BÁLINT TÓTH:
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SCALING LIMITS FOR SELF-INTERACTING RANDOM WALKS AND DIFFUSIONS WITH LONG MEMORY

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Based on joint work with:

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"True" self-avoiding random walk (TSAW), discrete time:

$$n \mapsto X(n) \in \mathbb{Z}^d,$$

It's local time (occupation time measure):

$$\ell(n, x) := \ell(0, x) + |\{0 < m \leq n : X(m) = x\}|$$

Self-interaction function:

$$w : \mathbb{Z} \to (0, \infty) \quad \text{increasing}$$

The law of the walk:

$$\mathbf{P}\Big(X(n+1) = y \,\Big|\, \mathcal{F}_n, X(n) = x \Big) =$$

$$1\!\!1_{\{\{|x-y|=1\}\}} \frac{w(\ell(n,x) - \ell(n,y))}{\sum_{z:|z-x|=1} w(\ell(n,x) - \ell(n,z))}$$

TSAW, continuous time:

$$t \mapsto X(t) \in \mathbb{Z}^d$$

Local time

$$\ell(t,x) := \ell(0,x) + |\{0 < s \leq t : X(s) = x\}\,|$$

Rate function:

$$w : \mathbb{R} \to (0,\infty), \qquad \inf_u w(u) = \gamma > 0 \quad \text{(unif. ellipticity)}$$

$$r(u) = \frac{w(u) - w(-u)}{2} \quad \text{increasing}, \qquad s(u) = \frac{w(u) + w(-u)}{2}$$

The law of the walk:

$$\mathbf{P}\Big(X(t+dt) = y \,\Big|\, \mathcal{F}_t, X(t) = x \Big) = 1\!\!1_{\{\{|x-y\,|=1\}\}} w(\ell(t,x) - \ell(t,y))dt$$

Self-repelling Brownian polymer (SRBP):

$$t \mapsto X(t) \in \mathbb{R}^d$$

Local time (occupation time measure):

$$\ell(t, A) := \ell(0, A) + |\{0 < s \leq t : X(s) \in A\}|$$

Self-interaction function: $V : \mathbb{R}^d \to \mathbb{R}$, C^∞, of fast decay and of positive type:

$$\widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x) dx \geq 0 \qquad (*)$$

E.g. $V(x) = e^{-|x|^2}$

The driving force:

$$F : \mathbb{R}^d \to \mathbb{R}^d, \qquad F(x) := -\text{grad}\, V(x).$$

The law of the process:

$$X(t) = B(t) + \int_0^t \int_0^s F(X(s) - X(u))du\, ds,$$

or:

$$dX(t) = dB(t) + \left(\int_0^t F(X(t) - X(u))du \right) dt.$$

or:

$$dX(t) = dB(t) - \text{grad}\left(V * \ell(t, \cdot) \right)(X(t))dt$$

Note: In all three cases **the position process is pushed by the negative gradient of its own occupation time measure**.

Roots:

TSAW, physics:

[D. Amit, G. Parisi, L. Peliti (1983)],
[S. Obukhov, L. Peliti (1983)],
[L. Peliti, L. Pietronero (1987)]
...

SRBP, probability:

[J. Norris, C. Rogers, D. Williams (1987)]
[R. Durrett, C. Rogers (1992)],
[M. Cranston, Y. Le Jan (1995)],
[M. Cranston, T. Mountford (1996)],
...

Conjectures, based on RG and scaling arguments ("physics"):

- $\mathbf{d = 1}:$ $X(t) \sim t^{2/3}$, intricate, non-Gausssian scaling limit. (Limit distributions not identified.)

- $\mathbf{d = 2}:$ $X(t) \sim t^{1/2}(\log t)^{\zeta}$, Gaussian scaling limit. (Controversy about the value of ζ.)

- $\mathbf{d \geq 3}:$ $X(t) \sim t^{1/2}$, Gaussian scaling limit.

