\,e^{h\gamma}-6\,\eta^4\mu_{x2}\,\mu_x\,\gamma^4\beta\,e^{h\eta}+6\,\eta^3\gamma^5\mu_{x2}\,\mu_x\,\beta\,e^{h\eta}- \\
& 3\,\mu_{x2}\,\mu_x\,\gamma^8\beta\,e^{h\gamma}+24\,\eta^6\mu_{x2}\,\mu_x\,\beta\,\gamma^2e^{h\eta}+15\,\eta^2\mu_{x2}\,\mu_x\,\gamma^6\beta\,e^{h\gamma}-3\,\gamma^7\mu_x{}^3\beta^2e^{h\gamma}
\end{aligned}$$

$$\begin{aligned}
T3 = {} & 18\,\eta^3\gamma^4\mu_x{}^3\beta^2e^{h\eta}-12\,\eta^4\gamma^3\mu_x{}^3\beta^2e^{h\eta}-6\,\eta^2\gamma^5\mu_x{}^3\beta^2e^{h\eta}+3\,\eta^2\gamma^5\mu_x{}^3\beta^2e^{h\gamma}-9\,\gamma^7e^{h(2\,\eta+\gamma)}\mu_x{}^3\beta^2+ \\
& 108\,\eta^4\gamma^5\mu_{x3}\,e^{h(\eta+\gamma)}+48\,\gamma^3e^{h(2\,\eta+\gamma)}\mu_{x3}\,\eta^6-72\,\eta^2\gamma^7\mu_{x3}\,e^{h(\eta+\gamma)}-48\,\eta^6\mu_{x3}\,\gamma^3e^{h(\eta+\gamma)}+84\,\gamma^2\mu_x{}^3e^{2\,h\eta}\eta^5\beta^2+ \\
& 18\,\gamma^4\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta^2\eta^3+24\,\mu_{x2}\,\mu_x\,\gamma^8\beta\,e^{h(\eta+\gamma)}+54\,\eta^5\gamma^5h\mu_{x3}\,e^{h(\eta+\gamma)}-24\,\eta^7h\mu_{x3}\,\gamma^3e^{h(\eta+\gamma)}-36\,\eta^3\gamma^7h\mu_{x3}\,e^{h(\eta+\gamma)}- \\
& 21\,\gamma^8e^{h(2\,\eta+\gamma)}\mu_{x2}\,\mu_x\,\beta+6\,\gamma^9e^{h(2\,\eta+\gamma)}h\mu_{x3}\,\eta+12\,\gamma^3\mu_x{}^3e^{h(\eta+\gamma)}\beta^2\eta^4+12\,\gamma^3\mu_x{}^3e^{2\,h\eta}\beta^2\eta^4-18\,\gamma^4\mu_x{}^3e^{2\,h\eta}\beta^2\eta^3- \\
& 24\,\gamma^2\mu_x{}^3h^2e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\eta^7\mu_c-12\,\gamma^3\mu_x{}^3e^{h(2\,\eta+\gamma)}\beta^2\eta^4-108\,\gamma^5e^{h(2\,\eta+\gamma)}\mu_{x3}\,\eta^4+6\,\gamma^8h\mu_x{}^3e^{h(\eta+\gamma)}\beta\,\lambda\,\mu_c- \\
& 4\,\gamma^3\mu_x{}^3h^3e^{h(2\,\eta+\gamma)}\lambda^2\eta^7\mu_c{}^2+108\,\mu_x\,\mu_{x2}\,\gamma^4he^{h(2\,\eta+\gamma)}\beta\,\eta^5+12\,\gamma^9\mu_{x3}\,e^{h(\eta+\gamma)}-24\,\mu_x\,\mu_{x2}\,\gamma^3h^2e^{h(2\,\eta+\gamma)}\lambda\,\eta^7\mu_c- \\
& 36\,\gamma^6\mu_x{}^3h^2e^{h(2\,\eta+\gamma)}\beta\,\lambda\,\mu_c\,\eta^3
\end{aligned}$$

and

$$\chi = \frac{\lambda\,\mu_c\,e^{-h(2\,\eta+\gamma)}}{(\eta^2+2\,\gamma\,\eta+\gamma^2)(\gamma^4-2\,\eta\,\gamma^3-3\,\eta^2\gamma^2+8\,\eta^3\gamma-4\,\eta^4)\gamma^3\eta^4}$$

the third order moment is:

$$E\left[\left(Y_i^{(h)}\right)^3\right] = \chi(T1+T2+T3)$$

D Appendix: 3rd-order continuous-time DD1 properties

We define:

