

Asymptotic analysis of transmission eigenvalues for perfect conducting bodies coated by a thin dielectric

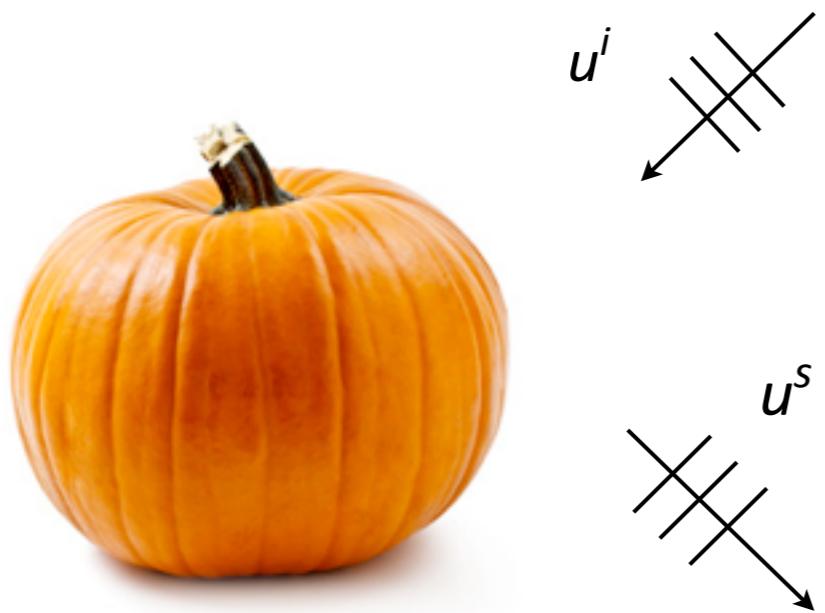
Fioralba Cakoni¹, Nicolas Chaulet², Houssem Haddar³

¹ University of Delaware, USA

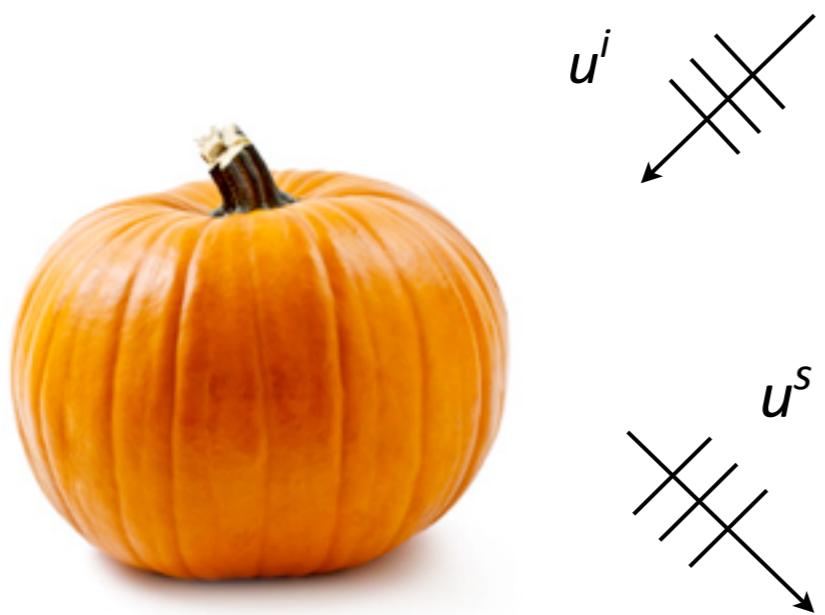
² University College London, UK

³ DEFI team, INRIA Sacalay/Ecole Polytechnique, France

The interior transmission eigenvalue problem

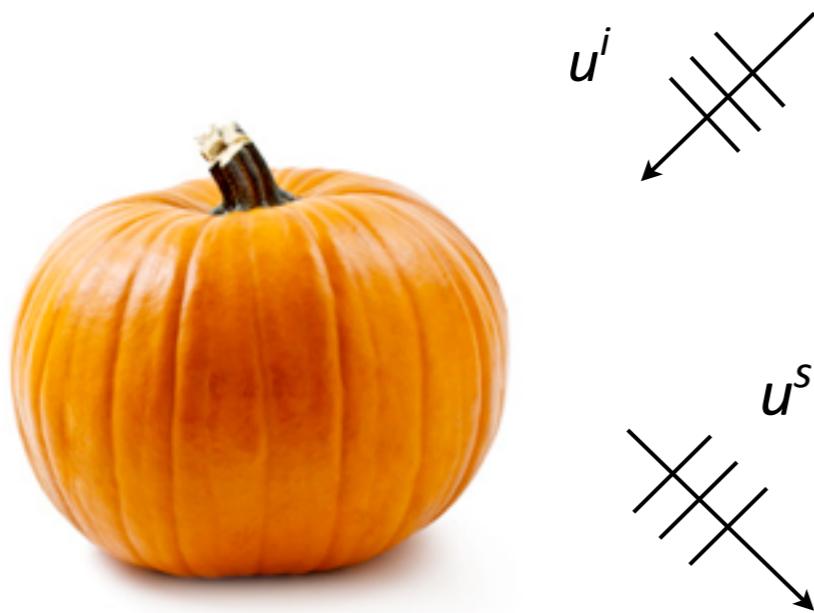


The interior transmission eigenvalue problem



Associated **Interior Transmission Eigenvalue Problem** related to invisibility questions

