Reconstruction of a perfectly conducting obstacle coated with a thin dielectric layer

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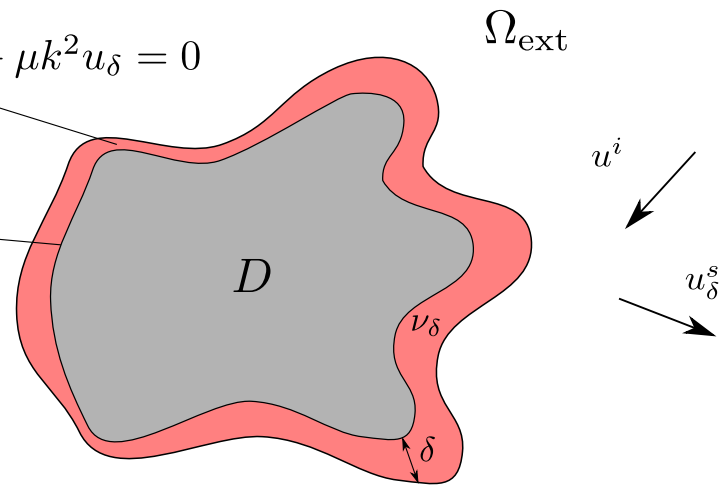
The scattering problem for coated obstacles in the harmonic regime

$$\mathrm{div}(\epsilon^{-1}\nabla u_\delta) + \mu k^2 u_\delta = 0$$

$$\frac{\partial u_\delta}{\partial \nu_\delta} = 0$$

$$\Delta u_\delta + k^2 u_\delta = 0$$



$$u_\delta = u_\delta^s + u^i$$

$$\lim_{R\to\infty} \int_{|x|=R} \left|\partial_r u_\delta^s - iku_\delta^s\right|^2 ds = 0$$

$D =$ Gray + Red

The inverse scattering problem for coated obstacles

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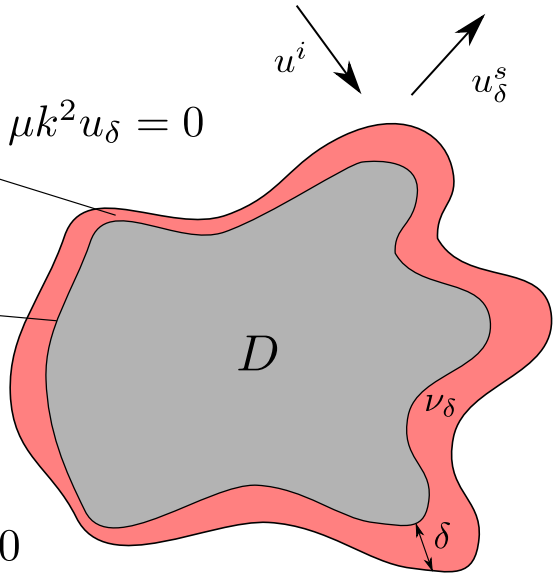

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Qualitative techniques

+ Fast
– Need a lot of data
– Not very accurate

Quantitative techniques

+ Provide accurate reconstruction
– Need a priori information
– Computational cost

Design an optimization procedure to find the shape and the thickness from scattered field data.

Outline

The inverse problem

<u>The far field map</u> For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$T^{\text{coat}} : (\epsilon, \mu, \delta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})$$

where u^∞ associated with u^s is defined in dimension d by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

The inverse problem

Given N far–fields $(u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j))_{j=1,\cdots,N}$, retrieve the geometry Γ and the properties of the layer.

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<u>Minimize</u>

$$F^{\text{coat}}(\epsilon, \mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \|T^{\text{coat}}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) - u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

T^{coat} is computationally expensive to evaluate!

