Condensation and metastability in stochastic particle systems

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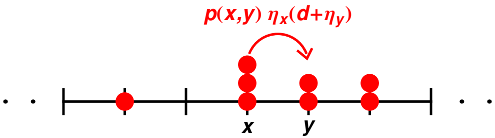
Warwick

in collaboration with Frank Redig, Kiamars Vafayi, Paul Chleboun

October 26, 2012

The inclusion process

Lattice: Λ of size L



State space: $X = \{0, 1, ..\}^\Lambda$

$$\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$$

Jump rates: $p(x,y)\eta_x(d+\eta_y)$, $\quad d > 0$

$p(x,y) \geq 0$ irreducible on Λ, finite range

Generator: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} p(x,y)\,\eta_x(d+\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

[Giardina, Kurchan, Redig, Vafayi (2009); G., Redig, Vafayi (2011); Cocozza-Thivent (1985)]

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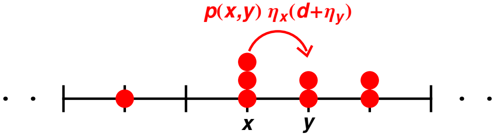

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ZRP: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} p(x,y)\,g(\eta_x)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

The inclusion process

Applications

- 2 sites, N particles: rates $d\,k + k(N-k)$

  $\rightarrow$ multi-species Moran model (related to Wright-Fisher)
- duality with Brownian energy/momentum process

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Condensation

- spatial heterogeneity $p(x, y)$

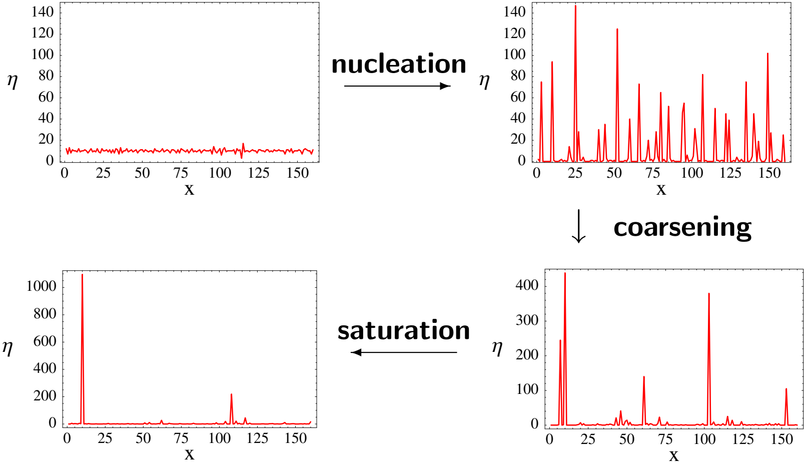
  $\Rightarrow$ condensation on the 'fittest' site

  [Evans (1996); Krug, Ferrari (1996); Benjamini, Ferrari, Landim (1996); Ferrari, Sisko (2007); G., Redig, Vafayi (2011)]
- effective attraction of particles (also ZRP with $g(k) \searrow$)

  $\Rightarrow$ condensation on a random site

  [Evans (2000); Jeon, March, Pittel (2000); G., Schütz, Spohn (2003); Ferrari, Landim, Sisko (2007); Armendáriz, Loulakis (2009); G., Chleboun (2010); Armendáriz, G., Loulakis (2012); Beltran, Landim (2010-12)]