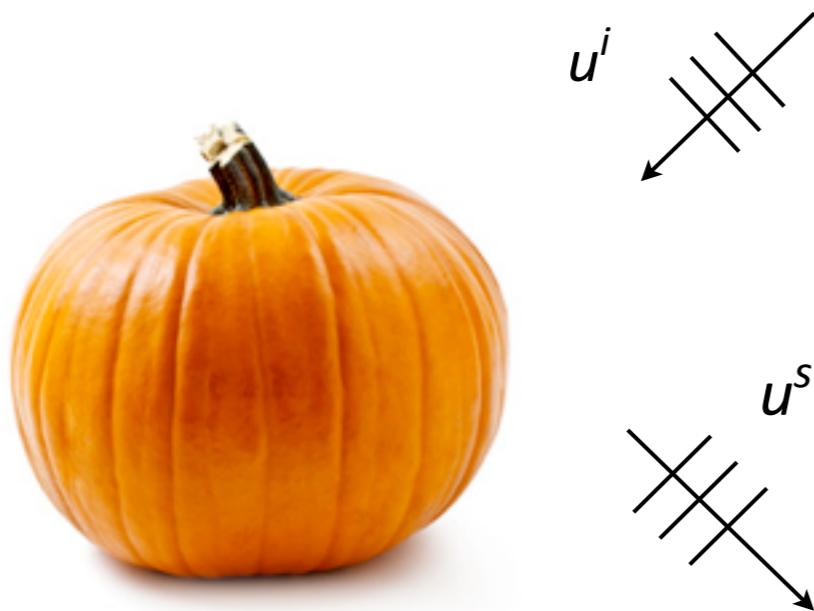
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Bonnet-Ben Dhia, Cakoni, Chesnel, Colton, Guzina, Haddar, Kirsch, Lakshtanov, Moskow, Païvarinta, Robiano, Sylvester...

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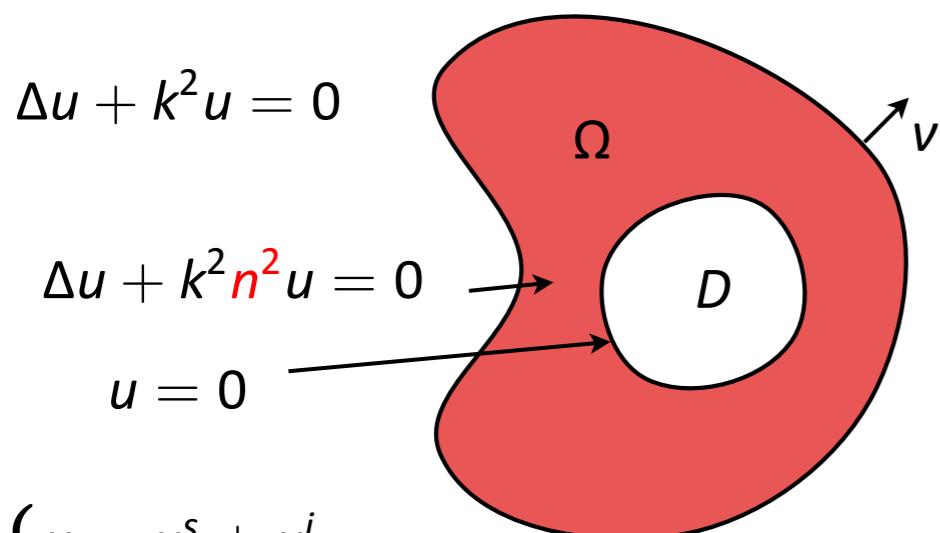
Motivations

- ITE can be determined from multi-static far field data [Cakoni-Colton-Haddar 10]
- ITE contain information on the scatterer (size, physical properties...) [Cakoni, Colton, Gintides]
- Very few results on the use of ITE for inverse problems

Main Difficulty

- Non classical eigenvalue problem
- Only a few general results concerning
existence, discreteness, asymptotic analysis...

