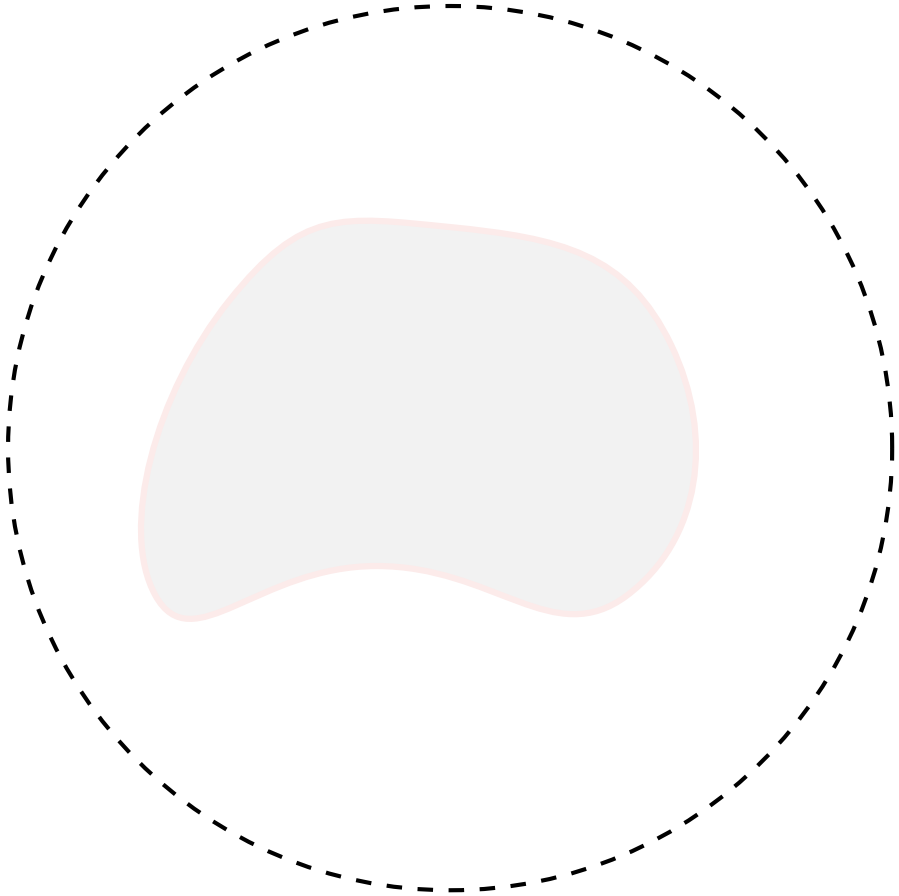
Generalized impedance models in inverse scattering

Nicolas Chaulet
Supervisors: L. Bourgeois and H. Haddar

PhD defense,
Palaiseau, November 27, 2012

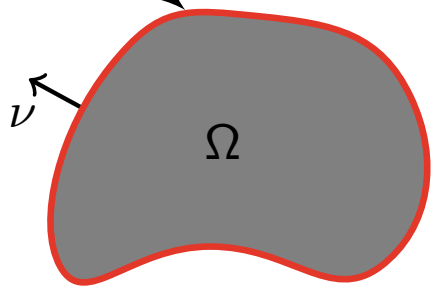
Inverse electromagnetic scattering problems

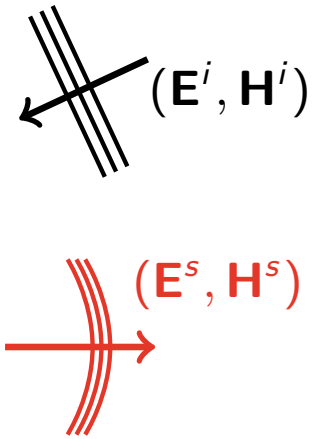
Radar imaging, non destructive testing, medical imaging...



Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

$$\nu \times \mathbf{E} + \mathbf{Z}\mathbf{H}_T = 0$$



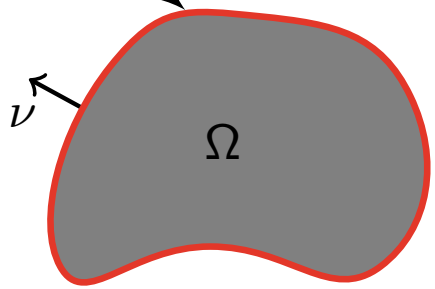


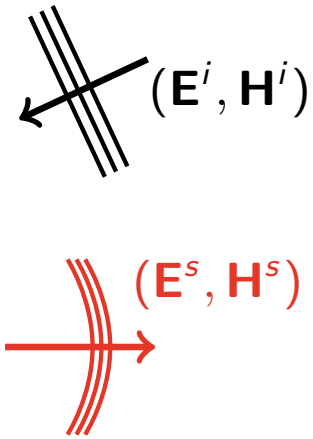
$$(\mathbf{E}, \mathbf{H}) := (\mathbf{E}^s, \mathbf{H}^s) + (\mathbf{E}^i, \mathbf{H}^i)$$