$$
\begin{aligned}
N_1 = {} & 1/6\,\mu\,\lambda\,F^3\eta^3\Big(-18\,\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma\,C^2\beta^2e^{\gamma\,z} \\
& +96\,\eta\,C^3\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} - 48\,\mu^2\lambda^2\eta^3\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2Ce^{\gamma\,y}e^{\gamma\,z} \\
& -12\,C^4\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y} + 16\,\eta\,C^3\beta^2e^{\gamma\,y}e^{\gamma\,z} \\
& -12\,\mu^2\lambda^2\eta^4\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2e^{\gamma\,y}e^{\gamma\,z} - 12\,\eta^4\left(e^{Cy}\right)^2 \\
& \left(e^{\eta\,y}\right)^2\beta^2e^{\gamma\,z} - 24\,\mu\,\lambda\,\eta^4\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,e^{\gamma\,z} - 12\,\mu^2\lambda^2\eta^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} \\
& -5\,\mu^2\lambda^2\gamma^2C^2e^{\gamma\,y}e^{\gamma\,z} + 18\,\eta^2\gamma\,C\beta^2e^{\gamma\,y}e^{\gamma\,z} + 6\,\gamma^2C^2\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y} \\
& -24\,\mu\,\lambda\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^4e^{\gamma\,z} + 16\,\mu^2\lambda^2\eta^3Ce^{\gamma\,y}e^{\gamma\,z} \\
& -96\,\mu\,\lambda\,\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^3e^{\gamma\,z} \\
& -48\,\mu^2\lambda^2\eta^3Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} - 12\,\mu\,\lambda\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^4e^{\gamma\,y}e^{\gamma\,z} \\
& -12\,\mu\,\lambda\,\eta^4\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,e^{\gamma\,y}e^{\gamma\,z} - 3\,\mu^2\lambda^2\left(e^{Cy}\right)^2 \\
& \left(e^{\eta\,y}\right)^2\gamma^4e^{\gamma\,y}e^{\gamma\,z} - 48\,\mu\,\lambda\,\eta\,\beta\,C^3e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} + 144\,\eta^2\beta^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} \\
& +96\,\mu\,\lambda\,\eta^3\beta\,Ce^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z} + 96\,\mu\,\lambda\,\eta\,\beta\,C^3e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z} \\
& -48\,\mu\,\lambda\,\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^3e^{\gamma\,y}e^{\gamma\,z} - 6\,\mu\,\lambda\,\gamma^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z} \\
& -36\,\mu\,\lambda\,\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma\,\beta\,Ce^{\gamma\,y}e^{\gamma\,z} + 24\,\mu\,\lambda\,\left(e^{\gamma\,y}\right)^2\beta\,C^4e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y} \\
& -48\,\mu\,\lambda\,\eta^3\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} - 12\,\mu\,\lambda\,\gamma\,C^3\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} \\
& +24\,\mu\,\lambda\,\beta\,C^4e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z} + 96\,\mu\,\lambda\,\eta\,\beta\,C^3e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} \\
& -72\,\mu\,\lambda\,\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^2e^{\gamma\,y}e^{\gamma\,z} + 2\,\gamma^2C^2\beta^2e^{\gamma\,y}e^{\gamma\,z} \\
& -12\,\mu\,\lambda\,\eta^3\gamma\,\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} - 12\,\mu\,\lambda\,\eta^3\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma\,\beta\,e^{\gamma\,y}e^{\gamma\,z} \\
& -12\,\eta\,\gamma^2C\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} + 24\,\mu\,\lambda\,\eta^4\left(e^{\gamma\,y}\right)^2\beta\,e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y} \\
& -12\,\mu\,\lambda\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma\,C^3\beta\,e^{\gamma\,y}e^{\gamma\,z} \\
& +24\,\mu\,\lambda\,\eta^4\beta\,e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z} \\
& +30\,\mu^2\lambda^2\eta\,\gamma^2Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} - 6\,\eta^2\gamma^2\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} \\
& +96\,\mu\,\lambda\,\eta\,\left(e^{\gamma\,y}\right)^2\beta\,C^3e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y} - 72\,\mu^2\lambda^2\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2C^2e^{\gamma\,y}e^{\gamma\,z} \\
& -36\,\mu\,\lambda\,\eta^2\gamma\,\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} + 3\,\mu\,\lambda\,\eta^2\gamma^2\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} \\
& +24\,\eta^4\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} + 144\,\mu\,\lambda\,\eta^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} \\
& +144\,\mu\,\lambda\,\eta^2\left(e^{\gamma\,y}\right)^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y} + 12\,\mu\,\lambda\,\eta\,\gamma^2\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y} \\
& -6\,\mu\,\lambda\,\gamma^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y} - 12\,\mu\,\lambda\,\eta\,\gamma^2\beta\,Ce^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z}
\end{aligned}
$$

$$-6\,\mu\,\lambda\,\eta\,\gamma^2\beta\,Ce^{\gamma\,y}e^{\gamma\,z} + 3\,\mu\,\lambda\,\eta\,\gamma^3\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z} + 3\,\mu\,\lambda\,\gamma^3\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}$$

