Stability results for the identification of generalized impedance boundary coefficients

Laurent Bourgeois, Nicolas Chaulet and Houssem Haddar

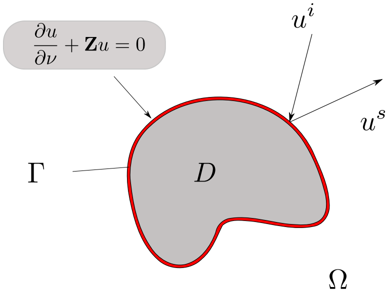
INRIA Saclay

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

centre de recherche SACLAY - ÎLE-DE-FRANCE

WAVES 2011

The Generalized Impedance Boundary Conditions in acoustic scattering



$$\Delta u + k^2 u = 0$$
$$u = u^s + u^i$$
$$\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0$$

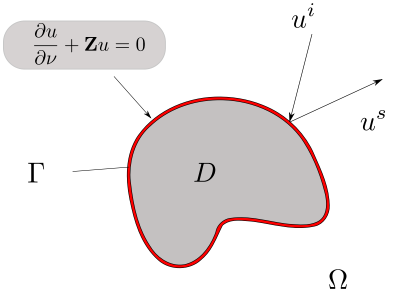
Context:

- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Advantages:

- Cheaper direct computation (no mesh refinement)

The Generalized Impedance Boundary Conditions in acoustic scattering



$$\Delta u + k^2 u = 0$$
$$u = u^s + u^i$$
$$\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0$$

Context:

- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Advantages:

- Cheaper direct computation (no mesh refinement)
- Inverse problem less unstable

Inverse problem: recover $\mathbf{Z}$ from the scattered field.

Example of generalized impedance boundary condition

Most commonly used impedance operator:

$$\mathbf{Z} = \lambda \quad \text{a function.}$$

A more general model:

$$\boxed{\mathbf{Z}u = \mathrm{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u.}$$

For example the first order approximation of the field given by sound hard obstacles with thin coatings involves

$$\mathbf{Z}u = \mathrm{div}_\Gamma(\delta \nabla_\Gamma u) + k_c^2 \delta u$$

where

- k_c^2 is the wave number inside the coating,
- and δ is the width of the coating (non necessarily constant).

The forward problem

Find $u = u^s + u^i$ such that

$$\boxed{u^s \in \left\{ v \in \mathcal{D}'(\Omega)\,,\ \varphi v \in\ H^1(\Omega)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\partial D}\ \in\ H^1(\partial D) \right\}}$$

and

$$(\mathcal{P}) \quad \left\{ \begin{array}{l} \Delta u + k^2 u = 0 \quad \text{in } \Omega := \mathbb{R}^d \setminus \overline{D} \\ \dfrac{\partial u}{\partial \nu} + \mathrm{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \partial D \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0. \end{array} \right.$$

u exists and is unique if

- $\Im m(\lambda) \geq 0,\ \Im m(\mu) \leq 0$ a.e. on ∂D (physical assumption)
- $\Re e(\mu) \geq c$ a.e. on ∂D for $c > 0$.

The inverse coefficient problem

The far field map

For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$T : (\lambda, \mu, \partial D, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})$$

where u^∞ associated with u^s is defined in dimension d by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

The inverse coefficient problem

Given a geometry ∂D and N incident directions $(\hat{\theta}_j)_{j=1,\cdots,N}$ and the corresponding far fields, retrieve λ and μ,

$$(u^\infty(\cdot, \hat{\theta}_j))_{j=1,\cdots,N} \mapsto (\lambda, \mu).$$

Uniqueness fails in general for constant μ and non–constant λ with a single incident wave!

The inverse coefficient problem

The inverse coefficient problem

Given a geometry ∂D and N incident directions $(\hat{\theta}_j)_{j=1,\cdots,N}$ and the corresponding far fields, retrieve λ and μ,

$$(u^\infty(\cdot,\hat{\theta}_j))_{j=1,\cdots,N} \mapsto (\lambda,\mu).$$

Main objective

Quantify the stability of this inversion with respect to an inexact knowledge of the geometry

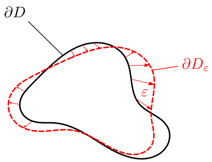
Motivations:

- Recover (λ,μ) on a geometry reconstructed using a qualitative method.
- Understand the convergence of the reconstruction of $(\partial D,\lambda,\mu)$ using an iterative procedure where (λ,μ) and the geometry ∂D are updated alternatively.