Some results: . . .

- $\mathbf{d=1}$: ∘ **Limit thm.** in some particular cases [B.T. (AP, 1995)], [B.T., B. Vető (ALEA, 2009)]:

$$\frac{X(t)}{t^{2/3}} \Rightarrow \mathcal{X}.$$

∘ Construction of the **scaling limit process** (TSRM, the Brownian Web, . . .) [B.T., W. Werner (PTRF, 1998)]

$$t \mapsto \mathcal{X}(t)$$

∘ **"Robust" superdiffusive bounds** [P. Tarrés, B.T., B. Valkó (AP, 2012)]:

$$C_1 t^{5/4} \leq \mathbf{E}\Big(X(t)^2 \Big) \leq C_2 t^{3/2}.$$

(and more bounds for more general self-interactions)

∘ **Missing:** fully robust proofs.

- $\mathbf{d=2}$: ∘ **Super diffusive lower bounds**
  [B.T., B. Valkó (JSP, 2012)]:

$$C_1 t \log\log t \le \mathbf{E}\Big(X(t)^2 \Big) \le C_2 t \log t.$$

  ∘ Expected order:

$$\mathbf{E}\Big(X(t)^2 \Big) \sim t\sqrt{\log t}$$

- $\mathbf{d \ge 3}$: ∘ **Diffusive bounds and CLT**
  [I. Horváth, B.T., B. Vető (PTRF, 2012)]:

$$\frac{X(t)}{t^{1/2}} \Rightarrow N(0,\sigma).$$

Precise conditions and statements later.

Some details for the SRBP:

Environment seen from the position of the moving particle:

$$\eta(t,x) := \eta(0, X(t)+x) + \int_0^t F(X(t)+x-X(u))\, \mathrm{d}u$$

$$= -\,\mathrm{grad}\left(V * \ell(t,\cdot)\right)(X(t)+x).$$

Remark on $d = 1, 2$ vs. $d \geq 3$ . . .

Note: $t \mapsto \eta(t,\cdot)$ is a **Markov process** in

$$\Omega := \left\{\omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}^d) \,:\, \omega \ \text{ grad-field}, \ \ ||\omega||_{k,m,r} < \infty\right\}$$

$$||\omega||_{k,m,r} := \sup_{x\in\mathbb{R}^d} \left(1 + |x\,|\right)^{-1/r} \left|\partial^{|m\,|}_{m_1,\ldots,m_d}\omega_k(x)\right|$$

Some operators acting on *smooth cylinder functions*

$$\nabla_j f(\omega) := \lim_{\varepsilon \to 0} \varepsilon^{-1}\Big(f(\tau_{\varepsilon e_j}\omega) - f(\omega)\Big),$$

$$\Delta f(\omega) := \sum_{j=1}^{d} \nabla_j^2 f(\omega),$$

$$\mathcal{D} f(\omega) := \lim_{\varepsilon \to 0} \varepsilon^{-1}\Big(f(\omega + \varepsilon F) - f(\omega)\Big)$$

The infinitesimal generator:

$$Gf(\omega) = \frac{1}{2}\Delta f(\omega) + \sum_{l=1}^{d} \omega_l(0)\nabla_l f(\omega) + \mathcal{D} f(\omega).$$

Stationary measure for the Markov process $t \mapsto \eta(t)$:

By some **"miracle"**: gradient of (mollified) massless free Gaussian field:

$$\langle \omega_k(x)\omega_l(y)\rangle = -\partial^2_{kl} V * \Delta^{-1}(y-x) =: K_{kl}(y-x)$$

$$\widehat{K}_{kl}(p) = \frac{p_k p_l}{|p|^2}\widehat{V}(p).$$

Proof 1: Itô-calculus. Proof 2: Functional analytic.

All results valid in this stationary/ergodic regime.

