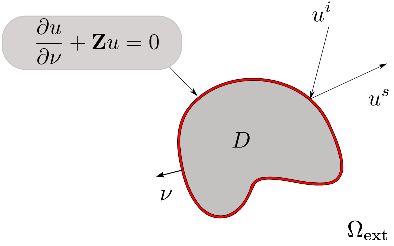
A factorization method for support characterization of an obstacle with a generalized impedance boundary condition

Mathieu Chamaillard, Nicolas Chaulet and Houssem Haddar

INRIA Saclay, France

Inverse problems: modeling and simulation,
Antalya, May 2012

The Generalized Impedance Boundary Conditions in acoustic scattering



$$\Delta u + k^2 u = 0$$

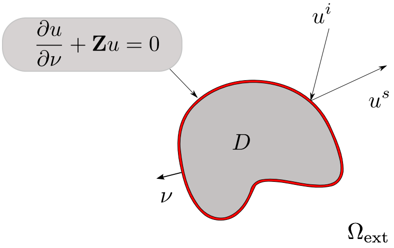
$$u = u^s + u^i$$

$$\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0$$

<u>Context</u>:

- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

The Generalized Impedance Boundary Conditions in acoustic scattering



$$\Delta u + k^2 u = 0$$
$$u = u^s + u^i$$
$$\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0$$

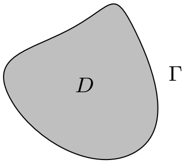
Context:

- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Inverse problem: recover D from the scattered field.

General notions in inverse scattering

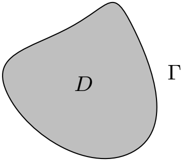
$$\begin{cases} \Delta u^s + k^2 u^s = 0 \\ \frac{\partial u^s}{\partial \nu} + \mathbf{Z} u = -\left(\frac{\partial u^i}{\partial \nu} + \mathbf{Z} u^i \right) \text{ on } \Gamma \\ \lim\limits_{R \to \infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0 \end{cases}$$



$$\boxed{u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty}$$

General notions in inverse scattering

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \\ \frac{\partial u^s}{\partial \nu} + \mathbf{Z}u = -\left(\frac{\partial u^i}{\partial \nu} + \mathbf{Z}u^i\right) \text{ on } \Gamma \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} \left|\frac{\partial u^s}{\partial r} - iku^s\right|^2 ds = 0 \end{cases}$$



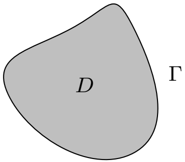
$$\boxed{u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty}$$

For incident plane waves $u^i(z,\hat{\theta}) = e^{ik\hat{\theta}\cdot z}$ we define

$$u^\infty(\hat{x},\hat{\theta}) \in L^2(S^d, S^d).$$

General notions in inverse scattering

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \\ \frac{\partial u^s}{\partial \nu} + \mathbf{Z} u = -\left(\frac{\partial u^i}{\partial \nu} + \mathbf{Z} u^i\right) \text{ on } \Gamma \\ \lim\limits_{R\to\infty} \int_{|x|=R} \left|\frac{\partial u^s}{\partial r} - iku^s\right|^2 ds = 0 \end{cases}$$



$$\boxed{u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty}$$

For incident plane waves $u^i(z, \hat{\theta}) = e^{ik\hat{\theta}\cdot z}$ we define

$$u^\infty(\hat{x}, \hat{\theta}) \in L^2(S^d, S^d).$$

Under minimal assumptions on $\mathbf{Z}$ design a method to recover D from u^∞ for all $(\hat{x}, \hat{\theta})$.

The factorization method: a sampling method

For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$(\mathbf{Z}, D) \longrightarrow u^\infty(\hat{x},\hat{\theta})$$

where u^∞ associated with $u^s(\mathbf{Z}, D)$ is defined in dimension d by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

$$\begin{aligned} F: \quad L^2(S^d) &\longrightarrow L^2(S^d) \\ g &\longmapsto \int_{S^d} u^\infty(\hat{x},\hat{\theta}) g(\hat{\theta})\, d\hat{\theta} \end{aligned}$$

Define the self-adjoint positive operator

$$F_\# := |\Re e(F)| + \Im m(F)$$

$$z \in D \Longleftrightarrow e^{-ik\hat{\theta}\cdot z} \in \mathcal{R}(F_\#^{1/2})$$