$$+3\,\mu\,\lambda\,\eta^2\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2\beta\,e^{\gamma\,y}e^{\gamma\,z}+3\,\mu\,\lambda\,\gamma^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}$$
$$-12\,\mu\,\lambda\,\eta\,(e^{\gamma\,y})^2\,\gamma^2\beta\,Ce^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}-96\,\mu\,\lambda\,\eta\,\beta\,C^3e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$+6\,\mu\,\lambda\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2\beta\,C^2e^{\gamma\,z}+36\,\mu\,\lambda\,\eta^2\gamma\,\beta\,Ce^{\gamma\,y}e^{\gamma\,z}$$
$$-96\,\mu\,\lambda\,\eta^3\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\beta\,Ce^{\gamma\,z}-12\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,C^4\beta^2e^{\gamma\,z}$$
$$-6\,\eta^3\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma\,\beta^2e^{\gamma\,z}+36\,\mu\,\lambda\,\eta\,\gamma\,\beta\,C^2e^{\gamma\,y}e^{\gamma\,z}$$
$$+15\,\mu^2\lambda^2\eta^2\gamma^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}+6\,\mu\,\lambda\,\eta^2\gamma^2\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$+6\,\mu^2\lambda^2\gamma^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}-6\,\mu\,\lambda\,\eta^2\gamma^2\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}$$
$$-72\,\eta^2\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\beta^2C^2e^{\gamma\,z}+15\,\mu^2\lambda^2\gamma^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}$$
$$+16\,\eta^3C\beta^2e^{\gamma\,y}e^{\gamma\,z}-6\,\mu\,\lambda\,\eta^2\gamma^2\beta\,e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z}$$
$$+3\,\mu\,\lambda\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2\beta\,C^2e^{\gamma\,y}e^{\gamma\,z}$$
$$+6\,\mu\,\lambda\,\eta\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2\beta\,Ce^{\gamma\,y}e^{\gamma\,z}$$
$$+6\,\mu\,\lambda\,\eta\,\gamma^2\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}+4\,\mu^2\lambda^2C^4e^{\gamma\,y}e^{\gamma\,z}$$
$$-6\,\mu\,\lambda\,(e^{\gamma\,y})^2\,\gamma^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}+30\,\mu^2\lambda^2\eta\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2Ce^{\gamma\,y}e^{\gamma\,z}$$
$$-6\,\gamma^2C^2\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}$$
$$+12\,\mu\,\lambda\,\eta\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2\beta\,Ce^{\gamma\,z}-12\,\mu\,\lambda\,\eta\,\gamma^2\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}$$
$$-36\,\mu\,\lambda\,\eta\,\gamma\,\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}$$
$$-48\,\eta^3\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,C\beta^2e^{\gamma\,z}+24\,\mu\,\lambda\,\beta\,C^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}$$
$$+24\,\mu^2\lambda^2\eta^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}$$
$$+24\,\mu\,\lambda\,\eta^4\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}+15\,\mu^2\lambda^2\eta^2\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2e^{\gamma\,y}e^{\gamma\,z}$$
$$-30\,\mu^2\lambda^2\eta^2\gamma^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}-12\,\eta^4\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$-30\,\mu^2\lambda^2\gamma^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}$$
$$-60\,\mu^2\lambda^2\eta\,\gamma^2Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}$$
$$-36\,\mu\,\lambda\,\eta\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma\,\beta\,C^2e^{\gamma\,y}e^{\gamma\,z}$$
$$+6\,\eta^2\gamma^2\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}+96\,\mu\,\lambda\,\eta^3\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}$$
$$-48\,\eta\,C^3\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}-72\,\eta^2\beta^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$-6\,\mu\,\lambda\,\eta^2\,(e^{\gamma\,y})^2\,\gamma^2\beta\,e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}$$
$$+15\,\mu^2\lambda^2\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma^2C^2e^{\gamma\,y}e^{\gamma\,z}$$
$$-48\,\mu\,\lambda\,\eta^3\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\beta\,Ce^{\gamma\,y}e^{\gamma\,z}-6\,\eta^3\gamma\,\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}-6\,\gamma\,C^3\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$+18\,\eta\,\gamma\,C^2\beta^2e^{\gamma\,y}e^{\gamma\,z}+48\,\mu\,\lambda\,\eta\,\beta\,C^3e^{\gamma\,y}e^{\gamma\,z}$$

$$-12\,\mu^2\lambda^2C^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}+6\,\mu\,\lambda\,\gamma^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}$$
$$+96\,\mu\,\lambda\,\eta^3\,(e^{\gamma\,y})^2\,\beta\,Ce^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}-12\,\mu\,\lambda\,\eta^4\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}$$
$$+24\,C^4\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}-6\left(e^{Cy}\right)^2(e^{\eta\,y})^2\,\gamma\,C^3\beta^2e^{\gamma\,z}$$

$$
\begin{aligned}
&-18\,\eta^2\gamma\,C\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}+12\,\eta\,\gamma^2C\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}\\
&+12\,\mu\,\lambda\,\beta\,C^4e^{\gamma\,y}e^{\gamma\,z}+6\,\mu\,\lambda\,\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma^2\beta\,e^{\gamma\,z}\\
&+96\,\mu^2\lambda^2\eta^3Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}\\
&-48\,\eta^3C\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}+3\,\mu\,\lambda\,\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma^3\beta\,e^{\gamma\,y}e^{\gamma\,z}\\
&+96\,\mu^2\lambda^2\eta\,C^3e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}+144\,\mu^2\lambda^2\eta^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}\\
&+24\,\mu^2\lambda^2\eta^2C^2e^{\gamma\,y}e^{\gamma\,z}+24\,\mu^2\lambda^2C^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}e^{\gamma\,z}\\
&+144\,\mu\,\lambda\,\eta^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{Cy}e^{\eta\,y}e^{\gamma\,z}-48\,\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2C^3\beta^2e^{\gamma\,z}\\
&-3\,\mu\,\lambda\,\gamma^3\beta\,Ce^{\gamma\,y}e^{\gamma\,z}-48\,\mu^2\lambda^2\eta\,C^3e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}\\
&-144\,\mu\,\lambda\,\eta^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}-12\,\mu^2\lambda^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2C^4e^{\gamma\,y}e^{\gamma\,z}\\
&+24\,\eta^2\beta^2C^2e^{\gamma\,y}e^{\gamma\,z}-3\,\mu^2\lambda^2\gamma^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}\\
&-24\,\mu\,\lambda\,\beta\,C^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}+4\,\eta\,\gamma^2C\beta^2e^{\gamma\,y}e^{\gamma\,z}\\
&-72\,\mu^2\lambda^2\eta^2C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}-10\,\mu^2\lambda^2\eta\,\gamma^2Ce^{\gamma\,y}e^{\gamma\,z}\\
&+96\,\eta^3C\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{Cy}e^{\eta\,y}-12\,\mu\,\lambda\,\beta\,C^4e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}\\
&+3\,\mu\,\lambda\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma^3\beta\,Ce^{\gamma\,y}e^{\gamma\,z}+16\,\mu^2\lambda^2\eta\,C^3e^{\gamma\,y}e^{\gamma\,z}\\
&-18\,\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\gamma\,C\beta^2e^{\gamma\,z}-18\,\eta\,\gamma\,C^2\beta^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}\\
&+48\,\mu\,\lambda\,\eta^3\beta\,Ce^{\gamma\,y}e^{\gamma\,z}+6\,\gamma\,C^3\beta^2e^{\gamma\,y}e^{\gamma\,z}+12\,\mu\,\lambda\,\gamma\,C^3\beta\,e^{\gamma\,y}e^{\gamma\,z}\\
&-3\,\mu\,\lambda\,\gamma^2\beta\,C^2e^{\gamma\,y}e^{\gamma\,z}-24\,\mu\,\lambda\,\eta^4\beta\,e^{Cz}e^{\eta\,z}e^{\gamma\,y}+72\,\mu\,\lambda\,\eta^2\beta\,C^2e^{\gamma\,y}e^{\gamma\,z}\\
&-48\,\mu^2\lambda^2\eta\,\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2C^3e^{\gamma\,y}e^{\gamma\,z}-144\,\mu\,\lambda\,\eta^2\left(e^{Cy}\right)^2\left(e^{\eta\,y}\right)^2\beta\,C^2e^{\gamma\,z}\\
&-72\,\mu\,\lambda\,\eta^2\beta\,C^2e^{Cz}e^{\eta\,z}e^{\gamma\,y}e^{\gamma\,z}-96\,\mu\,\lambda\,\eta^3\beta\,Ce^{Cz}e^{\eta\,z}e^{\gamma\,y}+4\,C^4\beta^2e^{\gamma\,y}e^{\gamma\,z}\\
&+4\,\eta^4\beta^2e^{\gamma\,y}e^{\gamma\,z}+6\,\eta^3\gamma\,\beta^2e^{\gamma\,y}e^{\gamma\,z}+2\,\eta^2\gamma^2\beta^2e^{\gamma\,y}e^{\gamma\,z}\\
&+4\,\mu^2\lambda^2\eta^4e^{\gamma\,y}e^{\gamma\,z}-5\,\mu^2\lambda^2\eta^2\gamma^2e^{\gamma\,y}e^{\gamma\,z}+\mu^2\lambda^2\gamma^4e^{\gamma\,y}e^{\gamma\,z}+12\,\mu\,\lambda\,\eta^4\beta\,e^{\gamma\,y}e^{\gamma\,z}\\
&-3\,\mu\,\lambda\,\eta^2\gamma^2\beta\,e^{\gamma\,y}e^{\gamma\,z}-3\,\mu\,\lambda\,\eta\,\gamma^3\beta\,e^{\gamma\,y}e^{\gamma\,z}+12\,\mu\,\lambda\,\eta^3\gamma\,\beta\,e^{\gamma\,y}e^{\gamma\,z}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
D_1 \;=\;& e^{Cy}e^{\eta\,y}e^{Cz}e^{\eta\,z}e^{\gamma\,z}e^{\gamma\,y}\big(-280\,C^5\eta^3\gamma^2+6\,\gamma^4C^5\eta\\
&-40\,\eta^7\gamma^2C-40\,\gamma^2C^7\eta+15\,\gamma^4C^4\eta^2+6\,\gamma^4C\eta^5-350\,C^4\eta^4\gamma^2\\
&+15\,\gamma^4C^2\eta^4-280\,C^3\eta^5\gamma^2-140\,C^2\eta^6\gamma^2-140\,\gamma^2C^6\eta^2\\
&+40\,\eta^9C+180\,C^2\eta^8-5\,C^8\gamma^2+\gamma^4C^6+840\,C^6\eta^4+40\,C^9\eta+480\,C^7\eta^3+4\,C^{10}+480\,C^3\eta^7\\
&+1008\,C^5\eta^5+20\,\gamma^4C^3\eta^3+840\,C^4\eta^6+180\,C^8\eta^2+\gamma^4\eta^6-5\,\eta^8\gamma^2+4\,\eta^{10}\big)
\end{aligned}
$$