An interior Transmission Eigenvalue Problem with a Dirichlet cavity



$$\begin{cases} u = u^s + u^i \\ \Delta u^i + k^2 u^i = 0 \text{ in } \mathbb{R}^d \\ u^s \text{ satisfies the radiation condition} \end{cases}$$

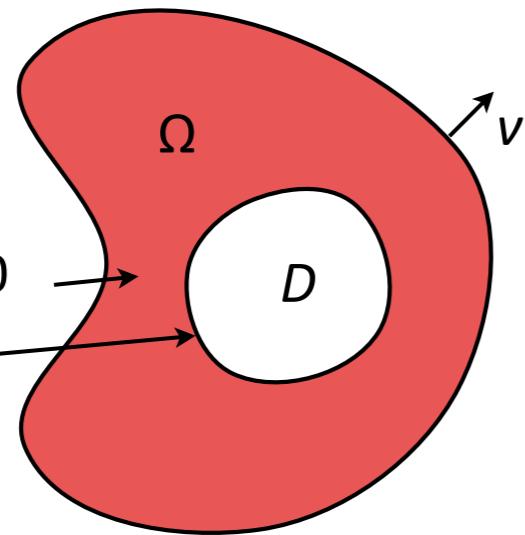
An interior Transmission Eigenvalue Problem with a Dirichlet cavity

$$\Delta u + k^2 u = 0$$

$$\Delta u + k^2 n^2 u = 0$$

$$u = 0$$

$$\begin{cases} u = u^s + u^i \\ \Delta u^i + k^2 u^i = 0 \text{ in } \mathbb{R}^d \\ u^s \text{ satisfies the radiation condition} \end{cases}$$



Def: Interior Transmission Eigenvalue Problem
Find (v, w) and $k^2 > 0$ such that:

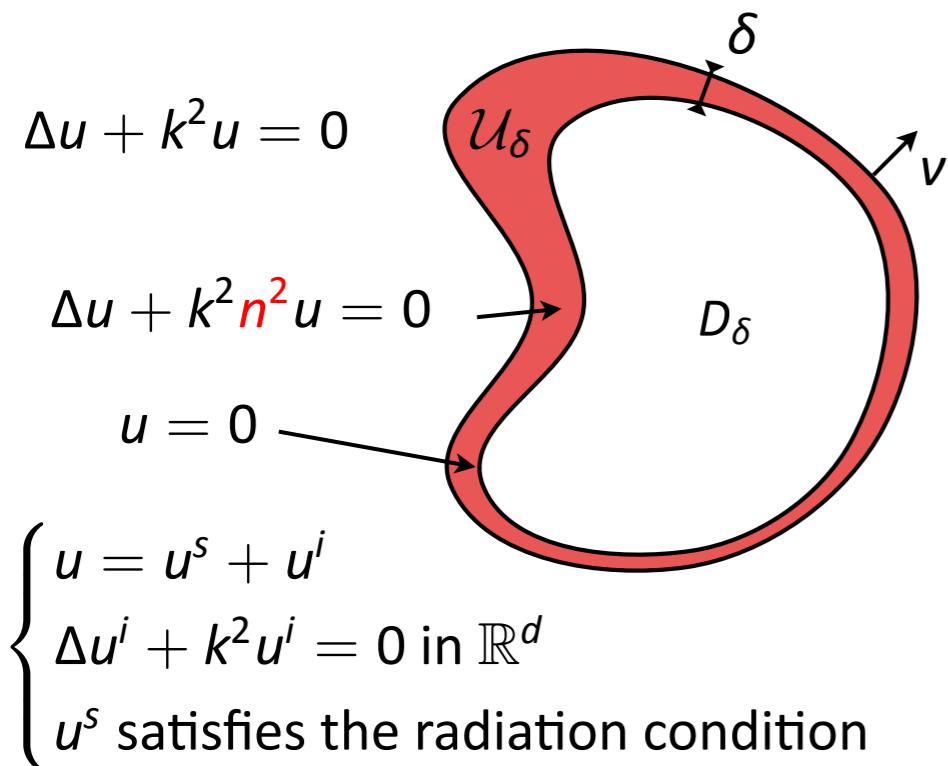
$$\begin{cases} \Delta v + k^2 v = 0 \text{ in } \Omega \\ \Delta w + k^2 n^2 w = 0 \text{ in } \Omega \setminus \bar{D} \\ w = 0 \text{ on } \partial D \\ v = w, \quad \partial_\nu v = \partial_\nu w \text{ on } \partial\Omega \end{cases}$$

k^2 : Interior Transmission Eigenvalue (ITE)
 (v, w) : Interior Transmission Eigenvector

Thm: [Cakoni - Cossonière - Haddar 12]

If $0 < n^2 < 1$ then the ITE exists and form a discrete set of \mathbb{R}^+ .

An interior Transmission Eigenvalue Problem with a Dirichlet cavity



Def: Interior Transmission Eigenvalue Problem

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k_δ^2 : Interior Transmission Eigenvalue (ITE)