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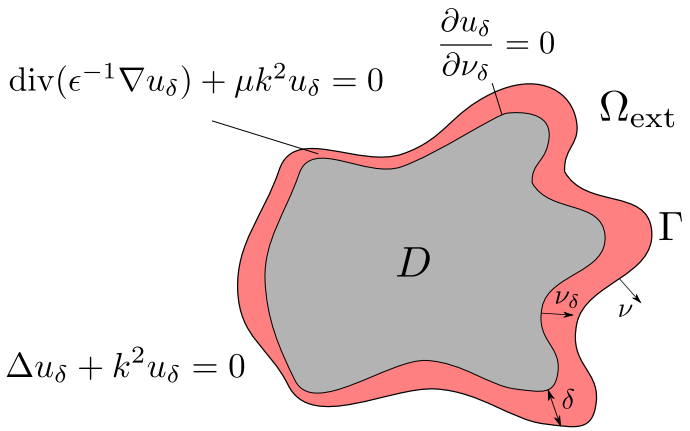
T^{coat} is computationally expensive to evaluate!

<u>Minimize</u>

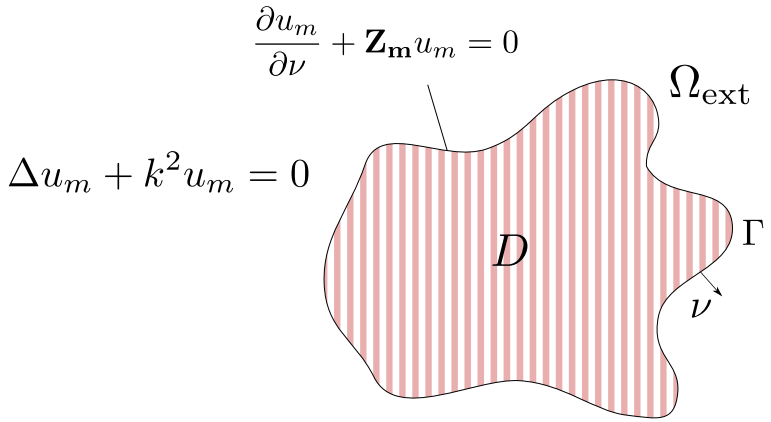
$$F^{equi}(\epsilon, \mu, \delta, \Gamma) := \frac{1}{2}\sum_{j=1}^{N} \|T^{equi}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) - u^{\infty}_{\text{coat}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

An approximate GIBC model

$$\|u_\delta - u_m\| \leq C\delta^{m+1/2}$$



Exact model



Equivalent model of order m

In dimension 2: *(Aslanyüreck, Haddar, Şahintürk [11])*

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}\delta\epsilon^{-1}\frac{\partial}{\partial s} + \delta k^2 \mu$$

$$\mathbf{Z}_2 = \frac{\partial}{\partial s}\left(\delta - \frac{\delta^2 c}{2}\right)\epsilon^{-1}\frac{\partial}{\partial s} + \left(\delta + \frac{\delta^2 c}{2}\right)k^2\mu.$$

The GIBC forward problem

Find $u = u^s + u^i$ such that

$$\boxed{u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}) \,,\ \varphi v \in\ H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma}\ \in\ H^1(\Gamma) \right\}}$$

and

$$(\mathcal{P}) \quad \begin{cases} \Delta u + k^2 u = 0 \quad \text{in } \Omega_{\text{ext}} \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \Gamma \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0. \end{cases}$$

u exists and is unique if

- $\Im m(\lambda) \geq 0,\ \Im m(\eta) \leq 0$ a.e. on Γ (physical assumption)
- $\Re e(\eta) \geq c$ a.e. on Γ for $c > 0$.