which yields the first of the 14 integrals as:

$$I_1 = N_1/D_1 \tag{74}$$

E Appendix: Probability dry approximation for LR model

E.1 Mean duration of a storm

Derivation of the approximation

The theoretical expression for the mean storm duration which is required for the estimation of the proportion of dry periods μ_T is (Onof, 1992):

$$\mu_T = \frac{\phi\nu}{\alpha - 1}\int_0^1 dv \int_0^1 dt\, v^{-1}t^{\phi-1}[1-(1-vt)e^{-\kappa v(1-t)}] + \frac{\phi^{-1}\nu}{\alpha-1} \tag{75}$$

Since approximating this integral involves expanding the exponential term as the sum of a series, a change of variable in the integral in t would be preferable (the new variable is $1-t$). This yields:

$$\mu_T = \frac{\phi\nu}{\alpha - 1}\int_0^1 dv \int_0^1 dt\, v^{-1}(1-t)^{\phi-1}[1-(1-v(1-t))e^{-\kappa vt}] + \frac{\phi^{-1}\nu}{\alpha-1} \tag{76}$$

By using the Taylor expansion of the exponential function, the double integral inside this expression can be written as:

$$\begin{aligned} I &= \int_0^1 dv \int_0^1 dt\, v^{-1}(1-t)^{\phi-1}\left[-\frac{\sum_{j=1}^{\infty}(-\kappa vt)^j}{j!} + v\frac{\sum_{j=0}^{\infty}(-\kappa vt)^j}{j!} - vt\frac{\sum_{j=0}^{\infty}(-\kappa vt)^j}{j!}\right] \\ &= \int_0^1 dv \int_0^1 dt\,(1-t)^{\phi-1}\left[-\frac{\sum_{j=1}^{\infty}(-\kappa vt)^j}{vj!} + \frac{\sum_{j=0}^{\infty}(-\kappa vt)^j}{j!} - t\frac{\sum_{j=0}^{\infty}(-\kappa vt)^j}{j!}\right] \end{aligned}$$

which is the sum of three terms:

$$\begin{aligned} I_1 &= -\sum_{j=1}^{\infty}\frac{(-\kappa)^j}{j!}\int_0^1 v^{j-1}\,dv\int_0^1(1-t)^{\phi-1}t^j\,dt \\ I_2 &= \sum_{j=0}^{\infty}\frac{(-\kappa)^j}{j!}\int_0^1 v^{j}\,dv\int_0^1(1-t)^{\phi-1}t^j\,dt \\ I_3 &= -\sum_{j=0}^{\infty}\frac{(-\kappa)^j}{j!}\int_0^1 v^{j}\,dv\int_0^1(1-t)^{\phi-1}t^{j+1}\,dt \end{aligned}$$