→ $\mathbf{Z}$ characterizes the inhomogeneity Ω

Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

$$\nu \times \mathbf{E} + \mathbf{Z}\mathbf{H}_T = 0$$





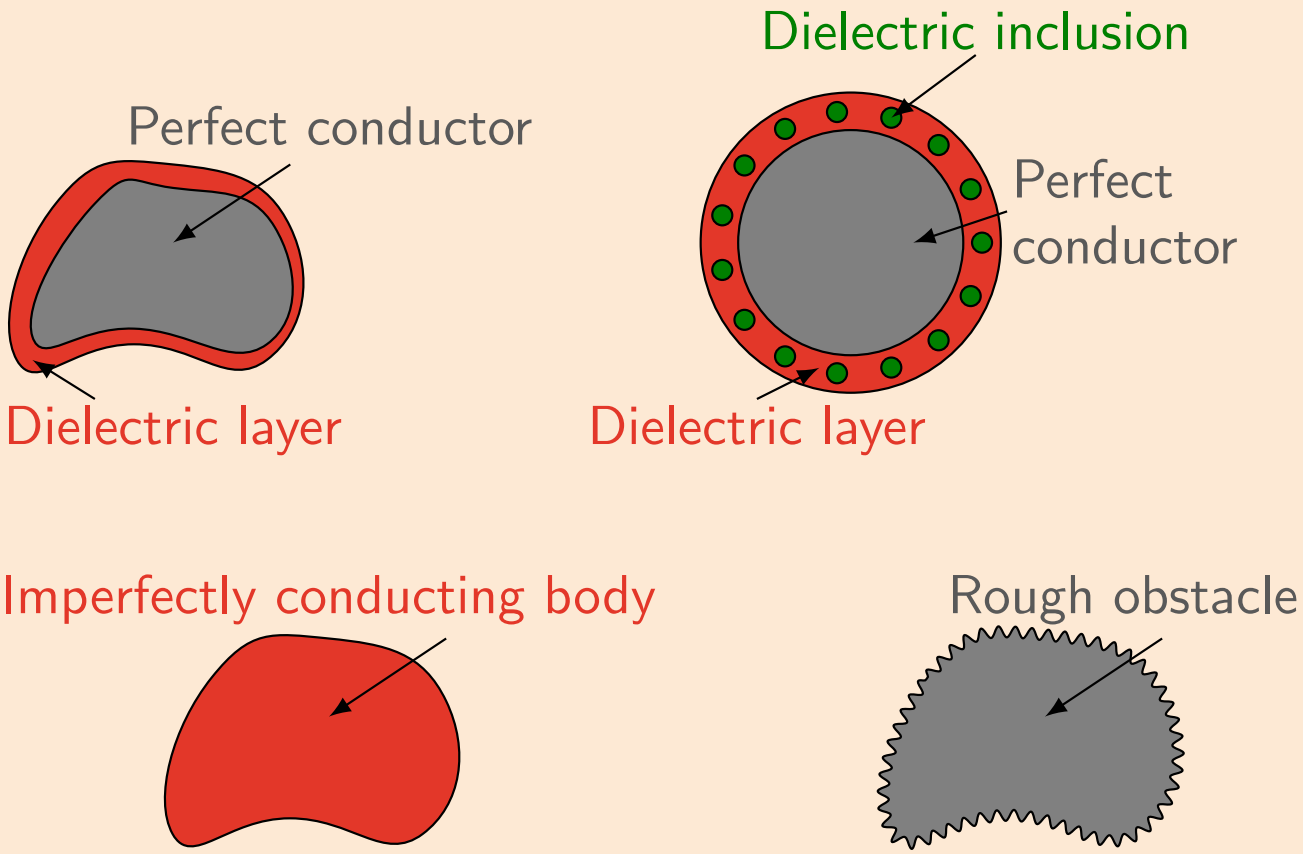
$$(\mathbf{E}, \mathbf{H}) := (\mathbf{E}^s, \mathbf{H}^s) + (\mathbf{E}^i, \mathbf{H}^i)$$

→ We consider GIBC: $\mathbf{Z}$ is local

Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

Examples for which **Z** is local

[Bouchitté 90, Engquist-Nédélec 93...]



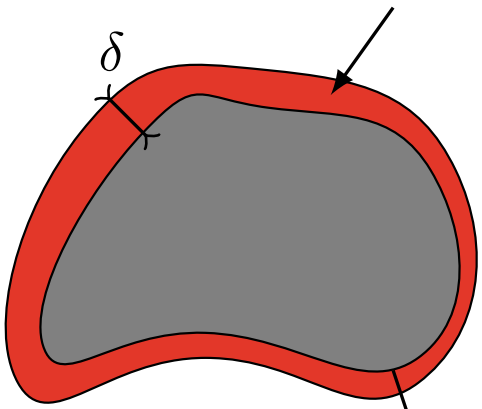
Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

$$\|u_\delta - u_{\text{app}}\| \leq C\delta^2$$

ex: scattering by thin coatings (TM electromagnetic mode)

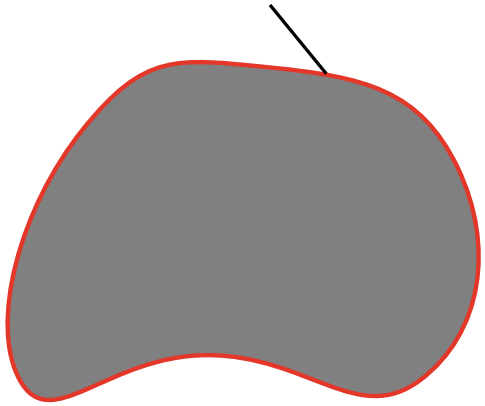
(TM mode)

$$\text{div}(\mu^{-1}\nabla u_\delta) + k^2\epsilon u_\delta = 0$$



$$\Delta u_\delta + k^2 u_\delta = 0 \qquad u_\delta = 0$$

$$\frac{\partial u_{\text{app}}}{\partial \nu} - \frac{1}{\delta} u_{\text{app}} = 0$$



$$\Delta u_{\text{app}} + k^2 u_{\text{app}} = 0$$

→ **Z** is the multiplication operator

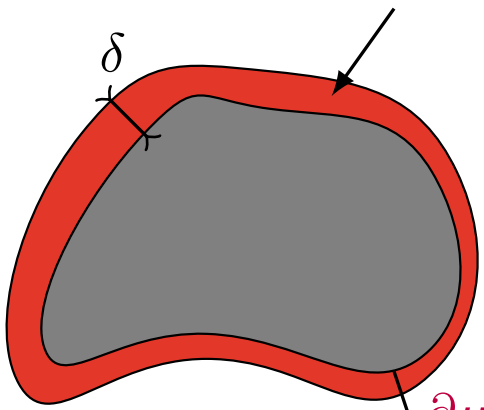
Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

$$\|u_\delta - u_{\text{app}}\| \leq C\delta^2$$

ex: scattering by thin coatings (TE electromagnetic mode)

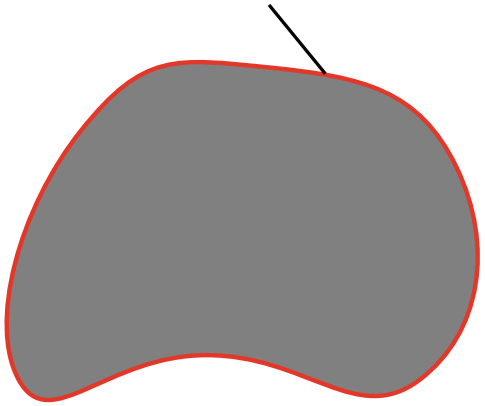
(TE mode)

$$\text{div}(\epsilon^{-1}\nabla u_\delta) + k^2\mu u_\delta = 0$$



$$\Delta u_\delta + k^2 u_\delta = 0 \qquad \frac{\partial u_\delta}{\partial \nu} = 0$$

$$\frac{\partial u_{\text{app}}}{\partial \nu} + \text{div}_\Gamma(\epsilon^{-1}\delta\nabla_\Gamma u_{\text{app}}) + k^2\mu\delta u_{\text{app}} = 0$$