The factorization method: a sampling method

For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$(\mathbf{Z}, D) \longrightarrow u^\infty(\hat{x},\hat{\theta})$$

where u^∞ associated with $u^s(\mathbf{Z}, D)$ is defined in dimension d by

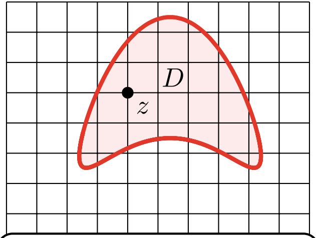
$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

$$F: \quad L^2(S^d) \longrightarrow L^2(S^d)$$
$$g \longmapsto \int_{S^d} u^\infty(\hat{x},\hat{\theta})g(\hat{\theta})\,d\hat{\theta}$$

Define the self-adjoint positive operator

$$F_\# := |\Re e(F)| + \Im m(F)$$

$$z \in D \Longleftrightarrow e^{-ik\hat{\theta}\cdot z} \in \mathcal{R}(F_\#^{1/2})$$



$\exists\, g$ s. t. $F_\#^{1/2} g = e^{-ik\hat{\theta}\cdot z}$

The factorization method: a sampling method

For $u^i(x,\hat\theta) = e^{ik\hat\theta\cdot x}$ define

$$(\mathbf{Z}, D) \longrightarrow u^\infty(\hat x, \hat\theta)$$

where u^∞ associated with $u^s(\mathbf{Z}, D)$ is defined in dimension d by

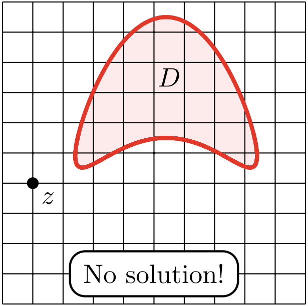
$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^\infty(\hat x) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

$$F: \quad L^2(S^d) \longrightarrow L^2(S^d)$$
$$g \longmapsto \int_{S^d} u^\infty(\hat x, \hat\theta) g(\hat\theta)\, d\hat\theta$$

Define the self-adjoint positive operator

$$F_\# := |\Re e(F)| + \Im m(F)$$

$$\boxed{z \in D \Longleftrightarrow e^{-ik\hat\theta\cdot z} \in \mathcal{R}(F_\#^{1/2})}$$



State of the art

- Factorization method for impenetrable scatterers:
  - Dirichlet and Neumann boundary condition: Kirsch 1998,
  - Impedance boundary condition ($\mathbf{Z} = \lambda$): Kirsch & Grinberg 2002,
- Inverse iterative methods with GIBC: Bourgeois, Chaulet & Haddar 2011–2012.

Outline

1. The GIBC forward problem
2. Characterization of scatterers via the factorization Theorem
3. Numerical examples

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset H^{1/2}(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous and

$$\mathbf{Z}^* u = \overline{\mathbf{Z}\overline{u}}$$

For example for complex functions $(\lambda, \mu) \in (L^\infty(\Gamma))^2$

$$\mathbf{Z} = div_\Gamma \mu \nabla_\Gamma + \lambda$$

$$V = H^1(\Gamma)$$

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset H^{1/2}(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous and

$$\mathbf{Z}^* u = \overline{\mathbf{Z}\overline{u}}$$

- $\Im m \langle \mathbf{Z}u, u \rangle_{V^*,V} \geq 0$ for uniqueness reasons

The GIBC problem writes:

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \qquad \left(f = -\dfrac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right) \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases}$$

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset H^{1/2}(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous and

$$\mathbf{Z}^* u = \overline{\mathbf{Z}\overline{u}}$$

- $\Im m \langle \mathbf{Z} u, u \rangle_{V^*,V} \geq 0$ for uniqueness reasons

The GIBC problem writes:

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \qquad \left(f = -\dfrac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right) \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases}$$

The sign of the real part of the impedance operator is imposed by the volume equation!