By Itô:

$$d\langle u_k, \eta_k(t)\rangle = -\langle \partial_l u_k, \eta_k(t)\rangle dB_l(t) + \Big(\frac{1}{2}\langle \partial^2_{ll} u_k, \eta_k(t)\rangle - \langle \partial_l u_k, \eta_k(t)\rangle \eta_l(t,0) + \langle \partial_k u_k, V\rangle\Big) dt$$

Hence:

$$\mathbf{E}\Big(d \exp\{\langle u_k, \eta_k(t)\rangle\} \,\Big|\, \mathcal{F}_t \Big) = \exp\{\langle u_k, \eta_k(t)\rangle\} \times \Big(\frac{1}{2}\langle \partial^2_{ll} u_k, \eta_k(t)\rangle + \frac{1}{2}\langle \partial_l u_k, \eta_k(t)\rangle \langle \partial_l u_m, \eta_m(t)\rangle - \langle \partial_l u_k, \eta_k(t)\rangle \eta_l(t,0) + \langle \partial_k u_k, V\rangle\Big) dt$$

Ansatz: $x \mapsto \eta(t,x)$ (with t fixed!) is Gaussian with covariance $C_{kl}(y-x)$.

$$\exp\{-\langle u_k, C_{kl} * u_l\rangle/2\}\frac{d\mathbf{E}\Big(\ \exp\{\langle u_k, \eta_k(t)\rangle\}\ \Big)}{dt}$$

$$= \quad \frac{1}{2}\langle\partial_{ll}^2 u_k, C_{km} * u_m\rangle + \frac{1}{2}\langle\partial_l u_k, C_{km} * \partial_l u_m\rangle$$
$$+ \frac{1}{2}\langle\partial_l u_k, C_{km} * u_m\rangle\langle\partial_l u_i, C_{ij} * u_j\rangle - \langle\partial_l u_k, C_{km} * u_m\rangle\langle u_i, C_{il}\rangle$$
$$+ \langle\partial_k u_k, V\rangle - \langle\partial_l u_k, C_{kl}\rangle.$$

$$\partial_k V + \partial_l C_{lk} = 0.$$

General sln:

$$C_{kl}(x) = K_{kl}(x) + D_{kl}(x)$$

where D_{kl} is covariance of div-free vector field.

Remark on the lattice model (TSAW, $d \geq 3$):

$$R : \mathbb{R} \to [0, \infty), \qquad R(u) := \int_0^u r(v)dv.$$

In the general case the stationary measure will be Gibbsian with specifications

$$d\pi\left(\omega_\Lambda \,\middle|\, \omega_{\mathbb{Z}^d \setminus \Lambda}\right) = Z_\Lambda^{-1} \exp\Big\{ -\frac{1}{2} \sum_{\substack{x,y \in \Lambda \\ |x-y|=1}} R(\omega(x) - \omega(y)) \Big\} \times$$

$$\exp\Big\{ - \sum_{\substack{x \in \Lambda, y \in \Lambda^c \\ |x-y|=1}} R(\omega(x) - \omega(y)) \Big\} d\omega_\Lambda.$$

This is the massless free Gaussian field for $r(u) = u$.

In the Hilbert space (Fock space / Wiener space)

$$\mathcal{L}^2(\Omega, \pi) =: \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

The infinitesimal generator acting on $\mathcal{L}^2(\Omega, \pi)$:

$$G = \Delta + \sum_{l=1}^{d} \left(\nabla_l a_l + a_l^* \nabla_l \right) = -S + A_- + A_+,$$

where

$$a_l^* :\omega_{k_1}(x_1) \cdots \omega_{k_n}(x_n): \; = \; :\omega_l(0) \omega_{k_1}(x_1) \cdots \omega_{k_n}(x_n):$$

$$a_l :\omega_{k_1}(x_1) \cdots \omega_{k_n}(x_n): \; = \sum_{m=1}^{n} K_{l k_m}(x_m) :\omega_{k_1}(x_1) \cdots \cancel{\omega_{k_m}(x_m)} \cdots \omega_{k_n}(x_n):$$

Proof: careful use of commutation relations, plus "directional derivative" identity (a la Malliavin calculus).