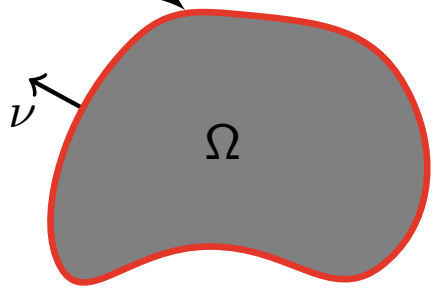


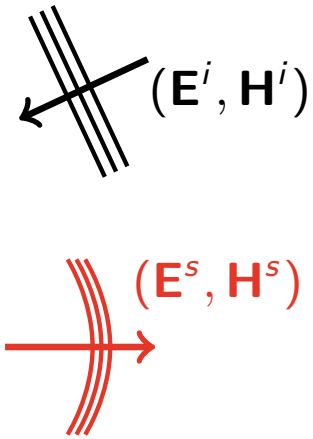
$$\Delta u_{\text{app}} + k^2 u_{\text{app}} = 0$$

→ **Z** is of the form $\text{div}_\Gamma \eta \nabla_\Gamma + \lambda$

Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions

$$\nu \times \mathbf{E} + \mathbf{Z}\mathbf{H}_T = 0$$





$$(\mathbf{E}, \mathbf{H}) := (\mathbf{E}^s, \mathbf{H}^s) + (\mathbf{E}^i, \mathbf{H}^i)$$

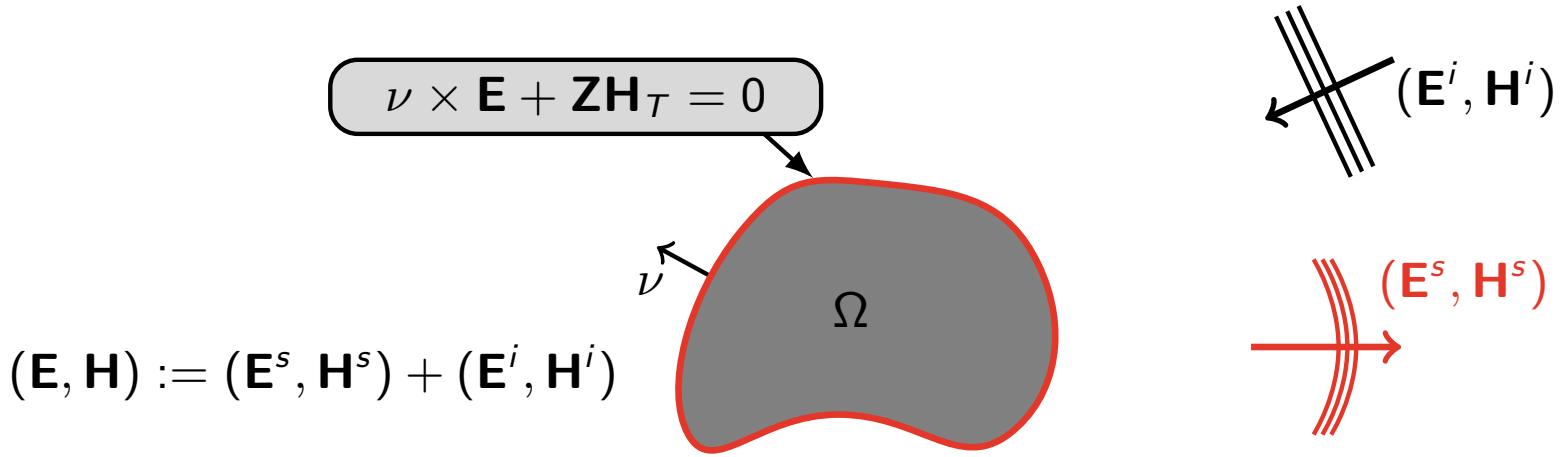
→ We consider GIBC: $\mathbf{Z}$ is local

Inverse problem: Find Ω and $\mathbf{Z}$ from the scattered fields

Difficulties

- The problem is non linear
- and ill-posed
  → uniqueness may fail, unstable w.r.t noise on the data

Inverse electromagnetic scattering with Generalized Impedance Boundary Conditions



→ We consider GIBC: $\mathbf{Z}$ is local

Inverse problem: Find Ω and $\mathbf{Z}$ from the scattered fields

Main inversion methods

- Qualitative methods or sampling methods (`Colton-Kirsch 96...`)
  → Few *a priori* information but a lot of data
- Quantitative methods (e.g. non-linear optimization methods)
  → Adapted to limited data but more *a priori* information and simple model

Outline of the talk

1. The GIBC forward problem
   - The scalar case
   - The Maxwell case

- We prove well-posedness for rather general impedance operators in the scalar case (with M. Chamaillard)
- We extend this to the $3D$ Maxwell's equations, theoretical difficulties arise from the variational spaces

Outline of the talk

1. The GIBC forward problem
   - The scalar case
   - The Maxwell case

- We prove well-posedness for rather general impedance operators in the scalar case (with M. Chamaillard)
- We extend this to the $3D$ Maxwell's equations, theoretical difficulties arise from the variational spaces

2. Use of qualitative methods in the scalar case
   - The factorization method
   - Application to a uniqueness proof

- We justify the factorization method for general impedance operators in the scalar case (with M. Chamaillard)
- We compute and justify an asymptotic development of the interior transmission eigenvalues for thin coating imaging (with F. Cakoni) *(not in this talk)*

Outline of the talk

3. Use of optimization methods
   - The scalar case
   - The Maxwell case

- Restriction to the case of

$$\mathbf{Z} = \mathrm{div}_\Gamma \eta \nabla_\Gamma + \lambda$$

- Stability for the reconstruction of (λ, η) on exact or inexact geometries
- We compute the shape and coefficients derivatives of the scattered field for non-constant impedances and second order surface operators
- We validate this method with numerical experiments for the $2D$ scalar case and the $3D$ Maxwell case

Outline

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset L^2(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous

For example for complex functions $(\lambda, \eta) \in (L^\infty(\Gamma))^2$

$$\mathbf{Z} = div_\Gamma \eta \nabla_\Gamma + \lambda$$

$$V = H^1(\Gamma)$$

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset L^2(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous
- $\Im m \langle \mathbf{Z} u, u \rangle_{V^*,V} \geq 0$ for uniqueness reasons

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset L^2(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous
- $\Im m \langle \mathbf{Z}u, u \rangle_{V^*,V} \geq 0$ for uniqueness reasons

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases} \qquad \left(f = -\frac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right)$$

The GIBC forward problem

A volume formulation

- V an Hilbert space such that $C^\infty(\Gamma) \subset V \subset L^2(\Gamma)$
- $\mathbf{Z} : V \longrightarrow V^*$ is linear and continuous
- $\Im m\langle \mathbf{Z}u, u\rangle_{V^*,V} \geq 0$ for uniqueness reasons

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z}u^s = f \text{ on } \Gamma, \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases} \qquad \left(f = -\frac{\partial u^i}{\partial \nu} - \mathbf{Z}u^i \right)$$

Theorem

If $\Re e(\mathbf{Z}) \geq 0$ then the forward problem is well posed and the continuity constant does not depend on $\mathbf{Z}$.