Outline

1. Stability with respect to an inexact geometry
2. A stability estimate on an exact geometry
3. Numerical experiments

The inverse coefficient problem on an inexact geometry



- The far–field data correspond to $(\lambda, \mu, \partial D)$ and we search $(\lambda_\varepsilon, \mu_\varepsilon)$ that approximate (λ, μ) on

$$\partial D_\varepsilon := f_\varepsilon(\partial D) \quad \text{with} \quad f_\varepsilon := \mathrm{Id} + \varepsilon.$$

The stability issue

For some small δ and ε if one finds $(\lambda_\varepsilon, \mu_\varepsilon) \in (L^\infty(\partial D_\varepsilon))^2$ such that

$$\|T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|_{L^2(S^{d-1})} \leq \delta$$

do we have

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\|_{L^\infty(\partial D)} + \|\mu_\varepsilon \circ f_\varepsilon - \mu\|_{L^\infty(\partial D)} \leq G(\delta, \varepsilon)$$

for some function $G(\delta, \varepsilon) \underset{\delta,\varepsilon \to 0}{\longrightarrow} 0$.

δ can be the tolerance of some reconstruction procedure.

A stable reconstruction on an inexact geometry

Assume that the inverse problem is stable for an exact geometry: for (λ, μ) in a compact set $K \subset (L^\infty(\partial D))^2$ and $(\widetilde{\lambda}, \widetilde{\mu}) \in K$, there exists C_K such that

$$\boxed{\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|.}$$

Theorem

For small ε and for all $(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon) \in K$ that satisfy

$$\|T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\| \leq \delta$$

we have

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\| + \|\mu_\varepsilon \circ f_\varepsilon - \mu\| \leq C_K(\delta + \|\varepsilon\|).$$

Proof

Main tools:

1. Stability for the exact geometry:

$$\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|,$$

2. continuity of the forward problem w.r.t. the geometry

$$\|T(\lambda \circ f_\varepsilon^{-1}, \mu \circ f_\varepsilon^{-1}, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\| \leq C_K \|\varepsilon\|$$

C_K does not depend on (λ, μ).

Proof

<u>Main tools:</u>

1. Stability for the exact geometry:

$$\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|,$$

2. continuity of the forward problem w.r.t. the geometry

$$\|T(\lambda \circ f_\varepsilon^{-1}, \mu \circ f_\varepsilon^{-1}, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\| \leq C_K \|\varepsilon\|$$

C_K does not depend on (λ, μ).

$\widetilde{\lambda} = \lambda_\varepsilon \circ f_\varepsilon$ and $\widetilde{\mu} = \mu_\varepsilon \circ f_\varepsilon$ in 1 gives

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\| + \|\mu_\varepsilon \circ f_\varepsilon - \mu\| \leq C_K \; \|T(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon, \partial D) - T(\lambda, \mu, \partial D)\|$$

Proof

Main tools:

1. Stability for the exact geometry:

$$\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|,$$

2. continuity of the forward problem w.r.t. the geometry

$$\|T(\lambda \circ f_\varepsilon^{-1}, \mu \circ f_\varepsilon^{-1}, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\| \leq C_K \|\varepsilon\|$$

C_K does not depend on (λ, μ).

$\widetilde{\lambda} = \lambda_\varepsilon \circ f_\varepsilon$ and $\widetilde{\mu} = \mu_\varepsilon \circ f_\varepsilon$ in 1 gives

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\| + \|\mu_\varepsilon \circ f_\varepsilon - \mu\| \leq C_K \boxed{\|T(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon, \partial D) - T(\lambda, \mu, \partial D)\|}$$

$$\leq$$

$$\boxed{\|T(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon, \partial D) - T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon)\|} + \boxed{\|T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|}$$

Proof

Main tools:

1. Stability for the exact geometry:

$$\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|,$$

2. continuity of the forward problem w.r.t. the geometry

$$\|T(\lambda \circ f_\varepsilon^{-1}, \mu \circ f_\varepsilon^{-1}, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\| \leq C_K \|\varepsilon\|$$

C_K does not depend on (λ, μ).