Having thus separated the variables of integration, the expressions simplify in terms of beta functions as in Richard's note (Chandler, 2003):

$$I_1 = -\sum_{j=1}^{\infty}\frac{(-\kappa)^j}{jj!}B(j+1,\phi) \tag{77}$$

$$I_2 = \sum_{j=0}^{\infty} \frac{(-\kappa)^j}{(j+1)!} B(j+1,\phi) \tag{78}$$

$$I_3 = -\sum_{j=0}^{\infty} \frac{(-\kappa)^j}{(j+1)!} B(j+2,\phi) \tag{79}$$

The sum $I = I_1 + I_2 + I_3$ can therefore be approximated by $I_M = I_{1_M} + I_{2_M} + I_{3_M}$ where:

$$I_{1_M} = -\sum_{j=1}^{M} \frac{(-\kappa)^j}{jj!} B(j+1,\phi) \tag{80}$$

$$I_{2_M} = \sum_{j=0}^{M} \frac{(-\kappa)^j}{(j+1)!} B(j+1,\phi) \tag{81}$$

$$I_{3_M} = -\sum_{j=0}^{M} \frac{(-\kappa)^j}{(j+1)!} B(j+2,\phi) \tag{82}$$

Note that, with computational efficiency in mind, I_M can be rewritten so as to minimise the calls a program has to make to the Beta function. Thus:

$$I_{1_M} = -\sum_{j=1}^{M} \frac{(-\kappa)^j}{jj!} B(j+1,\phi)$$

$$I_{2_M} = \sum_{j=0}^{M} \frac{(-\kappa)^j}{(j+1)!} B(j+1,\phi)$$

$$I_{3_M} = -\sum_{j=1}^{M+1} \frac{(-\kappa)^{j-1}}{j!} B(j+1,\phi)$$

this suggests we take the following as approximation:

$$I'_M = -\sum_{j=1}^{M} \frac{(-\kappa)^j}{jj!} B(j+1,\phi) + \sum_{j=0}^{M} \frac{(-\kappa)^j}{(j+1)!} B(j+1,\phi) - \sum_{j=1}^{M} \frac{(-\kappa)^{j-1}}{j!} B(j+1,\phi)$$

which yields:

$$I'_M = B(1,\phi) + \sum_{j=1}^{M} (-\kappa)^{j-1} \left(\frac{\kappa}{jj!} - \frac{\kappa}{(j+1)!} - \frac{1}{j!} \right) B(j+1,\phi)$$

or:

$$I'_M = \frac{1}{\phi} + \sum_{j=1}^{M} \frac{(-\kappa)^{j-1}(\kappa - j^2 - j)}{j(j+1)!} B(j+1,\phi) \tag{83}$$

where I'_M is related to I_M by:

$$I_M = I'_M - \frac{(-\kappa)^M}{(M+1)!} B(M+2,\phi) \tag{84}$$

Error estimation

To analyse the error on the approximation of I, let us return to the expression in terms of three integrals. The general term of the sequence which is being summed to compute I_M is:

$$u_j = -\frac{(-\kappa)^j}{j(j+1)!}(B(j+1,\phi) + jB(j+2,\phi))$$

which is alternatively negative (for j even) and positive. The sequence has the property that it is strictly decreasing in absolute value, therefore:

$$\begin{aligned} u_{2m+1} + u_{2m+2} > 0 &\Rightarrow \sum_{j=2m+1}^{n} u_j > 0 \text{ for any } n \geq 2m+1 \\ u_{2m} + u_{2m+1} < 0 &\Rightarrow \sum_{j=2m}^{n} u_j < 0 \text{ for any } n \geq 2m \end{aligned}$$

and therefore, the limit of the series (taking the sum from 0) is between $\sum_{j=0}^{M} u_j$ and $\sum_{j=0}^{M+1} u_j$ and upper bounds for the error made in approximating I with I_M are:

$$|I - I_M| < |u_{M+1}| = |\frac{(-\kappa)^M}{M(M+1)!}(B(M+1,\phi) + MB(M+2,\phi))| < \frac{(\kappa)^M}{MM!}B(M+1,\phi) \quad (85)$$

since $B(M+2,\phi) < B(M+1,\phi)$.

Consequently, the error on the computationally more efficient approximation I'_M can be bounded as follows:

$$\begin{aligned} |I - I'_M| &< |I - I_M| + |I_M - I'_M| \\ &< \frac{(\kappa)^M}{MM!}B(M+1,\phi) + \frac{(\kappa)^M}{(M+1)!}B(M+2,\phi) \\ &< 2\frac{(\kappa)^M}{MM!}B(M+1,\phi) \quad (86) \end{aligned}$$

from equation (84).

This is a coarse upper bound and in fact, more can be said about the errors involved in approximating with I_M or I'_M. In particular, we have the result:

Lemma 1 *I'_M is a better approximation of I than I_M for $\kappa > 1$.*

This can easily be seen by observing that, assuming M even, we have:

$$I'_M = I_M + \frac{\kappa^M}{(M+1)!}B(M+2,\phi)$$

thus:

$$I'_M > I_M$$

and

$$
\begin{aligned}
I_{M+1} = I_M + u_M &= I_M + \frac{\kappa^{M+1}}{M(M+1)!}\left(B(M=1,\phi) + MB(M+2,\phi)\right) \\
&= I_M + \frac{\kappa^M}{(M+1)!}\frac{\kappa\left(B(M=1,\phi) + MB(M+2,\phi)\right)}{M} \\
&> I_M + \kappa\left(1+\frac{1}{M}\right)B(M+2,\phi)
\end{aligned}
$$

so that, if $\kappa > 1$,

$$I_{M+1} > I'_M$$

Therefore, assuming $\kappa > 1$, we obtain:

$$
\begin{aligned}
&I'_M \in (I_M, I_{M+1}) \quad \text{for M even,} \\
&\quad \text{and similarly} \\
&I'_M \in (I_{M+1}, I_M) \quad \text{for M odd.}
\end{aligned}
\tag{87}
$$

This entails that

$$|I'_{M+1} - I'_M| < |I_{M+1} - I_M|$$

and, more generally:

$$
\begin{aligned}
|I'_{M+p} - I'_M| &< |I_{M+p} - I_M| \text{ for p odd} \\
|I'_{M+p} - I'_M| &< |I_{M+p+1} - I_M| \text{ for p even}
\end{aligned}
$$

so that, taking limits as $p \to \infty$, we have:

$$\forall M \, |I - I'_M| < |I - I_M| \tag{88}$$

q.e.d.