Condensation



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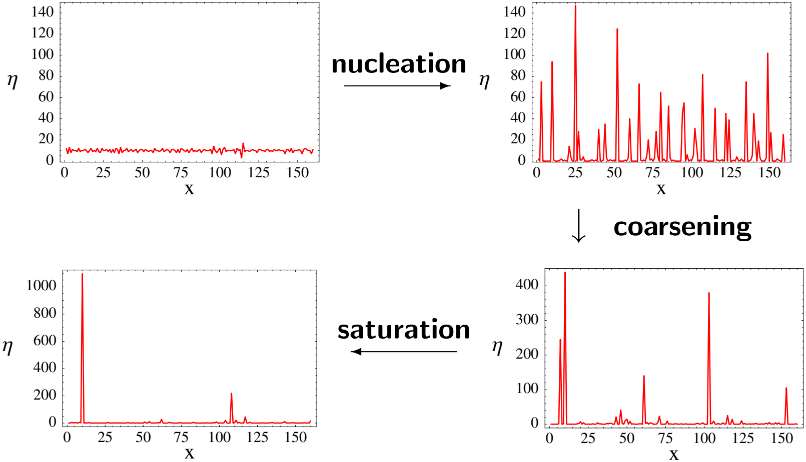
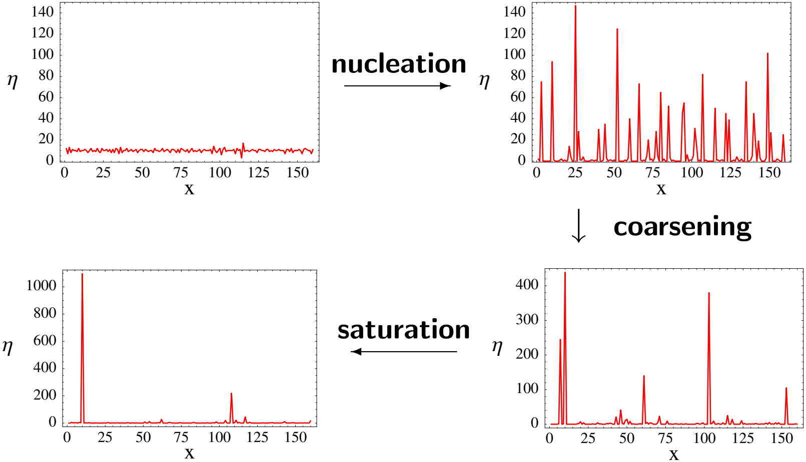


also for ZRP with rates $p(x,y)g(\eta_x)$, $g(k) = 1 + b/k^\gamma$

Condensation



inclusion process $p(x,y)\eta_x(d+\eta_y)$

I Stationary Results

S. G., F. Redig, K. Vafayi, J. Stat. Phys. 142(5), 952-974 (2011)

with Paul Chleboun (PhD thesis)

Stationary distributions

harmonic function $\lambda_x > 0$ $\quad \sum_{x\in\Lambda} \big(\lambda_x p(x,y) - \lambda_y p(y,x)\big) = 0$

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Stationary product measures (grand-canonical)

The IP has SPM $\nu_\phi^\Lambda(d\boldsymbol{\eta}) = \prod_{x\in\Lambda} \nu_\phi^x(\eta_x)\, d\boldsymbol{\eta}$ with

$$\nu_\phi^x(n) = \frac{1}{z_x(\phi)} (\lambda_x \phi)^n\, w(n) \quad \text{with} \quad w(n) = \frac{\Gamma(d+n)}{n!\,\Gamma(d)} \simeq n^{d-1}$$

with $\phi < \phi_c := \inf_{x\in\Lambda} 1/\lambda_x$,

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with $\phi < \phi_c := \inf_{x\in\Lambda} 1/\lambda_x$, provided that

$$\lambda_x p(x,y) = \lambda_y p(y,x) \quad \text{for all } x,y\in\Lambda \qquad (\Rightarrow\ \nu_\phi \text{ reversible})$$

OR

$$\sum_{z\in\Lambda} \big(p(x,z) - p(z,y)\big) = 0 \quad \text{for all } x,y\in\Lambda \qquad (\Rightarrow\ \lambda_x \equiv 1)\ .$$

Homogeneous condensation

Consider $|\Lambda| = L$ with $p(x,y)$ doubly stochastic ,

then $\quad \lambda_x \equiv 1 \,, \quad \phi_c = 1 \quad$ and $\quad z(\phi) = \sum_{n=0}^{\infty} \phi^n \, w(n) = (1-\phi)^{-d}$

density $\quad R(\phi) = \phi \, \partial_\phi \log z(\phi) = d \dfrac{\phi}{1-\phi}$