Well posedness of the forward problem

A surface equivalent formulation

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases} \qquad \left(f = -\frac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right)$$

- $S_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ the exterior DtN map

$$f \longmapsto \frac{\partial u_f}{\partial \nu}$$

where

$$\begin{cases} \Delta u_f + k^2 u_f = 0 \text{ in } \Omega_{\text{ext}}, \\ u_f = f \text{ on } \Gamma, \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u_f - iku_f|^2 = 0. \end{cases}$$

Well posedness of the forward problem

A surface equivalent formulation

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V \right\}$

$$(\mathcal{P}_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases} \qquad \left(f = -\frac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right)$$

- $S_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ the exterior DtN map

$$f \longmapsto \frac{\partial u_f}{\partial \nu}$$

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \cap H^{1/2}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma) u_\Gamma^s = f \end{cases}$$

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \cap H^{1/2}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma) u_\Gamma^s = f \end{cases}$$

Remark: $S_\Gamma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a Fredholm operator of index 0

Theorem

If one of the following holds

1. the embedding $H^{1/2}(\Gamma) \subset V$ is compact,
2. the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} : V \to V^*$ is Fredholm of index 0,

then $(\mathbf{Z} + S_\Gamma) : (V \cap H^{1/2}(\Gamma)) \to (V \cap H^{1/2}(\Gamma))^*$ is an isomorphism.

Application: $\mathbf{Z} = \lambda$.

Well posedness of the forward problem

A Fredholm operator

$$(\mathcal{P}_{\text{vol}}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } u_\Gamma^s \in V \cap H^{1/2}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma) u_\Gamma^s = f \end{cases}$$

Remark: $S_\Gamma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a Fredholm operator of index 0

Theorem

If one of the following holds

1. the embedding $H^{1/2}(\Gamma) \subset V$ is compact,
2. the embedding $V \subset H^{1/2}(\Gamma)$ is compact and $\mathbf{Z} : V \to V^*$ is Fredholm of index 0,

then $(\mathbf{Z} + S_\Gamma) : (V \cap H^{1/2}(\Gamma)) \to (V \cap H^{1/2}(\Gamma))^*$ is an isomorphism.

Application: $\mathbf{Z} = \text{div}_\Gamma \eta \nabla_\Gamma + \lambda$ with $\Re e(\eta) > 0$ or $\Re e(\eta) < 0$.

A forward model for Maxwell's equations

First order model for thin layers

Find $(\mathbf{E}^s, \mathbf{H}^s) \in \mathbf{H}^{\mathbf{rot}}_{\mathrm{loc}}(\Omega_{\mathrm{ext}}) \times \mathbf{H}^{\mathbf{rot}}_{\mathrm{loc}}(\Omega_{\mathrm{ext}})$ such that

$$(\mathcal{P}_{Max}) \quad \begin{cases} \mathbf{rot}\mathbf{H}^s + ik\mathbf{E}^s = 0 & \text{in } \Omega_{\mathrm{ext}}, \\ \mathbf{rot}\mathbf{E}^s - ik\mathbf{H}^s = 0 & \text{in } \Omega_{\mathrm{ext}}, \\ \nu \times \mathbf{E}^s + \mathbf{Z}\mathbf{H}^s_T = -(\nu \times \mathbf{E}^i + \mathbf{Z}\mathbf{H}^i_T) & \text{on } \Gamma, \\ \lim\limits_{R\to\infty} \displaystyle\int_{\partial B_R} |\mathbf{H}^s \times \hat{x} - (\hat{x} \times \mathbf{E}^s) \times \hat{x}|^2 ds = 0 \end{cases}$$

with

$$\mathbf{Z}\mathbf{H}_T = \mathbf{rot}_\Gamma(\eta \mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T$$

and

$$\mathbf{H}_T := (\nu \times \mathbf{H}) \times \nu$$
$$\mathrm{rot}_\Gamma = \nu \cdot \mathbf{rot}$$
$$\mathbf{rot}_\Gamma = -\nu \times \nabla_\Gamma$$

A forward model for Maxwell's equations

First order model for thin layers

$$(\mathcal{P}_{Max}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } \mathbf{H}_\Gamma \in \mathbf{H}_{\mathrm{rot}_\Gamma}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma)\mathbf{H}_\Gamma = f \end{cases}$$

with

$$\mathbf{Z}\mathbf{H}_T = \mathbf{rot}_\Gamma(\eta \mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T$$

and

$$S_\Gamma : \mathbf{H}^{-1/2}_{\mathrm{rot}_\Gamma} \longrightarrow \mathbf{H}^{-1/2}_{\mathrm{div}_\Gamma}$$

is the exterior Magnetic to Electric operator.

Notation: $\mathbf{H}_{\mathrm{rot}_\Gamma}(\Gamma) := \{\mathbf{v} \in (L^2(\Gamma))^3 \mid \mathbf{v} \cdot \nu = 0 \text{ and } \mathrm{rot}_\Gamma \mathbf{v} \in L^2(\Gamma)\}$

A forward model for Maxwell's equations

First order model for thin layers

$$(\mathcal{P}_{Max}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } \mathbf{H}_\Gamma \in \mathbf{H}_{\mathrm{rot}_\Gamma}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma)\mathbf{H}_\Gamma = f \end{cases}$$

with

$$\mathbf{Z}\mathbf{H}_T = \mathbf{rot}_\Gamma(\eta \mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T$$

and

$$S_\Gamma : \mathbf{H}^{-1/2}_{\mathrm{rot}_\Gamma} \longrightarrow \mathbf{H}^{-1/2}_{\mathrm{div}_\Gamma}$$

is the exterior Magnetic to Electric operator.