$\widetilde{\lambda} = \lambda_\varepsilon \circ f_\varepsilon$ and $\widetilde{\mu} = \mu_\varepsilon \circ f_\varepsilon$ in 1 gives

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\| + \|\mu_\varepsilon \circ f_\varepsilon - \mu\| \leq C_K \boxed{\|T(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon, \partial D) - T(\lambda, \mu, \partial D)\|}$$

$$\leq$$

$$\boxed{\|T(\lambda_\varepsilon \circ f_\varepsilon, \mu_\varepsilon \circ f_\varepsilon, \partial D) - T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon)\|} + \boxed{\|T(\lambda_\varepsilon, \mu_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|}$$

$\leq C_K \|\varepsilon\|$ by 2 $\qquad \leq \delta$

Outline

1. Stability with respect to an inexact geometry

2. A stability estimate on an exact geometry

3. Numerical experiments

Stability results for exact geometries

$$\|\lambda - \widetilde{\lambda}\| + \|\mu - \widetilde{\mu}\| \leq C_K \|T(\lambda, \mu, \partial D) - T(\widetilde{\lambda}, \widetilde{\mu}, \partial D)\|$$

- The case $\mu = 0$: Sincich [07].
- The case $\mu \neq 0$ and λ known: Bourgeois, Chaulet and Haddar (accepted).

Other stability results

- The case $\mu = 0$
  - Log–type global stability estimate for continuous λ: Sincich [06], Labreuche [99].
- The case $\mu \neq 0$
  - Local stability estimate for λ and μ: Bourgeois and Haddar [10].

Lipschitz stability for piecewise constant μ

- Let $(\partial D_i)_{i=1,\cdots,I}$ be a partition of ∂D,
- let K_I be a compact subset of $L^\infty(\partial D)^2$ such that if $(\lambda,\mu) \in K_I$,

$$\lambda(x) = \sum_{i=1}^{I} \lambda_i \chi_{\partial D_i}(x)\,, \qquad \mu(x) = \sum_{i=1}^{I} \mu_i \chi_{\partial D_i}(x)$$

  and assumptions for the forward problem are satisfied.
- $\forall i = 1, \cdots, I$ it exists $S_i \subset \partial D_i$ such that $\forall (\lambda,\mu) \in K_I$

$$(\mathcal{H}) \qquad \Delta_\Gamma u_{\lambda,\mu} \neq 0 \quad \text{on } S_i.$$

Stability for μ

There exists $C_{K_I} > 0$ such that for all (λ,μ_1) and (λ,μ_2) in K_I,

$$\|\mu^1 - \mu^2\| \leq C_{K_I} \|T(\lambda,\mu^1,\partial D) - T(\lambda,\mu^2,\partial D)\|.$$

Sketch of the proof for constant μ

Denote u^j the total field associated with $(\lambda, \mu^j, \partial D)_{j=1,2}$.

The auxiliary function

$$v := \frac{u^2 - u^1}{|\mu^2 - \mu^1|}$$

is a radiating solution of the Helmholtz equation with

$$\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v = \frac{\mu^1 - \mu^2}{|\mu^2 - \mu^1|} \Delta_\Gamma u^2.$$

Objective:

1. bound the left hand side from above by an increasing function of $\|v^\infty\|$,
2. bound the right hand side from below by $c_{K_I} > 0$,

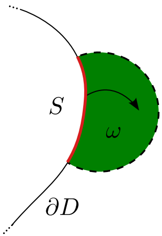
to obtain

$$\boxed{\|v^\infty\| \geq c_{K_I} > 0.}$$

Sketch of the proof for constant μ

$$\left|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right| = \left|\Delta_\Gamma u^2\right| \quad \text{on} \quad \partial D.$$

Upper bound for $N(\|v\|_S) := \left\|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)}$:



Propagation of the smallness:

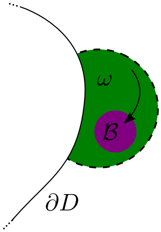
$$N(\|v\|_S) \le f_1\left(\|v\|_{H^1(\omega)}\right)$$

trace inequality

Sketch of the proof for constant μ

$$\left|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right| = \left|\Delta_\Gamma u^2\right| \quad \text{on} \quad \partial D.$$

Upper bound for $N(\|v\|_S) := \left\|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)}$:



Propagation of the smallness:

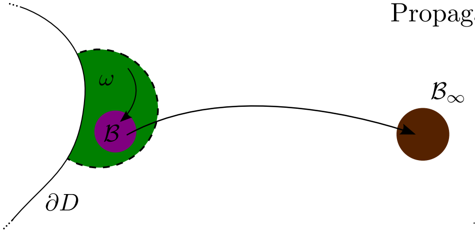
$$\begin{aligned} N(\|v\|_S) &\leq f_1\left(\|v\|_{H^1(\omega)}\right) \\ &\leq f_2\left(\|v\|_{H^1(\mathcal{B})}\right) \end{aligned}$$

Carleman estimate near the boundary

Sketch of the proof for constant μ

$$\left|\mathrm{div}_\Gamma(\mu^1\nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right| = \left|\Delta_\Gamma u^2\right| \quad \text{on} \quad \partial D.$$

Upper bound for $N(\|v\|_S) := \left\|\mathrm{div}_\Gamma(\mu^1\nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)}$:



Propagation of the smallness:

$$\begin{aligned} N(\|v\|_S) &\leq f_1\left(\|v\|_{H^1(\omega)}\right) \\ &\leq f_2\left(\|v\|_{H^1(\mathcal{B})}\right) \\ &\leq f_3\left(\|v\|_{H^1(\mathcal{B}_\infty)}\right) \end{aligned}$$

Interior Carleman estimate

and continuity of the near field to far field map:

$$\boxed{N(\|v\|_S) \leq f\left(\|v^\infty\|\right)}$$

Sketch of the proof for constant μ

$$\left|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right| = \left|\Delta_\Gamma u^2\right| \quad \text{on} \quad \partial D.$$

Upper bound for $N(\|v\|_S) := \left\|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)}$:

$$\boxed{\left\|\mathrm{div}_\Gamma(\mu^1 \nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)} \leq f\left(\|v^\infty\|\right)}$$

for f an increasing function independent of μ^1 and μ^2.

Sketch of the proof for constant μ

$$\left|\mathrm{div}_\Gamma(\mu^1\nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right| = \left|\Delta_\Gamma u^2\right| \quad \text{on} \quad \partial D.$$

Upper bound for $N(\|v\|_S) := \left\|\mathrm{div}_\Gamma(\mu^1\nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)}$:

$$\boxed{\left\|\mathrm{div}_\Gamma(\mu^1\nabla_\Gamma v) + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(S)} \leq f\left(\|v^\infty\|\right)}$$

for f an increasing function independent of μ^1 and μ^2.

Lower bound for $\|\Delta_\Gamma u^2\|_{L^2(S)}$:

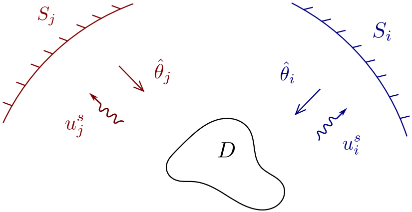
$$(\mathcal{H}) \quad \Longrightarrow \quad \min_{(\lambda,\mu)\in K_I} \|\Delta_\Gamma u_{\lambda,\mu}\|_{L^2(S)} \neq 0$$

$$\Longrightarrow \|\Delta_\Gamma u^2\|_{L^2(S)} \geq c_{K_I} > 0.$$

Outline

1. Stability with respect to an inexact geometry
2. A stability estimate on an exact geometry
3. Numerical experiments

A steepest descent method to solve the inverse coefficient problem



Assume that λ is known and minimize

$$F(\mu) := \frac{1}{2}\sum_{j=1}^{I} \|T(\mu,\hat{\theta}_j) - u_{\text{obs}}^{\infty}(\cdot,\hat{\theta}_j)\|_{L^2(S_j)}^2.$$

$\hookrightarrow$ A regularization procedure is needed!

A steepest descent method to solve the inverse coefficient problem

First strategy: regularization on the gradient

Minimize

$$F(\mu) := \frac{1}{2}\sum_{j=1}^{I} \|T(\mu,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)},$$

by taking $\mu_{n+1} = \mu_n + \delta\mu$ where for every ϕ in some finite dimensional space

$$\eta_\mu \int_{\partial D} \nabla_\Gamma(\delta\mu)\cdot\nabla_\Gamma\phi\, ds + \int_{\partial D} \delta\mu\phi\, ds = -\alpha_\mu\, F'(\mu_n)\cdot\phi.$$