Reformulation of the inverse problem

The far field map for the GIBC model

For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$T^{GIBC} : (\lambda, \eta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta}).$$

For thin layer we have:

$$\boxed{T^{GIBC}(\delta k^2 \mu, \delta \epsilon^{-1}, \Gamma, \hat{\theta}_j) = T^{\text{coat}}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) + \mathcal{O}(\delta^{3/2})}$$

New functional to minimize

Given N far–fields $(u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j))_{j=1,\cdots,N}$ corresponding to the coated obstacle, minimize

$$F^{GIBC}(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \|T^{GIBC}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

Outline

Difficulty related to the shape derivative

$$\boxed{\text{GIBC model: } \frac{\partial u}{\partial \nu} + \mathrm{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \Gamma}$$

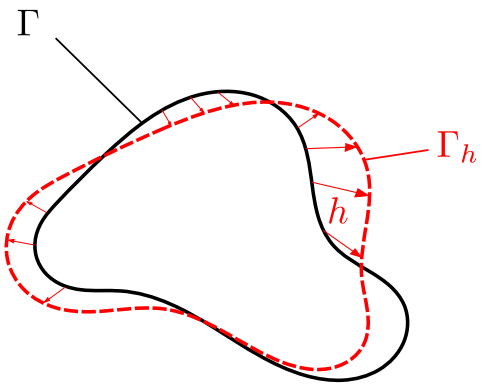
$$F^{GIBC}(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| T^{GIBC}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u^{\infty}_{\mathrm{coat}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

For minimizing F^{GIBC} we use a steepest descent method:

- we need partial derivatives of the far–field with respect to λ and η (quite standard),
- we need an appropriate derivative w.r.t. the obstacle.

Difficulty: the unknown impedances are supported by Γ.

Derivative of the cost function with respect to the obstacle



$h \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is "small"

$$f_h := \mathrm{Id} + h$$

$$\Gamma_h := f_h(\Gamma)$$

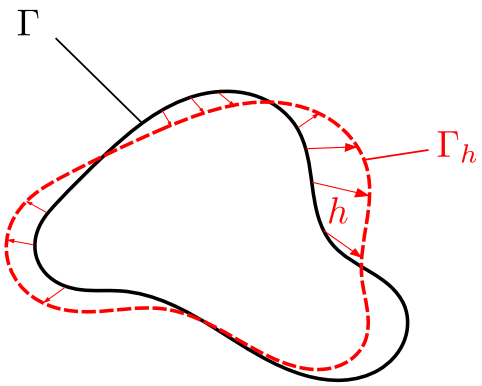
λ and η being constant

We define the derivative v_h of the scattered field with respect to the geometry at point (λ, η, Γ) by

$$u^s(\lambda, \eta, \Gamma_h) - u^s(\lambda, \eta, \Gamma) = v_h + o(||h||)$$

where $h \mapsto v_h$ is linear.

Derivative of the cost function with respect to the obstacle



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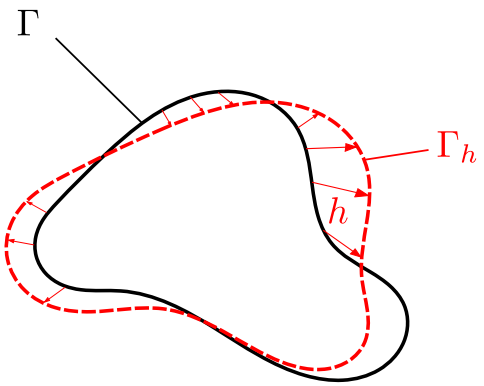

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where $h \mapsto v_h$ is linear.

One may find f_h such that $\Gamma = f_h(\Gamma)$ and

$$\boxed{F'_{\lambda,\eta}(\Gamma) \cdot h \neq 0.}$$

$\hookrightarrow$ $F'_{\lambda,\eta}(\Gamma)$ does not satisfy the classical shape derivative's properties!