Notation: $\mathbf{H}_{\mathrm{rot}_\Gamma}(\Gamma) := \{\mathbf{v} \in (L^2(\Gamma))^3 \mid \mathbf{v} \cdot \nu = 0 \text{ and } \mathrm{rot}_\Gamma \mathbf{v} \in L^2(\Gamma)\}$

Difficulty:

- $\lambda \mathbf{H}_T$ is not a compact perturbation of the $\mathbf{rot}_\Gamma \eta \mathrm{rot}_\Gamma : \mathbf{H}_{\mathrm{rot}_\Gamma} \to (\mathbf{H}_{\mathrm{rot}_\Gamma})^*$ operator
  $\longrightarrow$ introduce a Helmholtz' decomposition on the boundary

A forward model for Maxwell's equations

First order model for thin layers

$$(\mathcal{P}_{Max}) \Longleftrightarrow (\mathcal{P}_{\text{surf}}) \quad \begin{cases} \text{Find } \mathbf{H}_\Gamma \in \mathbf{H}_{\mathrm{rot}_\Gamma}(\Gamma) \text{ such that} \\ (\mathbf{Z} + S_\Gamma)\mathbf{H}_\Gamma = f \end{cases}$$

with

$$\mathbf{Z}\mathbf{H}_T = \mathbf{rot}_\Gamma(\eta \mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T$$

and

$$S_\Gamma : \mathbf{H}^{-1/2}_{\mathrm{rot}_\Gamma} \longrightarrow \mathbf{H}^{-1/2}_{\mathrm{div}_\Gamma}$$

is the exterior Magnetic to Electric operator.

Theorem (Well-posed)

If

$$\Re e(\lambda) \geq 0 \ , \quad \Re e(\eta) \geq 0,$$

$$\Im m(\lambda) < (>) 0 \ , \quad \Im m(\eta) < (>) 0,$$

then $(\mathcal{P}_{Max})$ has a unique solution.

Outline

The inverse problem with infinitely many data

The far-field pattern

The far field $u^{\infty}_{\mathbf{Z},\Gamma}$ associated with $u^{s}_{\mathbf{Z},\Gamma}$ is defined in dimension d by

$$u^{s}_{\mathbf{Z},\Gamma}(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^{\infty}_{\mathbf{Z},\Gamma}(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

for $\hat{x}$ in the unit sphere of $\mathbb{R}^d$.

The inverse problem with infinitely many data

The far-field pattern

The far field $u^{\infty}_{\mathbf{Z},\Gamma}$ associated with $u^{s}_{\mathbf{Z},\Gamma}$ is defined in dimension d by

$$u^{s}_{\mathbf{Z},\Gamma}(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^{\infty}_{\mathbf{Z},\Gamma}(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

for $\hat{x}$ in the unit sphere of $\mathbb{R}^d$.

The data:
For $u^i(x) = e^{ik\hat{\theta}\cdot x}$ we know

$$u^{\infty}_{\text{obs}}(\hat{x}, \hat{\theta})$$

for all $\hat{x}, \hat{\theta}$ on the unit sphere of $\mathbb{R}^d$.

Objective:
Find Ω without knowing $\mathbf{Z}$.

The inverse problem with infinitely many data

The far-field pattern

The far field $u^{\infty}_{\mathbf{Z},\Gamma}$ associated with $u^{s}_{\mathbf{Z},\Gamma}$ is defined in dimension d by

$$u^{s}_{\mathbf{Z},\Gamma}(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^{\infty}_{\mathbf{Z},\Gamma}(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

for $\hat{x}$ in the unit sphere of $\mathbb{R}^d$.

The data:
For $u^i(x) = e^{ik\hat{\theta}\cdot x}$ we know

$$u^{\infty}_{\text{obs}}(\hat{x}, \hat{\theta})$$

for all $\hat{x}, \hat{\theta}$ on the unit sphere of $\mathbb{R}^d$.

→

Objective:
Find Ω without knowing $\mathbf{Z}$.

Strategy:
Use a sampling method: the factorization method [Kirsch 98]
→ Does not need $\mathbf{Z}$!

The inverse problem with infinitely many data

The far-field pattern

The far field $u^{\infty}_{\mathbf{Z},\Gamma}$ associated with $u^{s}_{\mathbf{Z},\Gamma}$ is defined in dimension d by

$$u^{s}_{\mathbf{Z},\Gamma}(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(u^{\infty}_{\mathbf{Z},\Gamma}(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

for $\hat{x}$ in the unit sphere of $\mathbb{R}^d$.

The data:
For $u^i(x) = e^{ik\hat{\theta}\cdot x}$ we know

$$u^{\infty}_{\text{obs}}(\hat{x}, \hat{\theta})$$

for all $\hat{x}, \hat{\theta}$ on the unit sphere of $\mathbb{R}^d$.

→

Objective:
Find Ω without knowing $\mathbf{Z}$.

Strategy:
Use a sampling method: the factorization method [Kirsch 98]
→ Does not need $\mathbf{Z}$!

State of the art:

- Neumann, Dirichlet B.C.: Kirsch 98
- Impedance B.C. ($\mathbf{Z} = \lambda$): Grinberg & Kirsch 02

Characterization of the support of Ω

Use of the factorization theorem [Kirsch & Grinberg 2008]

1. First step: characterization of Ω

   Define the solution operator for the forward problem

$$\begin{aligned} G \ : \ V^* &\longrightarrow L^2(S^d) \\ f &\longmapsto u_f^\infty \end{aligned}$$

Then

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(G)} \qquad \text{where } \phi_y^\infty := e^{ik\hat{x}\cdot y}$$

$$\boxed{\begin{cases} \Delta u_f + k^2 u_f = 0 \text{ in } \Omega_{\text{ext}} \\ \dfrac{\partial u_f}{\partial \nu} + \mathbf{Z} u_f = f \text{ on } \Gamma \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} \left| \frac{\partial u_f}{\partial r} - iku_f \right|^2 ds = 0 \end{cases}}$$

Characterization of the support of Ω

Use of the factorization theorem [Kirsch & Grinberg 2008]

1. First step: characterization of Ω

   Define the solution operator for the forward problem

$$\begin{aligned} G \ : \ V^* &\longrightarrow L^2(S^d) \\ f &\longmapsto u_f^\infty \end{aligned}$$

   Then

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(G)} \qquad \text{where } \phi_y^\infty := e^{ik\hat{x}\cdot y}$$

2. Second step: link with the far field operator $Fg := \int_{S^d} g(\hat{\theta}) u_{\text{obs}}^\infty(\hat{\theta}, \hat{x}) d\hat{\theta}$

Characterization of the support of Ω

Use of the factorization theorem [Kirsch & Grinberg 2008]

1. First step: characterization of Ω

   Define the solution operator for the forward problem

$$\begin{aligned} G \ : \ V^* &\longrightarrow L^2(S^d) \\ f &\longmapsto u_f^\infty \end{aligned}$$

   Then

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(G)} \qquad \text{where } \phi_y^\infty := e^{ik\hat{x}\cdot y}$$

2. Second step: link with the far field operator $Fg := \int_{S^d} g(\hat{\theta}) u_{\text{obs}}^\infty(\hat{\theta}, \hat{x}) d\hat{\theta}$

   Prove that

$$\boxed{\mathcal{R}(G) = \mathcal{R}(F_\#^{1/2})} \qquad \text{with} \quad F_\# := |\Re e(F)| + |\Im m(F)|$$

   by factorizing F like

$$F = GT^*G^*$$

   for some appropriate linear operator T.