Back to the displacement:

$$X(t) = M(t) + \int_0^t \varphi(\eta_s) ds$$

$$\varphi_l(\omega) = \omega_l(0).$$

Goals:

$d \geq 3$: ∘ (partial) decorrelation of the two terms on the r.h.s. — easy
∘ diffusive limit (CLT) for the second term on the r.h.s. — try non-reversible Kipnis-Varadhan theory

$d = 2$: ∘ superdiffusive lower bound for variance of the second term — try Landim-Quastel-Salmhofer-Yau method

Theorem ([I. Horváth, B.T., B. Vető (PTRF, 2012)]). .
SRBP, $d \geq 3$*:*
(1)

$$\sigma^2 := d^{-1} \lim_{t\to\infty} t^{-1}\mathbf{E}\Big(\, |X(t)\,|^2 \, \Big) \quad \textit{exists, and}$$

$$1 \leq \sigma^2 \leq 1 + d^{-1} \int_{\mathbb{R}^d} |p|^{-2}\widehat{V}(p)dp < \infty.$$

(2)

$$\frac{X(Nt)}{\sigma\sqrt{N}} \overset{f.d.m.}{\Rightarrow} W(t).$$

Theorem ([I. Horváth, B.T. , B. Vető (PTRF, 2012)]). .
TSAW, $d \geq 3$*:*
(1) holds under very general conditions on the rate function $w(u)$*.*
(2) holds under more restrictive conditions:

$$r(u) = u, \qquad s(u) = a + bu^2 + cu^4.$$

Theorem ([B.T., B. Valkó (JSP, 2012)]). .
SRBP, $d = 2$*:*

$$C_1 t \log\log t \leq \mathbf{E}\Big(|X(t)|^2 \Big) \leq C_2 t \log t,$$

in the sense of Laplace transforms.
Same for TSAW with $r(u) = u$*.*

Proofs:

Diffusive limits in $\mathbf{d \geq 3}$:
Non-reversible Kipnis-Varadhan theory: H_{-1}-bound and *graded sector condition*
[S. Sethuraman, S.R.S. Varadhan, H-T. Yau (2000)]
— with improvement on conditions of applicability.

Superdiffusive lower bound in $\mathbf{d = 2}$:
Variational approach of
[C. Landim, J. Quastel, M. Salmhofer, H-T. Yau (2004)]
— with particularities . . .

Some details, $\mathrm{d} \geq 3$:

CLT for additive functionals of ergodic Markov processes, Kipnis-Varadhan theory:

η_t stationary+ergodic Markov process on $(\Omega, \mathcal{F}, \pi)$, $\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$.

The infinitesimal generator, the semigroup and the resolvent:

$$G = -S + A, \qquad P_t = \exp(tG), \qquad R_\lambda = \int_0^\infty e^{-\lambda s} P_s ds = (\lambda I - G)^{-1}.$$

Let

$$f \in \mathcal{H}, \quad \int_\Omega f d\pi = 0, \qquad u_\lambda := R_\lambda f$$

Wanted: CLT, invariance principle for

$$Y_t^{(N)} := N^{-1/2} \int_0^{Nt} f(\eta_s) ds$$

Theorem (KV). *If the following two limits hold in $\mathcal{H}$:*

$$\lim_{\lambda\to 0} \lambda^{1/2} u_\lambda = 0, \qquad \lim_{\lambda\to 0} S^{1/2} u_\lambda =: v \in \mathcal{H}, \qquad (**)$$

then

$$\sigma^2 := 2 \lim_{\lambda\to 0} (u_\lambda, f) \in [0, \infty),$$

exists and there exists a zero mean, $\mathcal{L}^2$-martingale M_t adapted to the filtration of the Markov process η_t, with stationary and ergodic increments and variance $\mathbf{E}\Big(M(t)^2 \Big) = \sigma^2 t$, such that

$$\lim_{N\to\infty} N^{-1} \mathbf{E}\Big(\Big(\int_0^N f(\eta(s)) ds - M(N) \Big)^2 \Big) = 0.$$

In particular, if $\sigma > 0$, then the finite dimensional marginal distributions of the rescaled process $t \mapsto \sigma^{-1} N^{-1/2} \int_0^{Nt} f(\eta(s))\, \mathrm{d}s$ converge to those of a standard Brownian motion.