Derivative of the scattered field with respect to the obstacle

Let (λ, η, Γ) be given and analytic, for some small $h \in C^{1,\infty}$ define

$$\Gamma_h = f_h(\Gamma)\,, \quad \lambda_h := \lambda \circ f_h^{-1} \text{ and } \eta_h := \eta \circ f_h^{-1}.$$

Let u_h^s $[u^s]$ be scattered field associated with $(\lambda_h, \eta_h, \Gamma_h)$ $[(\lambda, \eta, \Gamma)]$.

$$\boxed{u_h^s(x) - u^s(x) = v_h(x) + o(||h||),}$$

where $v_h(x)$ is the solution of the scattering problem with

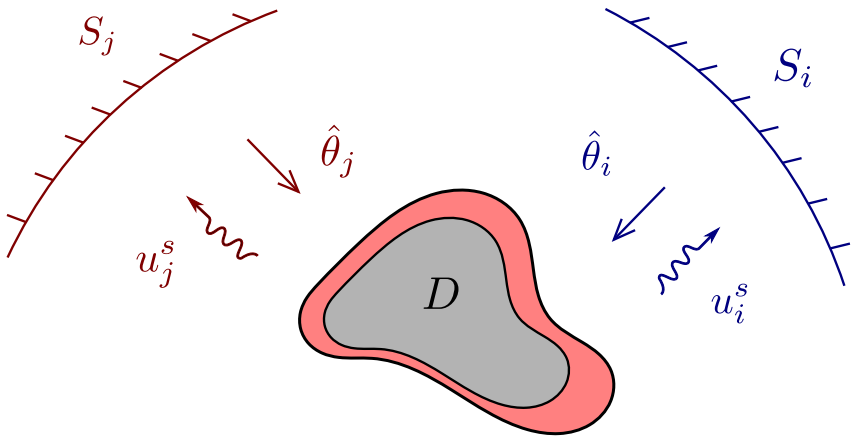
$$\frac{\partial v_h}{\partial \nu} + \mathbf{Z} v_h = B_h u \quad \text{on} \quad \Gamma$$

$$\begin{aligned} B_h u =& (h \cdot \nu)(k^2 - 2H\lambda)u + \operatorname{div}_\Gamma\left((Id + 2\eta(R - H\,Id))(h \cdot \nu)\nabla_\Gamma u\right) \\ &+ (\nabla_\Gamma \lambda \cdot h)u + \operatorname{div}_\Gamma\left((\nabla_\Gamma \eta \cdot h)\nabla_\Gamma u\right) \\ &+ \mathbf{Z}\left((h \cdot \nu)\mathbf{Z}u\right), \end{aligned}$$

with $2H := \operatorname{div}_\Gamma \nu$, $R := \nabla_\Gamma \nu$ and $\mathbf{Z}\cdot = \operatorname{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$

Outline

Numerical algorithm



$$F^{\mathrm{GIBC}}(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|T^{\mathrm{GIBC}}(\lambda,\eta,\Gamma,\hat{\theta}_j) - u^{\infty}_{\mathrm{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

<u>Numerical procedure</u>:

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

$$\lambda^{\mathsf{n}+1} = \lambda^{\mathsf{n}} - \alpha_{\mathsf{n}} \mathsf{F}'_{\eta,\Gamma}(\lambda^{\mathsf{n}})$$

The regularization procedure

$$F^{\mathrm{GIBC}}(\lambda,\eta,\Gamma) = \frac{1}{2}\sum_{j=1}^{I} \|T^{\mathrm{GIBC}}(\lambda,\eta,\Gamma,\hat{\theta}_j) - u^{\infty}_{\mathrm{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

We regularize the gradient, NOT the cost function, using a $H^1(\Gamma)$ regularization.