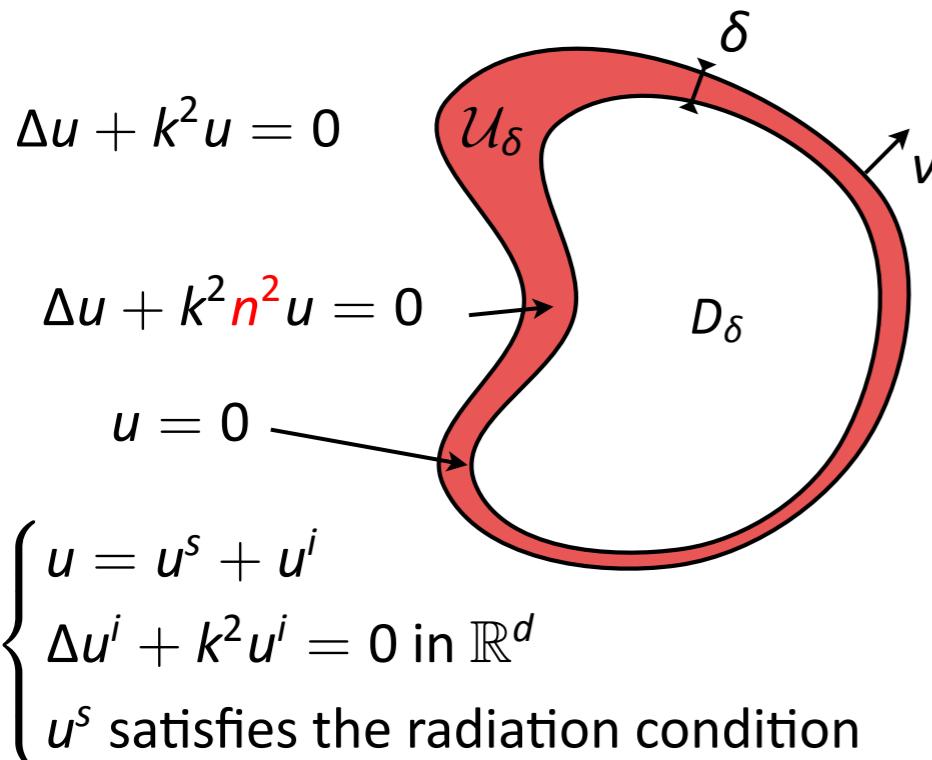
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What Happens if D almost fills in Ω ?

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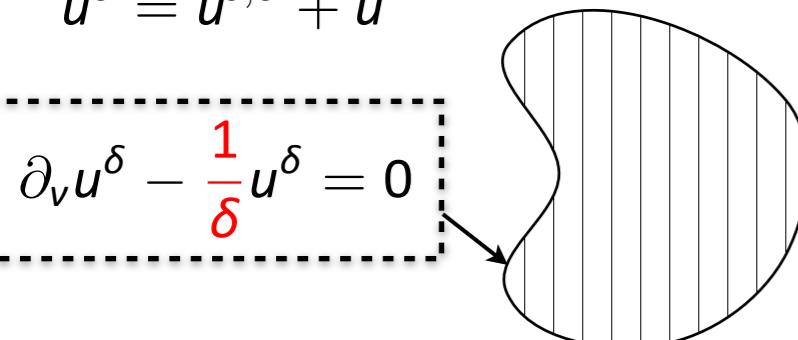
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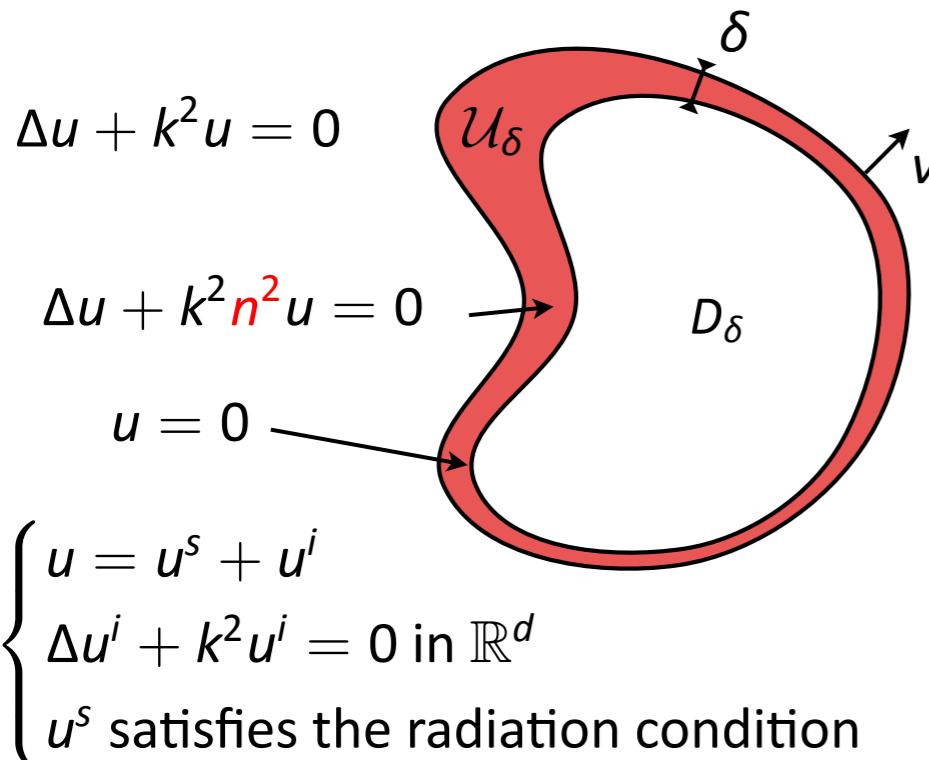
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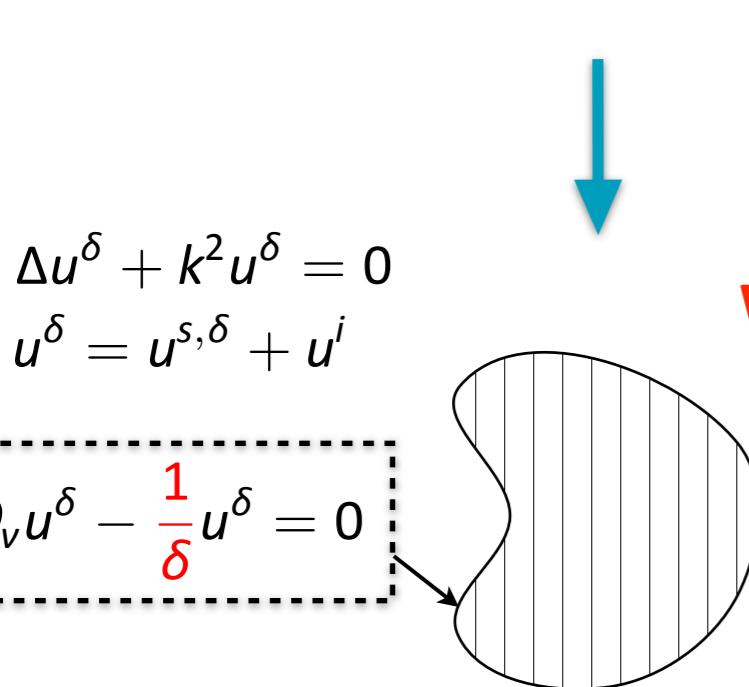
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What Happens if D almost fills in Ω ?