Results

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(F_\#^{1/2})}$$

Provided k^2 is not an eigenvalue of,

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mathbf{Z} u = 0 & \text{on } \Gamma \end{cases}$$

and

- The embedding $H^{1/2}(\Gamma) \subset V$ is compact, ✓

$$\mathbf{Z} = \lambda \cdot$$

$$V = L^2(\Gamma)$$

- The embedding $V \subset H^{1/2}(\Gamma)$ is compact, ✓

$$\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$$

$$V = H^1(\Gamma)$$

Results

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(F_\#^{1/2})}$$

Provided k^2 is not an eigenvalue of,

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mathbf{Z} u = 0 & \text{on } \Gamma \end{cases}$$

and

- The embedding $H^{1/2}(\Gamma) \subset V$ is compact, ✓

$$\mathbf{Z} = \lambda \cdot$$
$$V = L^2(\Gamma)$$

- The embedding $V \subset H^{1/2}(\Gamma)$ is compact, ✓

$$\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$$
$$V = H^1(\Gamma)$$

Results

$$\boxed{y \in \Omega \Longleftrightarrow \phi_y^\infty(\hat{x}) \in \mathcal{R}(F_\#^{1/2})}$$

Provided k^2 is not an eigenvalue of,

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mathbf{Z} u = 0 & \text{on } \Gamma \end{cases}$$

and

- The embedding $H^{1/2}(\Gamma) \subset V$ is compact, ✓
- The embedding $V \subset H^{1/2}(\Gamma)$ is compact, ✓
- Intermediate case: none of the compact embeddings hold. ✗

This is the case when $\mathbf{Z}$ is the interior DtN map!

→ Requires subtle hypothesis on $\mathbf{Z}$!

A uniqueness result

$$\begin{cases} \mathbf{Z}u = \mathrm{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u \\ V = H^1(\Gamma) \end{cases}$$

Regularity and sign assumptions:

- Γ is Lipschitz,
- λ is in $L^\infty(\Gamma)$ and $\Im m(\lambda) \geq 0$
- η is continuous, $\Im m(\eta) \leq 0$ and $\Re e(\eta) > 0$ (or < 0).

Theorem (Uniqueness)

Let $(\lambda_1, \eta_1, \Gamma_1)$ and $(\lambda_2, \eta_2, \Gamma_2)$ be such that

$$u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta}) \qquad \forall \quad (\hat{x}, \hat{\theta}) \in S^d \times S^d$$

then

$$\lambda_1 = \lambda_2, \quad \eta_1 = \eta_2 \quad \text{and} \quad \Gamma_1 = \Gamma_2.$$

Proof of uniqueness

$$\boxed{u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})}$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.

Proof of uniqueness

$$\boxed{u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta})}$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.
- For the impedance we have

$$\frac{\partial u_1}{\partial \nu} + \mathbf{Z}_1 u_1 = \frac{\partial u_2}{\partial \nu} + \mathbf{Z}_2 u_2 = 0 \text{ on } \Gamma$$

Proof of uniqueness

$$\boxed{u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta})}$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.
- For the impedance we have

$$\frac{\partial u_1}{\partial \nu} + \mathbf{Z}_2 u_1 = \frac{\partial u_1}{\partial \nu} + \mathbf{Z}_1 u_1 = \frac{\partial u_2}{\partial \nu} + \mathbf{Z}_2 u_2 = 0 \text{ on } \Gamma$$

hence

$$(\mathbf{Z}_1 - \mathbf{Z}_2) u_1(x,\hat{\theta}) = 0 \qquad \forall \quad (x,\hat{\theta}) \in \Gamma \times S^d$$

Proof of uniqueness

$$\boxed{u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta})}$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.
- For the impedance we have

$$\frac{\partial u_1}{\partial \nu} + \mathbf{Z}_2 u_1 = \frac{\partial u_1}{\partial \nu} + \mathbf{Z}_1 u_1 = \frac{\partial u_2}{\partial \nu} + \mathbf{Z}_2 u_2 = 0 \text{ on } \Gamma$$

hence

$$(\mathbf{Z}_1 - \mathbf{Z}_2) u_1(x,\hat{\theta}) = 0 \qquad \forall \quad (x,\hat{\theta}) \in \Gamma \times S^d$$

- But

Prop: $\{u_1(x,\hat{\theta})$ for $\hat{\theta} \in S^d\}$ is dense in $H^1(\Gamma)$

$$\rightarrow \quad \boxed{\mathrm{div}_\Gamma[(\eta_1 - \eta_2)\nabla_\Gamma \varphi] + (\lambda_1 - \lambda_2)\varphi = 0 \qquad \forall \varphi \in H^1(\Gamma)}$$

Proof of uniqueness

$$\boxed{u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta})}$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.
- For the impedance we have

$$\frac{\partial u_1}{\partial \nu} + \mathbf{Z}_2 u_1 = \frac{\partial u_1}{\partial \nu} + \mathbf{Z}_1 u_1 = \frac{\partial u_2}{\partial \nu} + \mathbf{Z}_2 u_2 = 0 \text{ on } \Gamma$$

hence

$$(\mathbf{Z}_1 - \mathbf{Z}_2) u_1(x,\hat{\theta}) = 0 \qquad \forall \quad (x,\hat{\theta}) \in \Gamma \times S^d$$

- But

Prop: $\{u_1(x,\hat{\theta}) \text{ for } \hat{\theta} \in S^d\}$ is dense in $H^1(\Gamma)$

$$\rightarrow \quad \boxed{\mathrm{div}_\Gamma[(\eta_1 - \eta_2)\nabla_\Gamma \varphi] + (\lambda_1 - \lambda_2)\varphi = 0 \qquad \forall \varphi \in H^1(\Gamma)}$$

- Good choices for φ gives

$$\lambda_1 = \lambda_2 \quad \text{and} \quad \eta_1 = \eta_2.$$

Proof of uniqueness

$$u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta})$$

- $\Gamma_1 = \Gamma_2$ by using the factorization theorem.
- For the impedance we have

Γ

hence

- But

Prop: $\{$

And with few incident waves?