- Descent direction for λ: find $\delta\lambda$ that solves for every ϕ in some finite dimensional space:

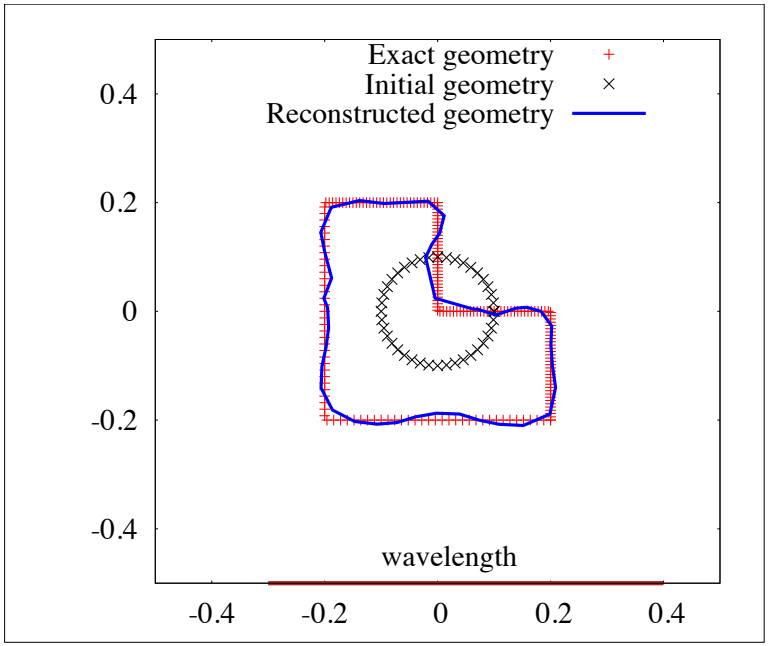
$$\beta_\lambda \int_\Gamma \nabla_\Gamma(\delta\lambda)\cdot\nabla_\Gamma\phi\, ds + \int_\Gamma \delta\lambda\phi\, ds = -\alpha_\lambda\, F'_{\eta,\Gamma}(\lambda)\cdot\phi$$

where β_λ is the regularization coefficient and α_λ is the descent coefficient.

- Do the same for $\delta\eta$ and $\delta\Gamma$.

Numerical reconstruction

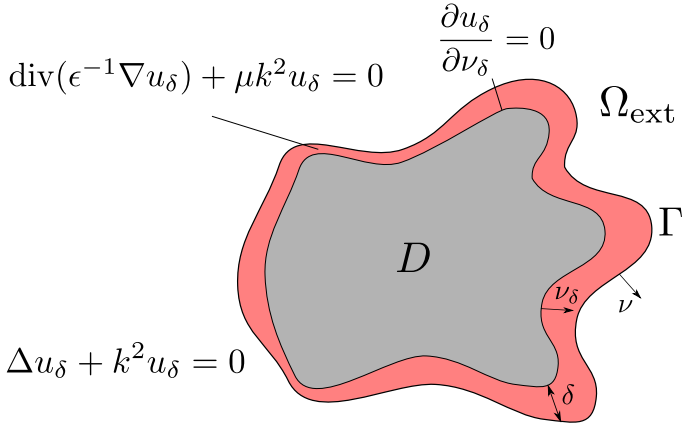
Finite elements method and remeshing procedure using FreeFem++



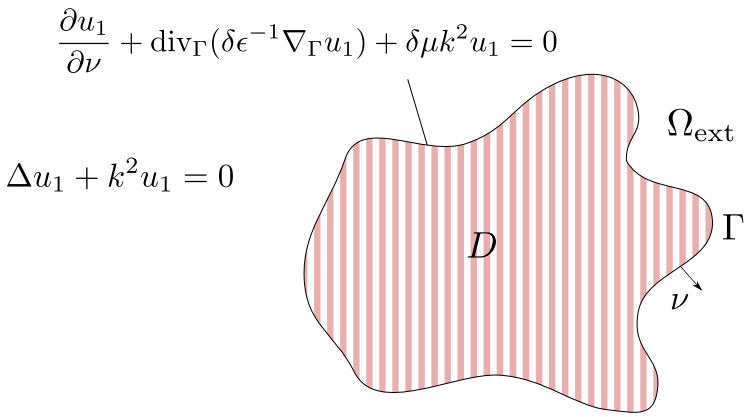
Reconstruction of the geometry with 2 incident waves and 1% noise on the far–field, $\lambda = ik/2$ and $\eta = 2/k$ being known.