Behaviour of the ITEP for big cavities

(Or thin coatings)

Main result:

Find $(v^\delta, w^\delta, k_\delta^2)$ such that:

$$(ITEP) \begin{cases} \Delta v^\delta + k_\delta^2 v^\delta = 0 \text{ in } \Omega \\ \Delta w^\delta + k_\delta^2 n^2 w^\delta = 0 \text{ in } \mathcal{U}_\delta \\ w^\delta = 0 \text{ on } \partial D_\delta \\ v^\delta = w^\delta, \quad \partial_\nu v^\delta = \partial_\nu w^\delta \text{ on } \partial\Omega \end{cases}$$

$$\delta \rightarrow 0$$

Find $(h^\delta, \Lambda_\delta)$ such that:

$$(RP) \begin{cases} \Delta h^\delta + \Lambda_\delta h^\delta = 0 \text{ in } \Omega \\ \partial_\nu h^\delta - \frac{1}{\delta} h^\delta = 0 \text{ on } \partial\Omega \end{cases}$$

1

Asymptotic analysis (ITEP)

2

Asymptotic analysis of (RP)

Proof of the result above

Formal asymptotic development

Let $(v^\delta, w^\delta, k_\delta^2)$ be an ITE:

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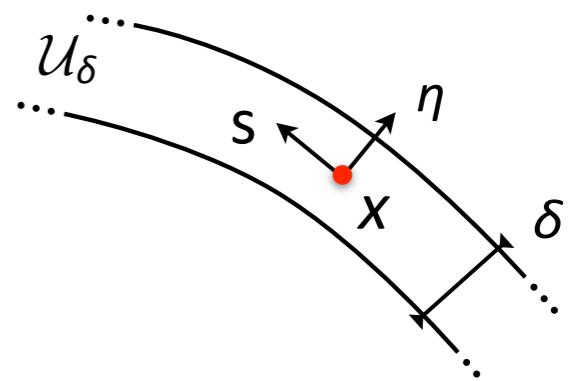
Ansatz for the eigenfunctions and eigenvalue:

$$v^\delta(x) = \sum_k \delta^k v_k(x) \quad \text{for } x \in \Omega$$

$$k_\delta^2 = \sum_k \delta^k \lambda_k$$

$$w^\delta(x) =: \hat{w}^\delta \left(s, \frac{\eta}{\delta} \right) = \sum_k \delta^k w_k(s, \xi) \quad \text{for } x \in \mathcal{U}_\delta$$

$\xi \in [0, 1]$



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Explicit expressions for the various terms:

- λ_0 is a **Dirichlet eigenvalue** for $-\Delta$ in the domain Ω
- λ_1 is the **shape derivative** of λ_0 in the normal direction
- $w_0 = 0$
- ...