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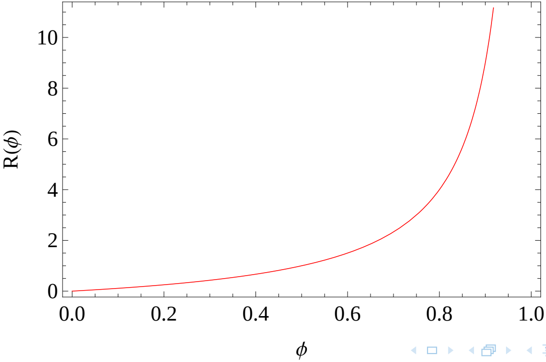

Homogeneous condensation

To get condensation

- rates $\eta_x^{\gamma}(d+\eta_y)^{\gamma}$, $\gamma > 2$ [Waclaw, Evans (2012)]

  then $w(n) \sim n^{(d-1)\gamma}$ and $R(\phi) \nearrow \rho_c < \infty$

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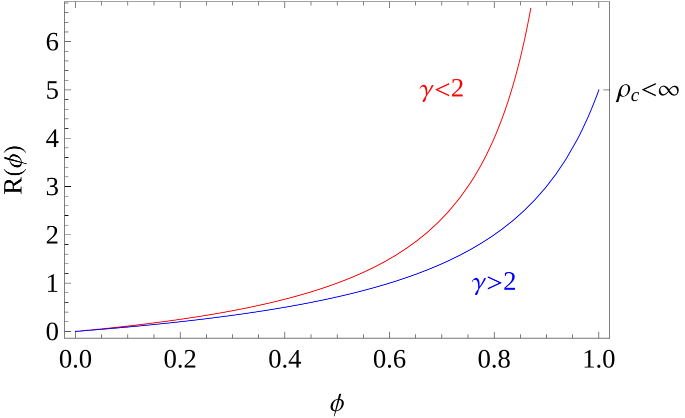

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- take $d = d_L \to 0$ (vanishing diffusion/mutation rate)

  fixing the density

$$R_L(\phi) = d_L \frac{\phi}{1-\phi} = \rho \quad \Rightarrow \quad \phi_L(\rho) = \frac{\rho}{d_L + \rho} \nearrow 1$$

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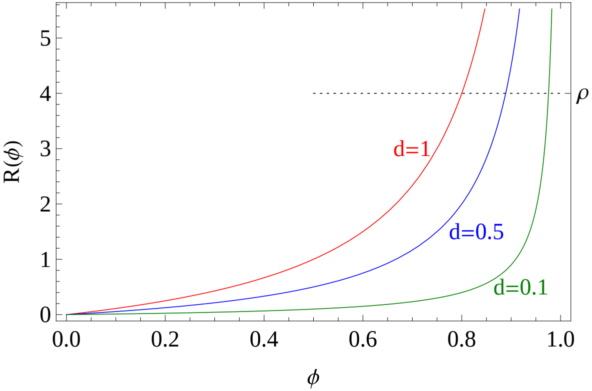

Homogeneous condensation

$$\eta_x^L \xrightarrow{d} 0\ , \quad \text{since} \quad \nu^x_{\phi_L(\rho)}(0) = \nu_\rho^{(L)}(0) \simeq (d_L/\rho)^{d_L} \nearrow 1$$

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Law of large numbers [G., Redig, Vafayi (2011)]

Consider $\eta_1^L, \ldots, \eta_L^L$ iidrvs with distribution $\nu_\rho^{(L)}$. Then, as $L \to \infty$,

$$\frac{1}{L}\sum_{x=1}^L \eta_x^L \to \begin{cases} 0 & , \; d_L \ll 1/L \\ X_\gamma & , \; d_L L \to \gamma \in (0,\infty) \\ \rho & , \; d_L \gg 1/L \end{cases} \;,$$

where $X_\gamma \sim \mathrm{Gamma}(\gamma/2, 2\rho/\gamma)$.