On the contrary, for small values of κ, the series which converges fastest is I_n if:

$$\kappa\left(\frac{n+\phi+1}{(n+2)n} + 1\right) < 1$$

which is true for n large enough (for given values of κ and ϕ). However, as the numerical experiments below show, small values of κ in any case lead to fast convergence of I'_n.

Numerical investigation

Using Maple which calculates sums of terms with Beta functions in terms of the Generalised Hypergeometric Function, we can however evaluate the exact relative error $\Delta I_M = |I - I'_M|/I$ for a range of values of M, for given values of parameters κ and ϕ.

The results are shown in table 2 below.

ϕ	κ	I	ΔI_M: $M=3$	$M=5$	$M=7$	$M=10$	$M=15$	$M=20$	$M=30$
0.01	0.01	1.98	$4.2\,10^{-7}$	$1.3\,10^{-12}$	$2.5\,10^{-18}$	$2.5\,10^{-27}$			
0.10	0.01	1.00	$3.8\,10^{-7}$	$1.2\,10^{-12}$	$2.1\,10^{-18}$	$2.1\,10^{-27}$			
1	0.01	0.50	$1.7\,10^{-7}$	$4.0\,10^{-13}$	$5.6\,10^{-19}$	$4.2\,10^{-28}$			
10	0.01	0.09	$4.6\,10^{-9}$	$1.9\,10^{-15}$	$6.3\,10^{-22}$	$7.9\,10^{-32}$			
0.01	0.1	10.63	$4.5\,10^{-5}$	$1.5\,10^{-8}$	$2.7\,10^{-11}$	$2.7\,10^{-17}$			
0.10	0.1	1.78	$4.0\,10^{-5}$	$1.3\,10^{-8}$	$2.3\,10^{-11}$	$2.2\,10^{-17}$			
1	0.1	0.54	$1.8\,10^{-5}$	$4.3\,10^{-9}$	$5.9\,10^{-13}$	$4.5\,10^{-18}$			
10	0.1	0.09	$4.8\,10^{-7}$	$2.0\,10^{-10}$	$6.6\,10^{-15}$	$8.3\,10^{-23}$			
0.01	1	79.75	$1.6\,10^{-1}$	$5.7\,10^{-3}$	$1.0\,10^{-4}$	$1.1\,10^{-6}$			
0.10	1	8.05	$1.4\,10^{-1}$	$4.8\,10^{-3}$	$8.7\,10^{-5}$	$8.8\,10^{-7}$			
1	1	0.86	$4.7\,10^{-2}$	$1.2\,10^{-3}$	$1.7\,10^{-5}$	$1.3\,10^{-7}$			
10	1	0.10	$9.3\,10^{-4}$	$4.0\,10^{-5}$	$1.3\,10^{-7}$	$1.7\,10^{-12}$			
0.01	5	217.81	1.4	1.8	0.9	0.1	$9.0\,10^{-4}$	$1.2\,10^{-6}$	$7.9\,10^{-13}$
0.10	5	20.98	1.3	1.5	0.8	0.1	$7.2\,10^{-4}$	$9.6\,10^{-7}$	$6.0\,10^{-14}$
1	5	1.66	0.6	0.5	0.2	$2.1\,10^{-2}$	$1.0\,10^{-4}$	$1.0\,10^{-7}$	$4.6\,10^{-15}$
10	5	.11	0.0	$2.7\,10^{-3}$	$2.4\,10^{-4}$	$4.0\,10^{-6}$	$1.7\,10^{-9}$	$2.6\,10^{-13}$	$6.6\,10^{-22}$
0.01	10	286.74	2.1	20.3	50.0	64.0	14.9	0.7	$4.7\,10^{-5}$
0.10	10	27.64	1.9	17.5	42.0	52.3	11.7	0.5	$3.4\,10^{-5}$
1	10	2.18	0.8	5.2	9.6	9.1	1.5	$5.1\,10^{-2}$	$2.5\,10^{-5}$
10	10	0.13	$3\,10^{-2}$	$2\,10^{-2}$	$1\,10^{-1}$	$1.4\,10^{-3}$	$2.0\,10^{-5}$	$1.1\,10^{-6}$	0.

Table 1: Relative errors on the estimation of μ_T

Conclusion

In conclusion, we note that although I_M (computed for instance as $I_M = I'_M - \frac{(-\kappa)^M}{(M+1)!}B(M+2,\phi)$) could be used as approximation,it is preferable to approximate I with I'_M, so that μ_T can be approximated as:

$$\mu_T \approx \frac{\nu}{\alpha - 1}\left(\phi I'_M + \phi^{-1}\right) \tag{89}$$

E.2 Integral term

For the random-parameter Bartlett-Lewis model, the exact probability that an arbitrary interval is dry depends on an integral of the form

$$I(\phi, \kappa) = \int_0^1 t^{\phi-1}\,(1-t)\,e^{\kappa t}dt \tag{90}$$

(see equation 46). This cannot be evaluated analytically, although it is (almost) a 'standard' integral, in the sense that it has a name, since $I(\phi,\kappa)/B(\phi,2)$ is a confluent hypergeometric function — $M(\phi, 2+\phi, \kappa)$ in the notation of Abramowitz and Stegun (1965, equation 13.2.1). $B(a,b)$ here is the beta function. There appear not to be any nice ways of evaluating $I(\phi,\kappa)$ or relating it to other special functions that can be calculated easily — I've checked everything in Abramowitz and Stegun (1965) and in Gradshteyn and Ryzhik (1980).