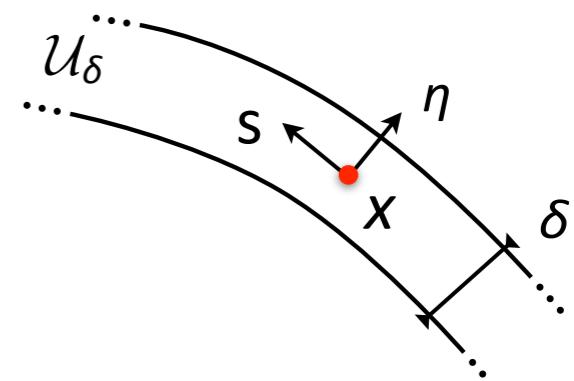
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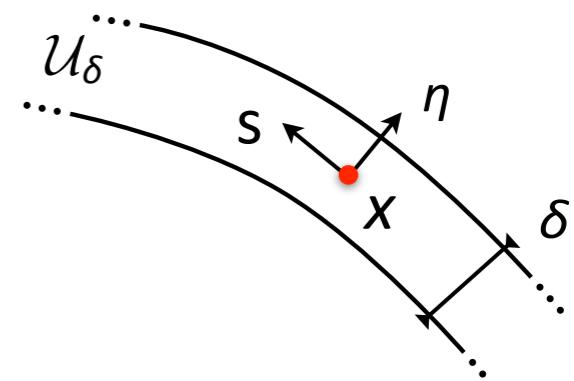
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- Explicit expressions for the various terms:*
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 - λ_1 is the **shape derivative** of λ_0 in the normal direction
 - $w_0 = 0$
 - ...



Goal: Prove that $|k_\delta^2 - \lambda_0 - \delta\lambda_1| \leq C\delta^2$

Establishing convergence rates

Main ideas



- ① Estimates for the solution to some source problem uniformly w.r.t. the layer

①

$$\begin{cases} \Delta v^\delta = f_1 \text{ in } \Omega \\ \Delta w^\delta = f_2 \text{ in } \mathcal{U}_\delta \\ w^\delta = 0 \text{ on } \partial D_\delta \\ v^\delta - w^\delta = f_3, \quad \partial_\nu v^\delta - \partial_\nu w^\delta = f_4 \text{ on } \partial \Omega \end{cases}$$

$$\|v^\delta\| \leq C(\delta)(\|f_1\| + \|f_2\| + \|f_3\| + \|f_4\|)$$

$$\|w^\delta\| \leq C(\delta)(\|f_1\| + \|f_2\| + \|f_3\| + \|f_4\|)$$

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- ② Let $A : H \rightarrow H$ be a compact self adjoint positive operator. If $(u, \lambda) \in H \times \mathbb{R}$ is such that

$$|(Au - \lambda u, v)_H| \leq \varepsilon \|v\|_H \quad \forall v \in H$$

then it exists an eigenvalue Λ and an eigenvector U of A such that

$$|\Lambda - \lambda| \leq \varepsilon \quad \text{and} \quad \|u - U\|_H \leq C\varepsilon.$$

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Main ideas



- ① Estimates for the solution to some source problem uniformly w.r.t. the layer
- ② Estimates for **quasi eigenvectors** for $-\Delta$ with Dirichlet B.C.
- ③ Variational characterisation of the interior transmission eigenvalues

③

$$\begin{cases} \Delta v^\delta + k_\delta^2 v^\delta = 0 & \text{in } \Omega \\ \Delta w^\delta + k_\delta^2 n^2 w^\delta = 0 & \text{in } \mathcal{U}_\delta \\ w^\delta = 0 & \text{on } \partial D_\delta \\ v^\delta = w^\delta, \quad \partial_\nu v^\delta = \partial_\nu w^\delta & \text{on } \partial \Omega \end{cases} \quad \begin{matrix} \times v^\delta \\ \times w^\delta \end{matrix}$$



Integration by part
and proper normalisation

$$k_\delta^2 = \int_{\mathcal{U}_\delta} k_\delta^2 (1 - n^2) |w^\delta|^2 dx + \int_{\mathcal{U}_\delta} |\nabla(v^\delta - w^\delta)|^2 dx + \int_{D_\delta} |\nabla v^\delta|^2 dx$$

Establishing convergence rates

Main ideas



- ① Estimates for the solution to some source problem uniformly w.r.t. the layer
- ② Estimates for **quasi eigenvectors** for $-\Delta$ with Dirichlet B.C.
- ③ Variational characterisation of the interior transmission eigenvalues

Ex: 1st order expansion

$$\Delta v^\delta + k_\delta^2 v^\delta = 0 \quad \text{in } \Omega$$

$$v^\delta = 0 + \mathcal{O}(\delta) \quad \text{on } \partial\Omega$$

$$w^\delta = \delta w^1 + \mathcal{O}(\delta^2) \quad \text{in } \mathcal{U}_\delta$$

Establishing convergence rates

Main ideas



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Establishing convergence rates

Main ideas



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- In the expression of k_δ^2 , replace (w^δ, v^δ) with δw^1 and v^0 \rightarrow $k_\delta^2 = \lambda_0 + \delta \lambda_1 + \mathcal{O}(\delta^2)$

Establishing convergence rates

Main ideas



- ① Estimates for the solution to some source problem uniformly w.r.t. the layer
- ② Estimates for **quasi eigenvectors** for $-\Delta$ with

- One can repeat the procedure to obtain convergence results for higher orders
- The index n^2 appears only in the 3rd order approximation
- δ can be determined by the first ITE