Homogeneous condensation

conditional (canonical) distribution: $\pi_{L,N} = \nu_\rho^{(L)}\big(\,.\, \big| \sum_{x\in\Lambda} \eta_x = N \big)$

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Condensation [Chleboun (PhD)]

Consider $\eta_1^L, \ldots, \eta_L^L$ with conditional distribution $\pi_{L,N}$. Then with $M_L := \max_{x\in\Lambda} \eta_x^L$ we have as $L \to \infty$

$$M_L/L \to \begin{cases} 1\,, & d_L \ll 1/L \\ 0\,, & d_L \gg 1/L \end{cases} .$$

Large deviations and condensation

large deviations of the maximum

$$I_\rho(m) = -\lim_{L\to\infty} \frac{1}{a_L} \log \pi_{L,N}\big(M_L/L \asymp m\big)$$

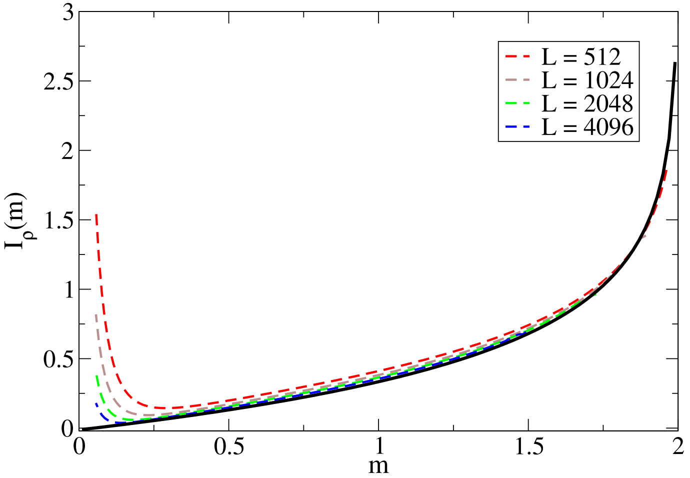
- for $d_L \gg 1/L$ we have $a_L = d_L L$ and

$$I_\rho(m) = \log \frac{\rho}{\rho - m} + \frac{m}{\rho}$$

- for $d_L \ll 1/L$ we have $a_L = \log L$ and

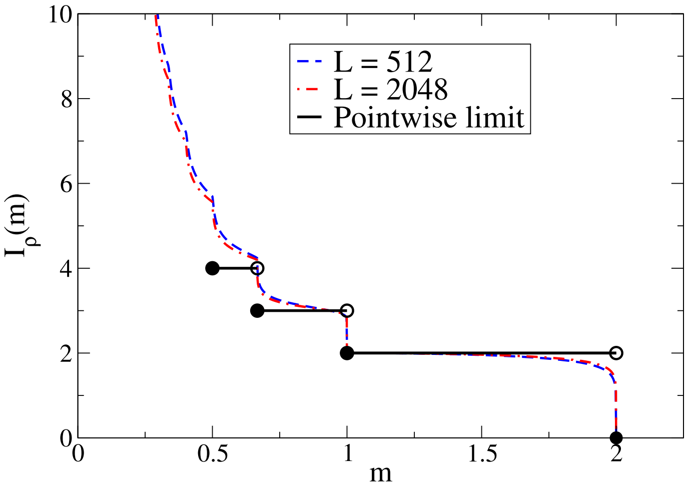
  $I_\rho(m)$ is a degenerate step function

Large deviations and condensation



Homogeneous for $d_L \gg 1/L$

Large deviations and condensation



Condensation for $d_L \ll 1/L \qquad (d_L = L^{-2})$

II Dynamics of condensation

S. G., F. Redig, K. Vafayi, arXiv:1210.3827

Dynamics of condensation

Λ fixed ; $N \to \infty$, $d_N \to 0$ such that $N\, d_N \to \infty$

time scale $\theta_N := 1/d_N$

$$\mathbf{u}^N(t) := \big(\eta_x(\theta_N t)/N : x \in \Lambda\big)$$

process on the simplex $E = \big\{\mathbf{u} \in [0,1]^\Lambda : \sum_{x\in\Lambda} u_x = 1\big\}$