Well posedness of the forward problem

A surface equivalent formulation

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \qquad \left(f = -\dfrac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right) \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases}$$

- $n_e\ :\ H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ the exterior DtN map
$$f \longmapsto \frac{\partial u_f}{\partial \nu}$$
where
$$\begin{cases} \Delta u_f + k^2 u_f = 0 \text{ in } \Omega_{\text{ext}}, \\ u_f = f \text{ on } \Gamma, \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} |\partial_r u_f - iku_s|^2 = 0. \end{cases}$$

Well posedness of the forward problem

A surface equivalent formulation

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \qquad \left(f = -\dfrac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right) \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases}$$

- $n_e : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ the exterior DtN map
  $$f \longmapsto \frac{\partial u_f}{\partial \nu}$$

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u^s_\Gamma \in V \text{ such that} \\ (\mathbf{Z} + n_e) u^s_\Gamma = f \end{cases}$$

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \iff (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \text{ such that} \\ (\mathbf{Z} + n_e)u_\Gamma^s = f \end{cases}$$

Theorem

If the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} = C_\mathbf{Z} + K_\mathbf{Z}$ with

- $C_\mathbf{Z} : V \to V^*$ isomorphism,
- $K_\mathbf{Z} : V \to V^*$ compact,

then $(\mathbf{Z} + n_e) : V \to V^*$ is an isomorphism.

Proof

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \qquad \begin{cases} \text{Find } u_\Gamma^s \in V \text{ such that} \\ (\mathbf{Z} + n_e) u_\Gamma^s = f \end{cases}$$

Theorem

If the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} = C_{\mathbf{Z}} + K_{\mathbf{Z}}$ with

- $C_{\mathbf{Z}} : V \to V^*$ isomorphism,
- $K_{\mathbf{Z}} : V \to V^*$ compact,

then $(\mathbf{Z} + n_e) : V \to V^*$ is an isomorphism.

<u>Proof</u>

- $n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous,

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \text{ such that} \\ (\mathbf{Z} + n_e) u_\Gamma^s = f \end{cases}$$

Theorem

If the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} = C_{\mathbf{Z}} + K_{\mathbf{Z}}$ with

- $C_{\mathbf{Z}} : V \to V^*$ isomorphism,
- $K_{\mathbf{Z}} : V \to V^*$ compact,

then $(\mathbf{Z} + n_e) : V \to V^*$ is an isomorphism.

Proof
- $n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous,
- hence $\mathbf{Z} + n_e : V \to V^*$ is Fredholm of index zero.

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \text{ such that} \\ (\mathbf{Z} + n_e) u_\Gamma^s = f \end{cases}$$

Theorem

If the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} = C_\mathbf{Z} + K_\mathbf{Z}$ with

- $C_\mathbf{Z} : V \to V^*$ isomorphism,
- $K_\mathbf{Z} : V \to V^*$ compact,

then $(\mathbf{Z} + n_e) : V \to V^*$ is an isomorphism.

Proof

- $n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous,
- hence $\mathbf{Z} + n_e : V \to V^*$ is Fredholm of index zero.
- Since $\Im m \langle \mathbf{Z}u, u \rangle_{V,V^*} \geq 0$, $(\mathcal{P}_{\text{vol}})$ is injective and so is $\mathbf{Z} + n_e$.

Outline

Implementation of the factorization method

1. First step: formal factorization
   Find two bounded operators $G : \Lambda^* \to L^2(S^d)$ and $T : \Lambda \to \Lambda^*$ such that

$$F = GT^*G^*,$$

and

$$\boxed{z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(G)}$$

where $\phi_z^\infty(\hat{x}) := e^{-ikz\cdot\hat{x}}$.

Implementation of the factorization method

1. First step: formal factorization
   Find two bounded operators $G : \Lambda^* \to L^2(S^d)$ and $T : \Lambda \to \Lambda^*$ such that

$$F = GT^*G^*,$$

   and

$$\boxed{z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(G)}$$

   where $\phi_z^\infty(\hat{x}) := e^{-ikz\cdot\hat{x}}$.

2. Second step: justification
   Find the space Λ and prove that

$$\mathcal{R}(G) = \mathcal{R}(F_\#^{1/2}).$$

$$\boxed{F_\# = |\Re e(F)| + \Im m(F)}$$

Formal factorization

Support characterization

Define the solving operator for the forward problem

$$\begin{aligned} G \ : \ V^* &\longrightarrow L^2(S^d) \\ f &\longmapsto u_f^\infty \end{aligned}$$

where u_f^∞ is the far field associated with the solution to $(\mathcal{P}_{\mathrm{vol}})$ with f in the second hand side.