The synthetic data come from the GIBC model

Application to the reconstruction of a coated obstacle



Exact model



Equivalent model of order 1

Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

$$F^{GIBC}(\mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \|T^{GIBC}(\mu, \delta, \Gamma, \hat{\theta}_j) - u^{\infty}_{\mathrm{coat}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

with $\epsilon = 0.1$ known.

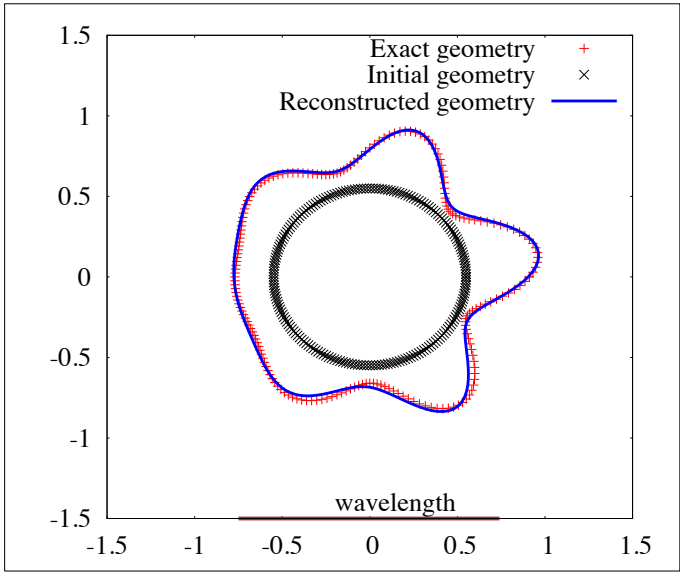
Application to the reconstruction of a coated obstacle

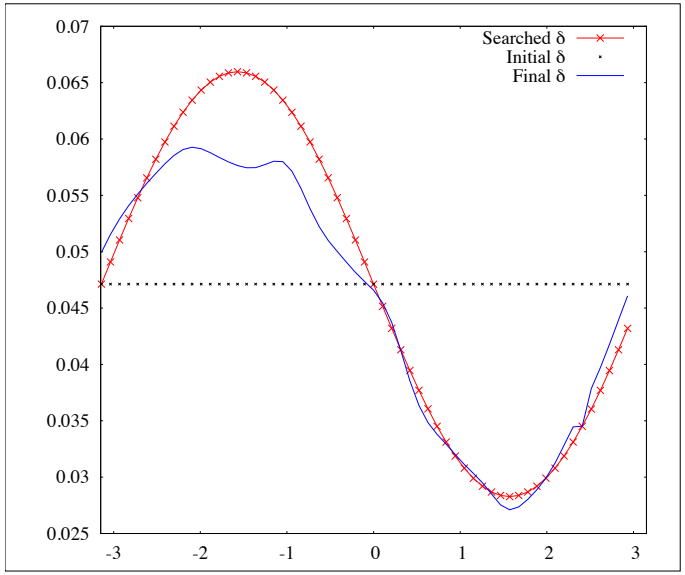
Numerical results

Artificial data created with

- $\epsilon = 0.1$ is known,
- $\delta = 0.04l(1 - 0.4\sin(\theta))$ is unknown; l being the wavelength,
- $\mu = 2.5$ is unknown.

Reconstructed μ: 2.3.





Fails with a classical impedance boundary condition model!

Conclusion

- The quantitative reconstruction of coated obstacles can be quite fast and accurate.
- The optimization methods also provide the thickness of the coating as well as its physical parameters.

- Extension to the $3D$ Maxwell equations.
- Can be applied for any complex structure that can be approximated by a GIBC.

On simultaneous identification of a scatterer and its generalized impedance boundary condition. L. Bourgeois, N. Chaulet and H. Haddar. INRIA Research Report n 7645, 2011. *Submitted*