Rodriguez-Iturbe et al. (1987) approximated the integral using a third-order series expansion that is valid when ϕ and κ are both small (i.e. substantially less than 1). This approach runs through all subsequent developments of the Bartlett-Lewis model, and is still used in our fitting programs. However, the requirement for κ and ϕ to be small seems to have gone largely unnoticed (or been forgotten) in our fitting work. Now that we're thinking about fitting models to lots of datasets, it may be worth examining. For example, some recent problems with the SCE fitting code 'blowing up' for some datasets appear to be caused solely by the failure of this approximation in a region of the parameter space that the algorithm was exploring. Moreover, the exact magnitude of the approximation error for any given κ and ϕ is not known, which makes me feel a bit uncomfortable ...

In view of this, it may be worth exploring alternative means of evaluating the integral. A possible solution is to use standard quadrature methods; however, since the integrand becomes infinite at $t = 0$ for $\phi < 1$, this may be delicate. Instead, consider expanding the $e^{\kappa t}$ term, to get

$$\begin{aligned} I(\phi,\kappa) = \int_0^1 t^{\phi-1}(1-t)\sum_{j=0}^{\infty}\frac{(\kappa t)^j}{j!}dt &= \sum_{j=0}^{\infty}\frac{\kappa^j}{j!}\int_0^1 t^{j+\phi-1}(1-t)\,dt \\ &= \sum_{j=0}^{\infty}\frac{\kappa^j}{j!}B(j+\phi,2)\ . \end{aligned}$$

This suggests truncating the infinite sum at a suitably large value, say M, and approximating the integral by

$$\tilde{I}_M(\phi,\kappa) = \sum_{j=0}^{M}\frac{\kappa^j}{j!}B(j+\phi,2)\ . \tag{91}$$

The point about this is that standard algorithms exist for calculating the Beta function to a high degree of accuracy (it can be expressed in terms of gamma functions — $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$), and these are readily available in both R and FORTRAN (there's a FORTRAN routine to evaluate gamma functions in file `OURPROGS/rec_math.f` on argos).

The error in approximating I with $\tilde{I}_M$ is

$$I(\phi,\kappa) - \tilde{I}_M(\phi,\kappa) = \int_0^1 t^{\phi-1}(1-t)\left[e^{\kappa t} - \sum_{j=0}^{M}\frac{(\kappa t)^j}{j!}\right]dt = \int_0^1 t^{\phi-1}(1-t)\sum_{j=M+1}^{\infty}\frac{(\kappa t)^j}{j!}dt\ . \tag{92}$$

Each term in the infinite sum is non-negative and increasing in t. Therefore, over the range of the integral, it takes its maximum value at $t = 1$. The maximum value of the sum is

$$\delta_M(\kappa) = \sum_{j=M+1}^{\infty} \frac{\kappa^j}{j!} = e^{\kappa} - \sum_{j=0}^{M} \frac{\kappa^j}{j!} \tag{93}$$

which is, again, easily evaluated providing M is not too large. We therefore have

$$0 < I(\phi,\kappa) - \tilde{I}_M(\phi,\kappa) < \delta_M(\kappa) \int_0^1 t^{\phi-1}(1-t)\,dt \ = \delta_M(\kappa)\,B(\phi,2)\ . \tag{94}$$

Therefore, for any value of M we can calculate an upper bound on the approximation error. This enables us to find a value of M that will approximate the integral to any desired accuracy. A pragmatic criterion, for example, may be to choose M such that $\delta_M(\kappa)\,B(\phi,2) < 0.01 \times \tilde{I}_M(\kappa,\phi)$.

From equation (94), it is clear that $\tilde{I}_M$ will always underestimate I. It is natural to ask whether a correction can be made for this, to improve the approximation. From equation (92) we have

$$I(\phi,\kappa) - \tilde{I}_M(\phi,\kappa) = \int_0^1 \sum_{j=M+1}^{\infty} \frac{\kappa^j}{j!} t^{j+\phi-1} dt \ - \int_0^1 \sum_{j=M+1}^{\infty} \frac{\kappa^j}{j!} t^{j+\phi} dt\ .$$

Each of the integrands here increases monotonically from 0 to $\delta_M(\kappa)$ as t ranges from 0 to 1. Moreover, since $t < 1$ throughout the range of integration, the $j = M+1$ term is the largest in each sum. This suggests approximating the error by taking just the $M+1$ term from each sum and scaling it to match the correct value at each end of the range of integration. This yields the approximation

$$\delta_M(\kappa) \int_0^1 \left(t^{M+\phi} - t^{M+\phi+1}\right) dt = \frac{\delta_M(\kappa)}{(M+\phi+1)(M+\phi+2)}$$

which, in turn, suggests that

$$\hat{I}_M(\phi,\kappa) = \tilde{I}_M(\phi,\kappa) + \frac{\delta_M(\kappa)}{(M+\phi+1)(M+\phi+2)} \tag{95}$$

will be an improved estimate of $I(\phi,\kappa)$. Note that the improvement is obtained almost 'free of charge' — it depends only on M (which is known) and upon $\delta_M(\kappa)$ (which has already been calculated to determine the accuracy of $\tilde{I}_M$).