$$\boxed{z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(G)}$$

<u>Hint:</u>

$$G = G_{\mathrm{Dir}} \circ (\mathbf{Z} + n_e)^{-1}$$

where G_{Dir} is the solving operator for the Dirichlet problem and

$$z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(G_{\mathrm{dir}})$$

Formal factorization

Definition of the central operator

For $G_k(x)$ ($= e^{ik|x|}/|x|$ in dimension 3) the radiating Green function for $\Delta + k^2$ define

$$\mathrm{SL}_k(q)(x) = \int_\Gamma G_k(x-y)q(y)ds(y),\ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\mathrm{DL}_k(q)(x) = \int_\Gamma \frac{\partial G_k(x-y)}{\partial \nu(y)} q(y)ds(y),\ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\begin{cases} \mathcal{S}_k := \mathrm{SL}_k|_\Gamma, \\ \mathcal{D}_k := \mathrm{DL}_k|_\Gamma, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{S}_k' := \partial_\nu \mathrm{SL}_k|_\Gamma, \\ \mathcal{D}_k' := \partial_\nu \mathrm{DL}_k|_\Gamma. \end{cases}$$

Formal factorization

Definition of the central operator

For $G_k(x)$ ($= e^{ik|x|}/|x|$ in dimension 3) the radiating Green function for $\Delta + k^2$ define

$$\mathrm{SL}_k(q)(x) = \int_\Gamma G_k(x-y)q(y)ds(y),\ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\mathrm{DL}_k(q)(x) = \int_\Gamma \frac{\partial G_k(x-y)}{\partial \nu(y)} q(y)ds(y),\ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\begin{cases} \mathcal{S}_k := \mathrm{SL}_k|_\Gamma, \\ \mathcal{D}_k := \mathrm{DL}_k|_\Gamma, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{S}'_k := \partial_\nu \mathrm{SL}_k|_\Gamma, \\ \mathcal{D}'_k := \partial_\nu \mathrm{DL}_k|_\Gamma. \end{cases}$$

Thus we can deduce

$$\boxed{T := \mathbf{Z}\mathcal{S}_k\mathbf{Z}^* + \mathcal{D}'_k + \mathbf{Z}\mathcal{D}_k + \mathcal{S}'_k\mathbf{Z}^*}$$

T has to be defined from Λ to Λ^* for some Hilbert space Λ.
$\Lambda = V$ does not fit because $\mathbf{Z}\mathcal{S}_k\mathbf{Z}^*$ not symmetric!

Formal factorization

Difficulty in the definition of T

$$\boxed{T := \mathbf{Z}\mathcal{S}_k\mathbf{Z}^* + \mathcal{D}_k' + \mathbf{Z}\mathcal{D}_k + \mathcal{S}_k'\mathbf{Z}^*}$$

$$\mathcal{S}_k \ : \ H^s(\Gamma) \to H^{s+1}(\Gamma)$$

We want $T \ : \ \Lambda \to \Lambda^*$.

Consider

$$\mathbf{Z} = \Delta_\Gamma,$$

$$V = H^1(\Gamma),$$

then by taking $\Lambda = V$ *we have:*

$$T \ : \ H^1(\Gamma) \to H^{-2}(\Gamma).$$

Right space : $\Lambda = H^{3/2}(\Gamma) = \Delta_\Gamma^{-1}(H^{-1/2}(\Gamma))$

Careful definition of T and rigorous factorization

If V is compactly embedded into $H^{1/2}(\Gamma)$ define

$$\boxed{\Lambda := \left\{u \in V,\, \mathbf{Z}^* u \in H^{-1/2}(\Gamma)\right\}}$$

with

$$(u, v)_\Lambda := (u, v)_{H^{1/2}(\Gamma)} + (\mathbf{Z}^* u, \mathbf{Z}^* v)_{H^{-1/2}(\Gamma)}.$$

Proposition

- $\mathbf{Z} + n_e : \Lambda \to H^{-1/2}(\Gamma)$ is an isomorphism,
- $G : \Lambda^* \to L^2(S^d)$ is continuous,
- $T : \Lambda \to \Lambda^*$ is continuous,

$$T = \mathbf{Z}\mathcal{S}_k\mathbf{Z}^* + \mathcal{D}_k' + \mathbf{Z}\mathcal{D}_k + \mathcal{S}_k'\mathbf{Z}^*$$

- $F = -GT^*G^*$.

Application of the factorization Theorem

Theorem [Grinberg 2002]

If $F = -GT^*G^*$ with

1. G compact with dense range,
2. $\Re e(T) = C + K$ with C coercive and K compact,
3. $-\Im m(T^*)$ compact and strictly positive on $\overline{\mathcal{R}(G^*)}$,

then $\mathcal{R}(G) = \mathcal{R}(F_{\#}^{1/2})$.