For the impedance coefficients:
`Bourgeois-Haddar 10, Cakoni-Kress 13`

$$\rightarrow \quad \mathrm{div}_\Gamma[(\eta_1 - \eta_2)\nabla_\Gamma \varphi] + (\lambda_1 - \lambda_2)\varphi = 0 \qquad \forall \varphi \in H^1(\Gamma)$$

- Good choices for φ gives

$$\lambda_1 = \lambda_2 \quad \text{and} \quad \eta_1 = \eta_2.$$

Numerical framework

- $\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot)$ with $\eta = 1$,
- For N=100, the synthetic data are

$$\left\{ u^\infty \left(\frac{2i\pi}{N}, \frac{2j\pi}{N} \right) \right\}_{i,j=1,\cdots,N}$$

- For each z in a given sampling grid we solve a discrete version of

$$F_\#^{1/2} g_z = \phi_z^\infty$$

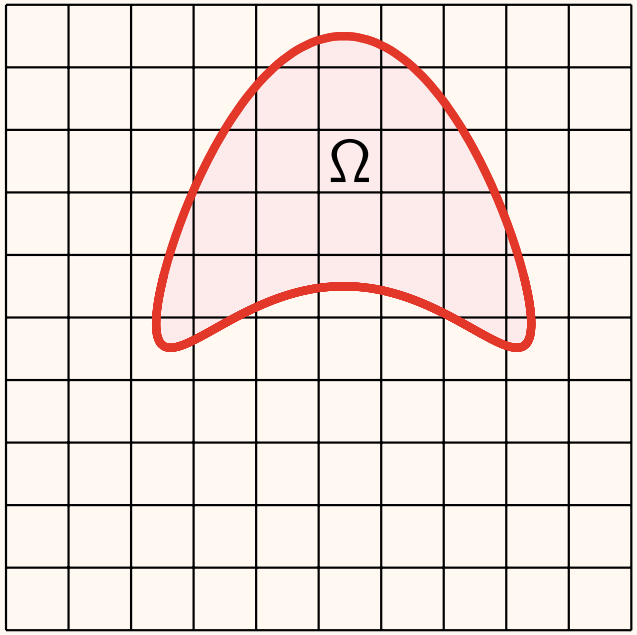
with Tikhonov-Morozov regularization and plot