Numerical investigation

To assess the adequacy of these approximations, some numerical experiments have been carried out for several values of κ and ϕ. It is of particular interest to determine how large M needs to be to achieve a specified degree of accuracy. Define $\tilde{M}_\alpha$ and $\hat{M}_\alpha$ to be the values required to obtain a relative error of less than $100\alpha\%$ for particular values of κ and ϕ, using

ϕ	κ	$I(\phi,\kappa)$	$\tilde{M}_{0.01}$	$\hat{M}_{0.01}$	$\tilde{M}_{10^{-6}}$	$\hat{M}_{10^{-6}}$
0.01	0	99.0099	0	0	0	0
0.10	0	9.0909	0	0	0	0
1	0	0.5000	0	0	0	0
10	0	0.0091	0	0	0	0
0.01	0.01	99.0148	0	0	1	0
0.10	0.01	9.0952	0	0	1	1
1	0.01	0.5017	0	0	2	1
10	0.01	0.0092	0	0	2	1
0.01	0.10	99.0600	0	0	2	1
0.10	0.10	9.1350	0	0	3	2
1	0.10	0.5171	1	0	3	2
10	0.10	0.0099	1	0	4	2
0.01	1	99.6013	0	0	5	4
0.10	1	9.6160	1	1	6	5
1	1	0.7183	3	2	7	6
10	1	0.0210	4	2	8	6
0.01	10	385.9201	15	13	26	23
0.10	10	288.2351	16	13	26	23
1	10	220.1547	16	13	26	23
10	10	56.3963	17	13	27	24

Table 2: Exact values of $I(\phi,\kappa)$, together with values of M required to achieve relative errors of less than 10^{-2} and 10^{-6} respectively.

$\tilde{I}_M$ and $\hat{I}_M$ respectively. Table 2 shows the values of $\tilde{M}_{0.01}$, $\hat{M}_{0.01}$, $\tilde{M}_{10^{-6}}$ and $\hat{M}_{10^{-6}}$ for values of ϕ and κ between 0 and 10. In all cases, the 'exact' expression was calculated as $\tilde{I}_{100}(\phi,\kappa)$. As a check on the adequacy of this (and on the overall accuracy of the theory and programming!), the results for $\phi = 0.1, \kappa = 0.1$ and for $\phi = 0.1, \kappa = 1$ have also been evaluated manually using Table 13.1 of Abramowitz and Stegun (1965).

Table 2 shows that for the values of ϕ and κ likely to be encountered in rainfall modelling applications, a small value of M yields very high accuracy. In such applications it would be unusual to find values in excess of 1) For example, using $\hat{I}_M$, when $\phi = \kappa = 1$ a relative error of less than 1% can be achieved with $M = 2$. Indeed, $M = 6$ is sufficient to ensure a relative error of less than 10^{-6} in this case. As expected, $\hat{I}_M$ is more accurate than $\tilde{I}_M$ and hence is preferable (since it is no more expensive to compute). The magnitude of this improvement can be illustrated by comparing $\tilde{I}_0(1,1) = 0.6667$ and $\hat{I}_0(1,1) = 0.7265$ (not shown in Table 2) with the actual value of $I(1,1)$, which is 0.7183. In this case, $\hat{I}$ improves considerably over $\tilde{I}$. The magnitude of the error here suggests that taking M as small as zero may adequate for some applications, if using $\hat{I}$.

F Appendix: Std. deviation of number of cells/storm for QR model

Conditional upon η, the mean number of cells N_c in a storm is geometrically distributed with mean $E[N_c|\eta] = 1 + \frac{\kappa_1+\kappa_2\eta}{\phi}$. Let $a = 1 - E[N_c|\eta]^{-1}$. The distribution is thus given by:

$$Pr\{N_c = n|\eta\} = (1-a)a^{n-1} \text{ for } n > 0$$

The variance of N_c is:

$$\text{var}[N_c] = E[\text{var}[N_c|\eta]] \tag{96}$$

Let us first calculate the conditional variance:

$$\text{var}[N_c|\eta] = E[N_c^2|\eta] - E[N_c|\eta]^2$$

and

$$E[N_c^2|\eta] = (1-a)\sum_{n=1}^{\infty} n^2 a^{n-1}$$

The following sums are useful:

$$\begin{aligned}\sum_{n=1}^{\infty} n a^{n-1} &= \frac{\partial}{\partial a}\sum_{n=0}^{\infty} a^n \\ &= \frac{1}{(1-a)^2} \\ \sum_{n=1}^{\infty} n(n-1)a^{n-2} &= \frac{\partial^2}{\partial a^2}\sum_{n=0}^{\infty} a^n \\ &= \frac{2}{(1-a)^3}\end{aligned}$$

Hence:

$$\sum_{n=1}^{\infty} n^2 a^{n-1} = \frac{a+1}{(1-a)^3}$$

thus,

$$E[N_c^2|\eta] = \frac{a+1}{(1-a)^2}$$

and:

$$\begin{aligned}\text{var}[N_c|\eta] &= \frac{2a}{(1-a)^2} \\ &= 2\left[\left(\frac{1}{1-a}\right)^2 - \frac{1}{1-a}\right]\end{aligned}$$

Therefore:

$$\text{var}[N_c|\eta] = 2\left[\left(1 + \frac{\kappa_1+\kappa_2\eta}{\phi}\right)^2 - 1 - \frac{\kappa_1+\kappa_2\eta}{\phi}\right]$$

or,

$$\mathrm{var}[N_c|\eta] = 2\left[\left(\frac{\kappa_1 + \kappa_2\eta}{\phi}\right)^2 + \frac{\kappa_1 + \kappa_2\eta}{\phi}\right] \tag{97}$$

From equation (96), the unconditional variance is:

$$\mathrm{var}[N_c] = \frac{2}{\phi^2}\left[(\kappa_1^2 + \kappa_1\phi) + (2\kappa_1\kappa_2 + \kappa_2\phi)E[\eta] + \kappa_2^2 E[\eta^2]\right]$$

Since:

$$\begin{aligned} E[\eta] &= \alpha/\nu \\ E[\eta^2] &= (\alpha+1)\alpha/\nu^2 \end{aligned}$$

we finally obtain:

$$\mathrm{var}[N_c] = \frac{2}{\phi^2}\left[(\kappa_1^2 + \kappa_1\phi) + (2\kappa_1\kappa_2 + \kappa_2\phi)\frac{\alpha}{\nu} + \kappa_2^2\frac{\alpha(\alpha+1)}{\nu^2}]\right] \tag{98}$$

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