Application of the factorization Theorem

Theorem [Grinberg 2002]

If $F = -GT^*G^*$ with

1. G compact with dense range,
2. $\Re e(T) = C + K$ with C coercive and K compact,
3. $-\Im m(T^*)$ compact and strictly positive on $\overline{\mathcal{R}(G^*)}$,

then $\mathcal{R}(G) = \mathcal{R}(F_\#^{1/2})$.

Conclusion: if k^2 is not an eigenvalue for the interior GIBC problem

$$\boxed{z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(F_\#^{1/2})}$$

V is compactly embedded into $H^{1/2}(\Gamma)$,

$\mathbf{Z}$ is an admissible impedance boundary operator.

What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,

$$\boxed{\begin{aligned} \mathbf{Z} &= \mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot \\ V &= H^1(\Gamma) \end{aligned}}$$

What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,

$$\mathbf{Z} = \mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot$$
$$V = H^1(\Gamma)$$

- Symmetric case: the embedding $H^{1/2}(\Gamma) \subset V$ is compact,

$$\mathbf{Z} = \lambda \cdot$$
$$V = L^2(\Gamma)$$

What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,

$$\mathbf{Z} = \mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot$$
$$V = H^1(\Gamma)$$

- Symmetric case: the embedding $H^{1/2}(\Gamma) \subset V$ is compact,

$$\mathbf{Z} = \lambda \cdot$$
$$V = L^2(\Gamma)$$

- Intermediate case: none of the compact embeddings hold.

$\Re e(T)$ fails to be signed!

Outline

Numerical framework

- $\mathbf{Z} = \mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot,$
- For N=50, the synthetic data are

$$\left\{ u^\infty_{i,j}\left(\frac{2i\pi}{N}, \frac{2j\pi}{N}\right)\right\}_{i,j=1,\cdots,N}$$

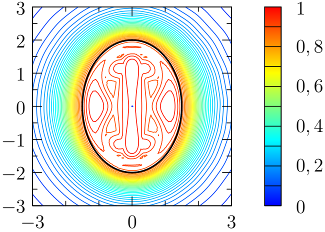
- The wavelength is equivalent to the size of the scatterer
- For each z in a given sampling grid we solve a discrete version of

$$F_\#^{1/2} g_z = \phi_z^\infty$$

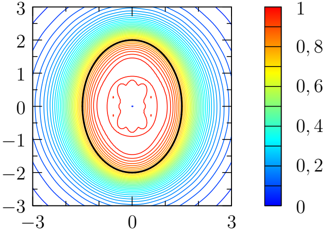
with Tikhonov-Morozov regularization and plot

$$z \longmapsto \frac{1}{\|g_z\|}.$$

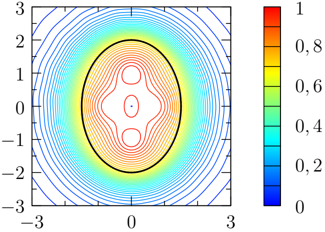
Influence of μ



(a) $\mu = 100$

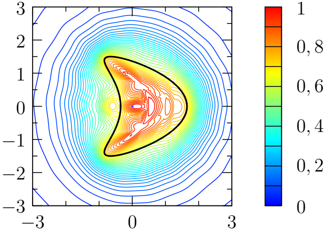


(b) $\mu = 1$

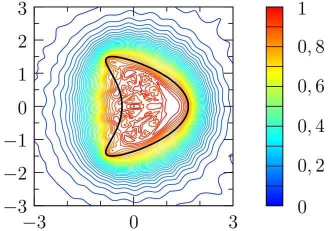


(c) $\mu = 0.1$

Influence of the wavelength

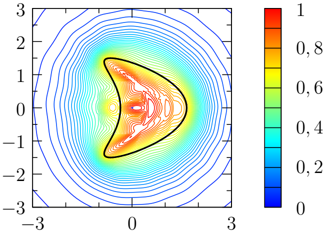


(d) $\mu = 1$, wavelength $= 3$

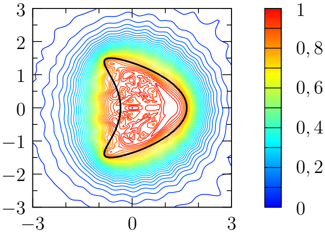


(e) $\mu = 1$, wavelength $= 1.5$

Influence of the wavelength



(d) $\mu = 1$, wavelength $= 3$



(e) $\mu = 1$, wavelength $= 1.5$

Thank you for your attention!