Ex: 1st order ex

$$\begin{aligned}\Delta v^\delta + \lambda_0 v^\delta &= 0 \\ v^\delta &= 0 +\end{aligned}$$
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- In the expression of k_δ^2 , replace (w^δ, v^δ) with δw^1 and v^0



$$k_\delta^2 = \lambda_0 + \delta \lambda_1 + \mathcal{O}(\delta^2)$$

Behaviour of the ITEP for big cavities

(Or thin coatings)

Main result:

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$$\delta \rightarrow 0 \longrightarrow$$

Find $(h^\delta, \Lambda_\delta)$ such that:

$$(RP) \begin{cases} \Delta h^\delta + \Lambda_\delta h^\delta = 0 \text{ in } \Omega \\ \partial_\nu h^\delta - \frac{1}{\delta} h^\delta = 0 \text{ on } \partial\Omega \end{cases}$$

1 Asymptotic analysis (ITEP)

Proof of the result above

2 Asymptotic analysis of (RP)

The Robin eigenvalue problem

Some preliminary comments

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Find $(h^\delta, \Lambda_\delta)$ such that $\forall v \in H^1(\Omega)$

$$\int_{\Omega} \nabla h^\delta \cdot \nabla v dx - \frac{1}{\delta} \int_{\partial\Omega} h^\delta v ds = \Lambda_\delta \int_{\Omega} h^\delta v dx$$

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Non positive definite!

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Shifted variational formulation

$$\int_{\Omega} \nabla h^\delta \cdot \nabla v dx - \frac{1}{\delta} \int_{\partial\Omega} h^\delta v ds + \frac{\alpha}{\delta^2} \int_{\Omega} h^\delta v dx = \left(\Lambda_\delta + \frac{\alpha}{\delta^2} \right) \int_{\Omega} h^\delta v dx$$

Definite positive for some α independent of δ

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Definite positive for some α independent of δ

$\forall \delta$ it exists $(\Lambda_\delta^k)_{k=1,\dots,\infty}$ such that

$$-\frac{\alpha}{\delta^2} \leq \Lambda_\delta^1 \leq \cdots \leq \Lambda_\delta^k \leq \cdots \rightarrow +\infty$$

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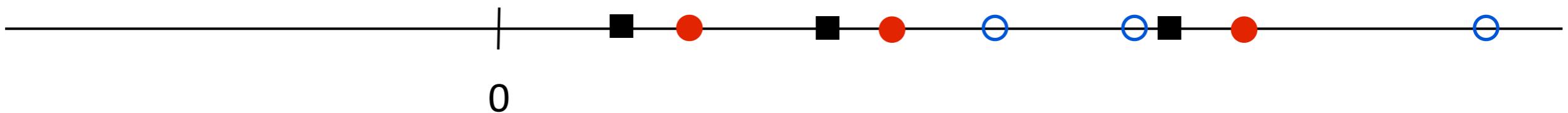
And the k^{th} eigenvalue satisfies [Danners - Kennedy 2010]:

$$\Lambda_\delta^k \underset{\delta \rightarrow 0}{\sim} -\frac{1}{\delta^2}$$

Behaviour of the Robin eigenvalues

$$\delta \rightarrow 0$$

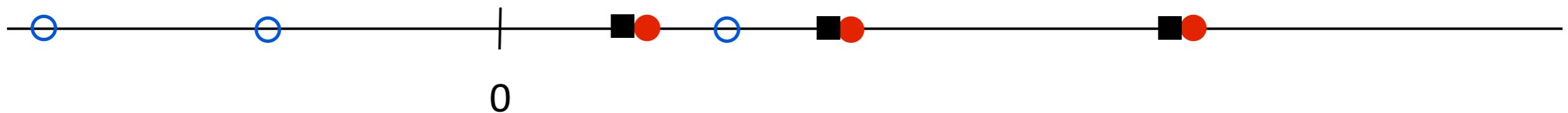
- Unbounded sequences of eigenvalues for the RP
- Bounded sequences of eigenvalues for the RP
- Dirichlet eigenvalues for Ω