Second strategy: total variation regularization

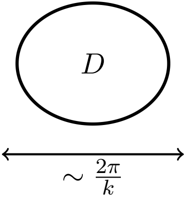
Minimize

$$F_{\eta_\mu}(\mu) := \frac{1}{2}\sum_{j=1}^{I} \|T(\mu,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)} + \eta_\mu |\nabla_\Gamma \mu|_{L^1(\partial D)}$$

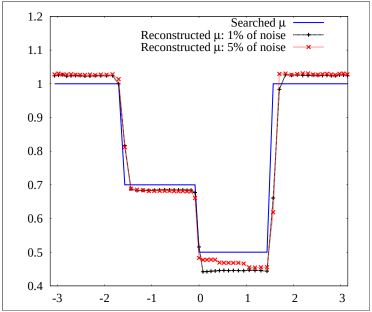
and $\mu_{n+1} = \mu_n + \delta\mu$ with $\displaystyle\int_{\partial D} \delta\mu\phi\, ds = -\alpha_\mu\, F'_{\eta_\mu}(\mu_n)\cdot\phi.$

Numerical reconstruction

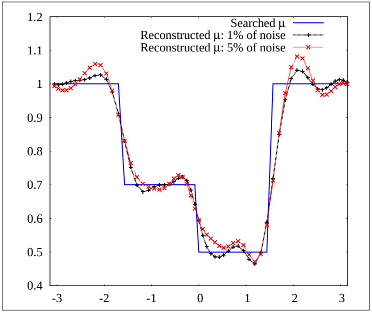
Total variation VS H^1 regularization



Reconstruction of μ, from $\mu_{\text{init}} = 0.7$ with 10 incident waves uniformly distributed on the unit circle and an aperture of $\pi/5$.



TV regularization



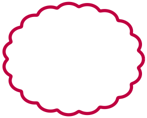
$H^1(\partial D)$ regularization

Reconstruction on an inexact geometry

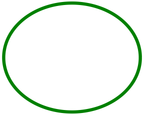
A fast oscillating boundary

Minimize

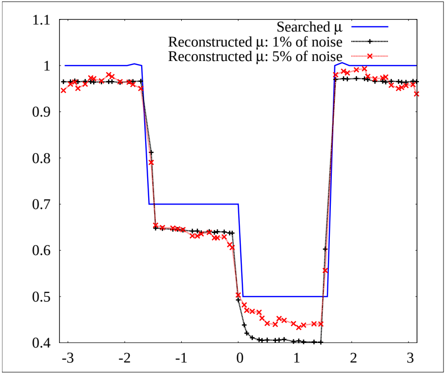
$$\boxed{F_\varepsilon(\mu) := \frac{1}{2}\sum_{j=1}^{I} \|T(\mu, \hat{\theta}_j, \partial D_\varepsilon) - T(\mu_{\text{searched}}, \hat{\theta}_j, \partial D)\|^2_{L^2(S_j)}.}$$



Exact geometry ∂D



Inexact geometry ∂D_ε



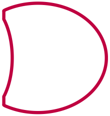
Reconstruction on an inexact geometry

Partially reconstructed boundary

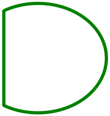
Minimize

$$\boxed{F_\varepsilon(\mu) := \frac{1}{2}\sum_{j=1}^{I} \|T(\mu, \hat{\theta}_j, \partial D_\varepsilon) - T(\mu_{\text{searched}}, \hat{\theta}_j, \partial D)\|^2_{L^2(S_j)}}$$

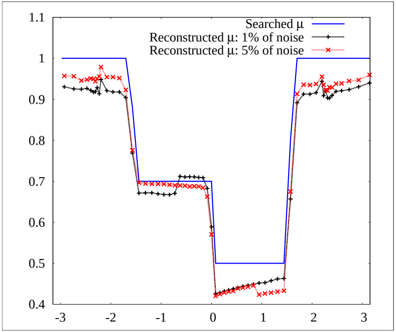
with $\hat{\theta}_j$ uniformly distributed in $[-\pi/2; \pi/2]$.



Exact geometry ∂D



Inexact geometry ∂D_ε



Conclusion

Stable recovery of piecewise constant μ on exact and inexact geometries.

Open questions

- Global stability for constant λ and μ.
- Uniqueness for piecewise constant λ and μ with a few incident waves.
- $\dots$

Stable reconstruction of generalized impedance boundary conditions. L. Bourgeois, N. Chaulet and H. Haddar. Inverse Problems (accepted).