Checking (**) is not easy!

Sufficient condition: $\mathbf{H_{-1}}$**-bound + graded sector condition**:

$$(f, S^{-1} f) < \infty \qquad (H_{-1})$$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \qquad S_n : \mathcal{H}_n \to \mathcal{H}_n, \qquad A_{n,\pm} : \mathcal{H}_n \to \mathcal{H}_{n\pm 1}$$

$$\left\| S_{n\pm 1}^{-1/2} A_{n,\pm} S_n^{-1/2} \right\| \le c_n, \quad \text{with} \quad \sum_{n=1}^{\infty} c_n^{-1} = \infty. \qquad \text{(GSC)}$$

. . . , [S. Sethuraman, S.R.S. Varadhan, H-T. Yau (2000)], . . . ,
this version: [I. Horváth, B.T., B. Vető (Bull. IMAS, 2012+)]

Check H_{-1}:

SRBP: estimates for "diffusion in random scenary"

TSAW: estimates for "rw in random environment" + Brascamp-Lieb inequality

Check GSC:

SRBP: computation . . .

$$\left\| \, |\Delta|^{-1/2} a_j^* \restriction_{\mathcal{H}_n} \right\| \leq C n^{1/2}$$

valid only in $d \geq 3$!!!

TSAW: Improvement on GSC + similar computations.

Some details $\mathbf{d}=2$:

The variational problem:

$$(f, R_\lambda f) = \sup_{g\in\mathcal{H}} \Big\{2(f,g) - (g,(\lambda+S)g) - (Ag,(\lambda+S)^{-1}Ag)\Big\}$$

$$\geq \sup_{g\in\mathcal{H}_1} \Big\{2(f,g) - (g,(\lambda+S)g) - (A_+g,(\lambda+S)^{-1}A_+g)\Big\}$$

$$=: \sup_{g\in\mathcal{H}_1} \Big\{2J_1 - J_2 - J_3\Big\}.$$

Let

$$g(\omega) = \sum_{l=1}^{2} \int_{\mathbb{R}^2} u_l(x)\omega_l(x)dx, \qquad \hat{v}(p) := p\cdot\hat{u}(p)$$

Then

$$J_1(\widehat{v}) = \int_{\mathbb{R}^2} \widehat{V}(p) \frac{p_1}{|p|^2} \widehat{v}(p) dp$$

$$J_2(\widehat{v}) = \int_{\mathbb{R}^2} \widehat{V}(p) \frac{\lambda + |p|^2}{|p|^2} \widehat{v}(p)^2 dp$$

$$J_3(\widehat{v}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \widehat{V}(p) \widehat{V}(q) \frac{(p \cdot q)^2}{|p|^2 |q|^2} \frac{1}{\lambda + |p-q|^2} (\widehat{v}(p) - \widehat{v}(q))^2 \, dq dp$$

First two: "diagonal", easy. Third: unpleasant.

$$J_3 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \dots \mathbb{1}_{\{|p-q| \geq |p|/2\}} dq dp + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \dots \mathbb{1}_{\{|p-q| \leq |p|/2\}} dq dp$$

$$=: J_{31} + J_{32}$$

Estimate J_{31} by Schwarz, J_{32} by $(\widehat{v}(p) - \widehat{v}(q))^2 \approx |p-q|^2 |\nabla \widehat{v}(p)|^2$ . . . . and compute.