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with **generator** ($p(x,y)$ symmetric)

$$\mathcal{L}_N f(\mathbf{u}) = -\frac{1}{2}\sum_{x,y\in\Lambda} p(x,y)(u_x - u_y)(\partial_{u_x} - \partial_{u_y})f(\mathbf{u})$$
$$+\frac{1}{2}\sum_{x,y\in\Lambda} p(x,y)u_x u_y \theta_N (\partial_{u_x} - \partial_{u_y})^2 f(\mathbf{u}) + O(\theta_N/N) =$$
$$= L\, f(\mathbf{u}) + \theta_N\, L'\, f(\mathbf{u}) + O(\theta_N/N)$$

two-scale system with drift and fast Wright-Fisher diffusion

Dynamics of condensation

WF-diffusion has absorbing set

$$\mathcal{A} := \big\{ \mathbf{u} \in E \,:\, p(x,y)\, u_x u_y = 0 \text{ for all } x, y \in \Lambda \big\} \,.$$

corner points $\mathcal{C} := \big\{ \mathbf{e}_x : x \in \Lambda \big\} \subset \mathcal{A}$

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Theorem 1

Assume $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in \mathcal{C}$. Then $(\mathbf{u}^N(t) : t \geq 0)$ converges weakly on path space to $(\mathbf{u}(t) : t \geq 0)$ on $\mathcal{C}$ with $\mathbf{u}(0) = \mathbf{u}^0$ and generator

$$Af(\mathbf{e}_x) = \sum_{y \in \Lambda} p(x,y)\big(f(\mathbf{e}_y) - f(\mathbf{e}_x)\big) \ .$$

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$$Af(\mathbf{e}_x) = \sum_{y \in \Lambda} p(x,y) \big(f(\mathbf{e}_y) - f(\mathbf{e}_x) \big) \ .$$

If $p(x,y) > 0$ for all $x,y \in \Lambda$ the same holds (with $t > 0$) for general initial conditions $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ with $\mathbb{P}(\mathbf{u}(0) = e_x) = u_x^0$.

Dynamics of condensation

Theorem 2

Let $p(x,y) \in \{0,1\}$, $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ and write

$$\hat{p}(x,y) = \big(1 - p(x,y)\big) \sum_{z\in\Lambda} p(x,z)p(z,y) \geq 0 \ .$$

Dynamics of condensation

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Then $\big(\mathbf{u}^N(t) : t > 0\big)$ converges weakly on path space to $\big(\mathbf{u}(t) : t > 0\big)$ on $\mathcal{A}$ with initial condition $\mathbf{u}(0) \sim \nu_{\mathbf{u}^0}$ and generator

$$\mathcal{A}f(\mathbf{u}) = \sum_{x,y\in\Lambda} \frac{1}{2}\,\hat{p}(x,y)\, u_x u_y \left(\partial_{u_x} - \partial_{u_y}\right)^2 f(\mathbf{u}) + \sum_{y\in\Lambda} \delta_{u_y,0} \Big(\sum_{x\in\Lambda} p(x,y) u_x\Big)\Big(f\Big(\mathbf{u} + \sum_{x\in\Lambda} p(x,y) u_x (\mathbf{e}_y - \mathbf{e}_x)\Big) - f(\mathbf{u})\Big)$$

Method of proof

convergence of the semigroups $e^{t\mathcal{L}_N}$ and $e^{(L+\theta_N L')t}$

Central lemma. For all $t > 0$

$$p(x,y) \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t) u_y^N(t)\big] \to 0 \quad \text{as } N \to \infty ,$$

$$p(x,y) \limsup_{N\to\infty} \theta_N \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t) u_y^N(t)\big] \le C .$$

from **Gronwall**-type estimate due to two-scale structure

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from **Gronwall**-type estimate due to two-scale structure

- **tightness** of $(\mathbf{u}^N(t) : t > 0)$ with Lemma
  for $t = 0$ use right-continuity of paths
- for Theorem 1, characterize through **martingale problem** on $\mathcal{C}$

$$M_x(t) := u_x(t) - u_x(0) - \sum_{y\in\Lambda} \int_0^t p(x,y)\big(u_y(s) - u_x(s)\big)\, ds$$

Method of proof

for Theorem 2 (general initial condition)

- **harmonic projection** $Pf(x) := \int_{\mathcal{A}} f(a)\nu_x(da)$ [Kurtz (1973)]