Behaviour of the Robin eigenvalues

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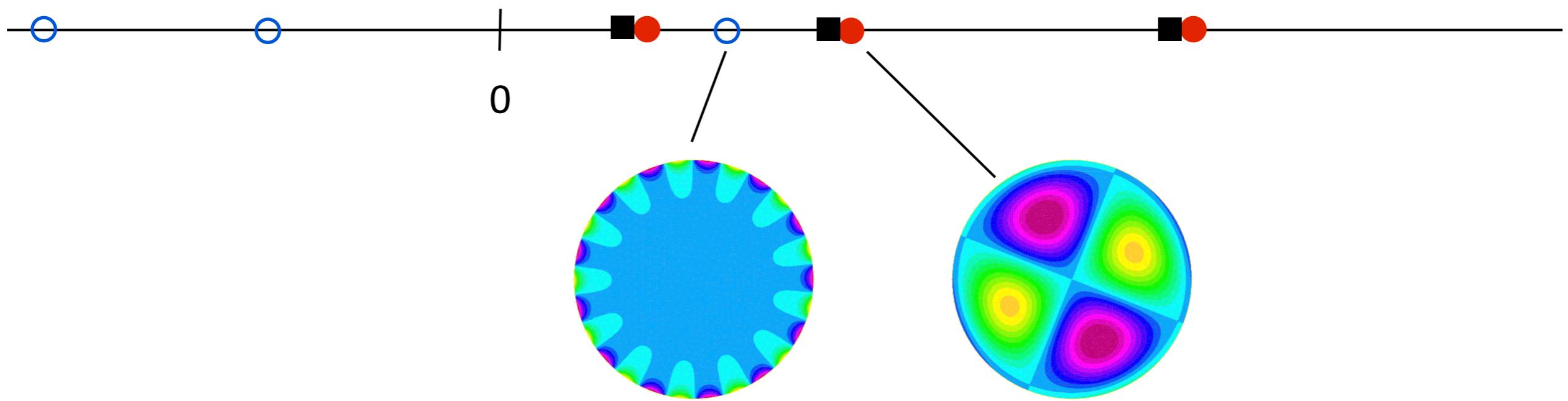
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- Dirichlet eigenvalues for Ω



Behaviour of the Robin eigenvalues

$$\delta \rightarrow 0$$

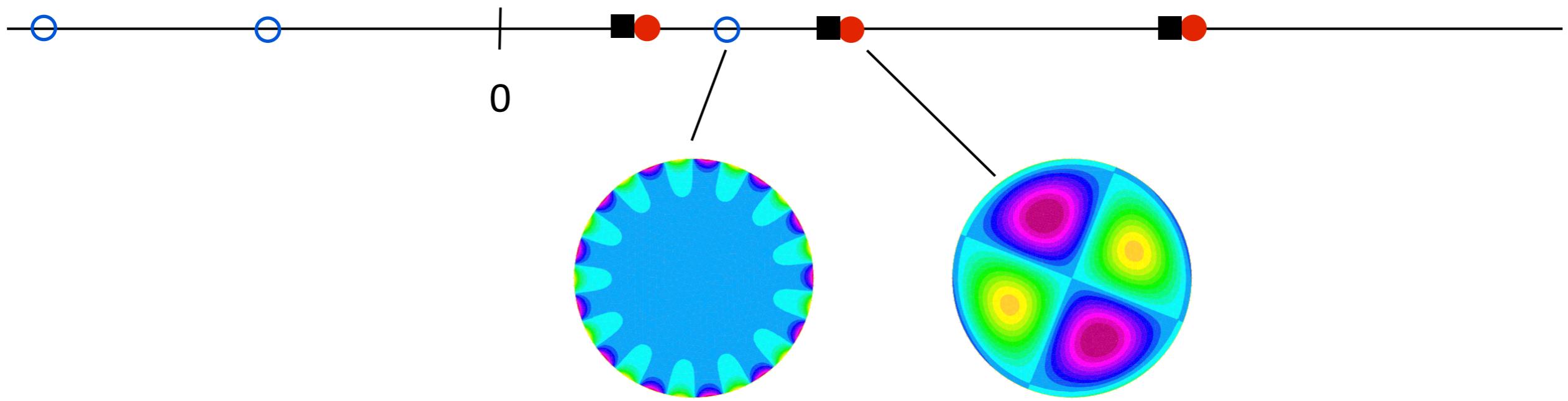
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Two very different behaviours for the eigenvectors / eigenvalues

- layer modes / unbounded eigenvalues
- non localised mode / bounded eigenvalue

Link with the Interior Transmission Problem

A formal approach

Take $(h^\delta, \Lambda_\delta)$ such that

$$\begin{cases} \Delta h^\delta + \Lambda_\delta h^\delta = 0 \text{ in } \Omega \\ \partial_\nu h^\delta - \frac{1}{\delta} h^\delta = 0 \text{ on } \partial\Omega \end{cases}$$

Ansatz for the eigenfunctions and eigenvalue:

$$\Lambda_\delta = \sum_{k=0} \delta^k \Lambda_k$$

$$h^\delta = \sum_{k=0} \delta^k h_k$$

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- Λ_0 is a **Dirichlet eigenvalue** for $-\Delta$ in the domain Ω
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This expansion only captures the non localised modes!

What can be proven?

Link with the ITEP

Prop: [Cakoni- N.C. - Haddar]

- For every $\delta > 0$ it exists an eigenvalue Λ_δ to the RP such that

$$|\Lambda_\delta - \lambda_0 - \delta\lambda_1| \leq C\delta^2$$

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- Can we have a development for the unbounded sequences

$$\Lambda_\delta = -\frac{1}{\delta^2} + ??$$

- Expansion of the eigenvectors?

Concluding remarks and future work

- Asymptotic development of the first ITE with convergence estimates
- Explicit formula to obtain the thickness of a layer from the first ITE:

$$\delta \simeq \frac{k_\delta^2 - \lambda_0}{\lambda_1}$$

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- * Use the (RP) to obtain more from the ITE?

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Thank you!