  $P : C(E,\mathbb{R}) \to \mathcal{H}(E,\mathbb{R})$, $L'(Pf) = 0$ with BC $f(a)$

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  $P : C(E,\mathbb{R}) \to \mathcal{H}(E,\mathbb{R})$, $L'(Pf) = 0$ with BC $f(a)$

- convergence $e^{(L+\theta_N L')t} \to S(t)$ with $S(0) = P$

  semigroup on $\mathcal{H}(E,\mathbb{R})$ with **generator** $Af := (PL)f$

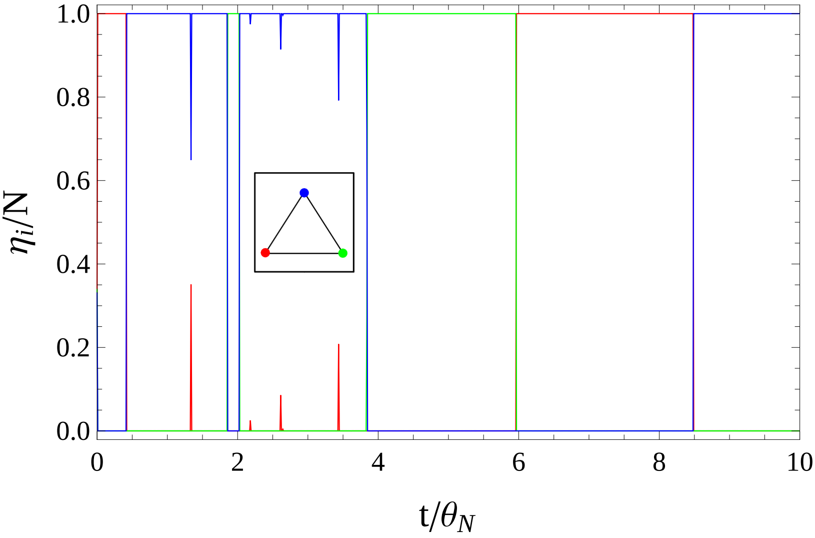
  process on $C(\mathcal{A},\mathbb{R})$ by uniqueness of harmonic functions

- computation $PL = \lim_{h\searrow 0} P\left(\frac{e^{hL}-I}{h}\right)$

  use **martingales** $u_x(t)$, $u_x(t)u_y(t)$ if $p(x,y) = 0$

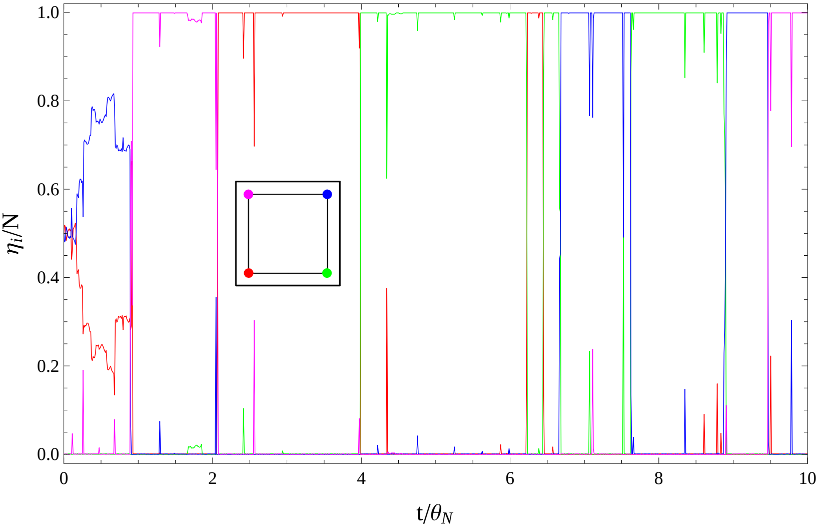
Illustration

3-site ring $N = 10000,\ d_N = 0.001$



Illustration

4-site ring $N = 1000,\ d_N = 0.01$



Conclusion

- stationary results, large deviations
- condensation due to vanishing diffusion rate
- dynamics of condensation on finite lattices

Open questions.

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Thank you!