$$z \longmapsto \frac{1}{\|g_z\|}.$$

Numerical framework

- Z
- Fo
- Fo te version of

wi

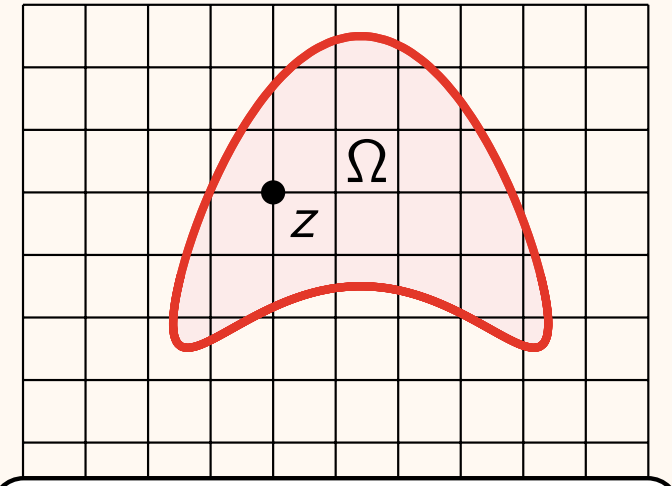


$$z \longmapsto \frac{1}{\|g_z\|}.$$

Numerical framework

- Z
- Fo
- Fo te version of

wi



$$\exists\, g \text{ s. t. } F_{\#}^{1/2} g = e^{-ik\hat{\theta}\cdot z}$$

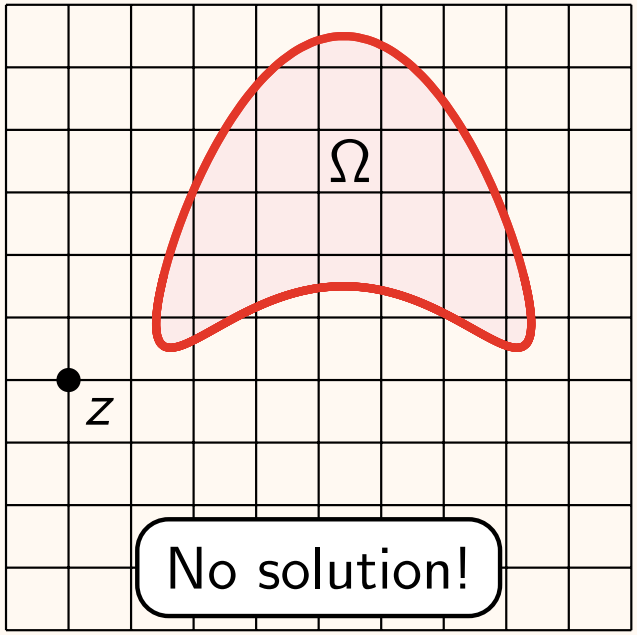
$$z \longmapsto \frac{1}{\|g_z\|}.$$

Numerical framework

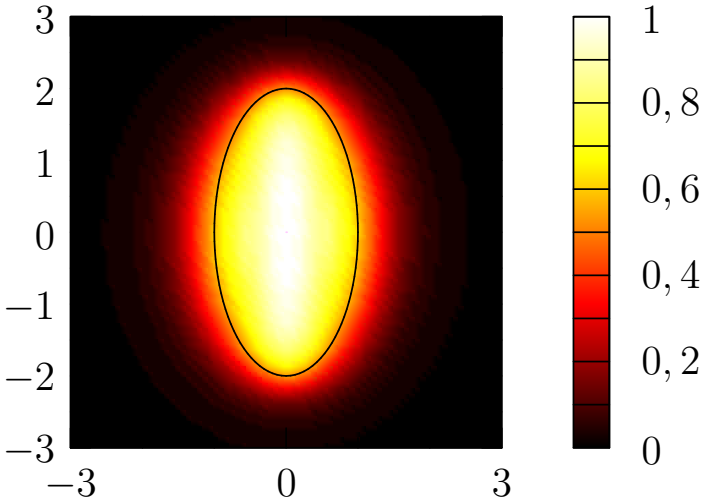
- Z
- Fo
- Fo te version of

wi

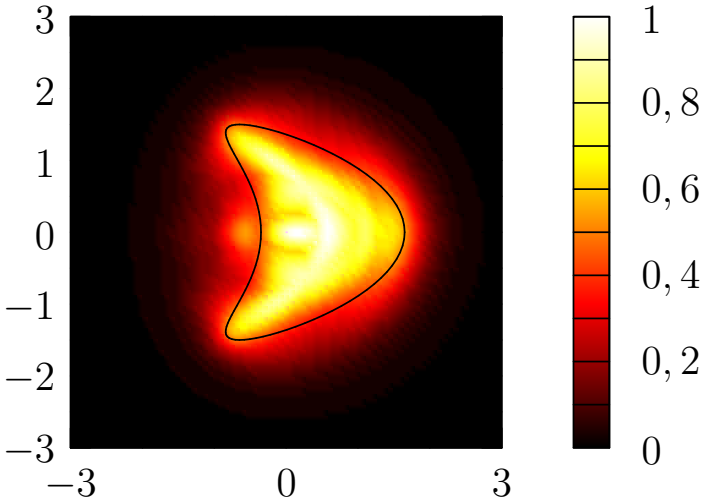
$$z \longmapsto \frac{1}{\|g_z\|}.$$



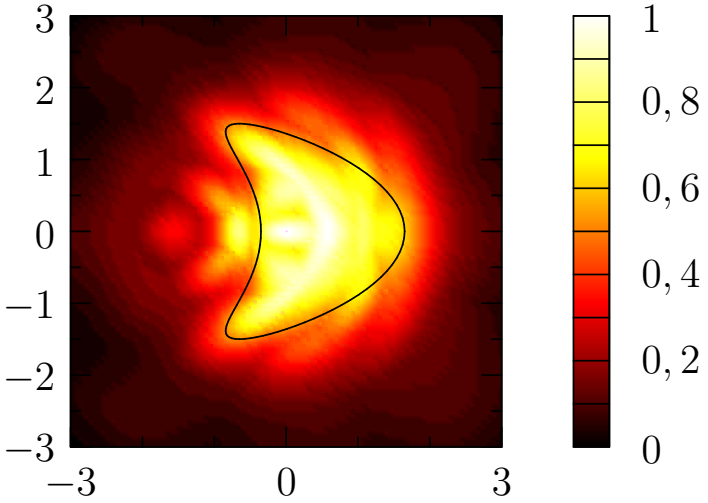
Numerical reconstructions



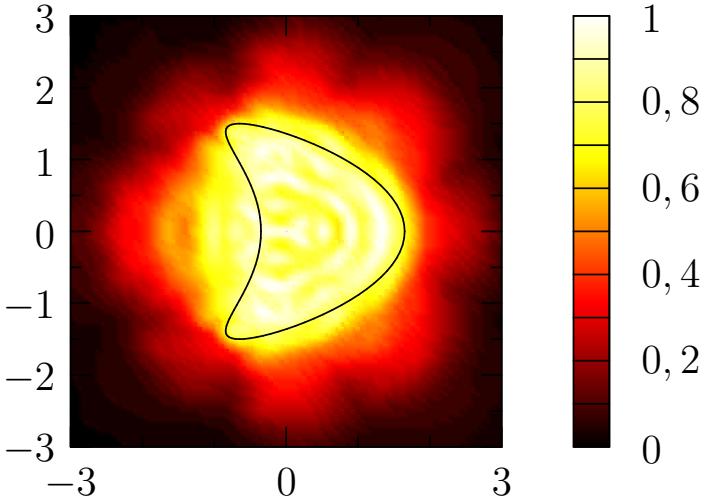
(a) no noise, ellipse, $k = 2$



(b) no noise, kite, $k = 2$



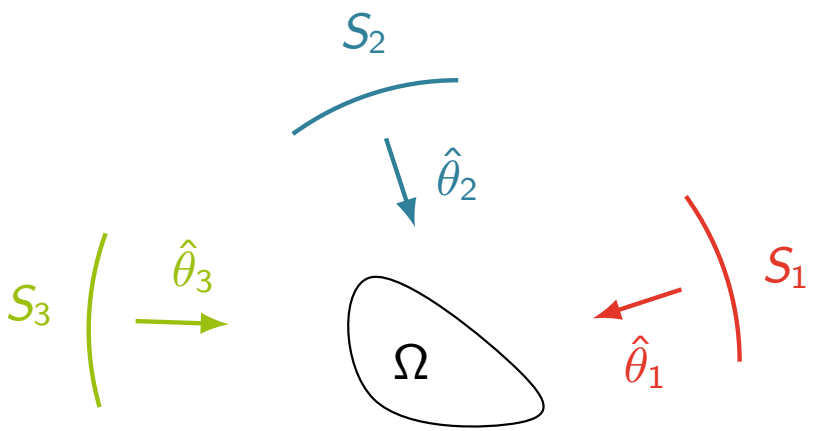
(a) 1% noise, $k = 2$



(b) 1% noise, $k = 5$

Outline

Solving the inverse problem with few incident waves



The data:
For $u^i(x) = e^{ik\hat{\theta}_i \cdot x}$ we know

$$u^{\infty}_{\text{obs}}(\hat{x}, \hat{\theta}_i)$$

for few $\hat{\theta}_j$ and for $\hat{x} \in S_j$ a portion of the unit circle.

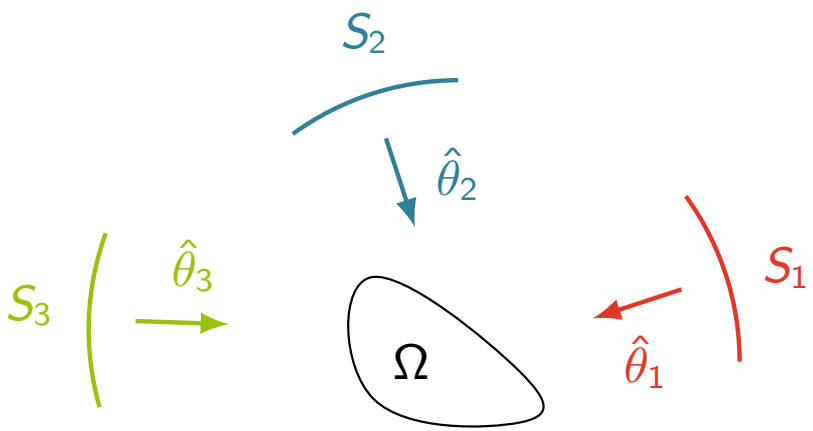
The unknowns:

Γ and $\mathbf{Z}$.

The use of the sampling methods is not appropriate anymore
→ Use non linear optimization methods!

Solving the inverse problem with few incident waves

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} \\ u = u^s + u^i(\cdot, \hat{\theta}) \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0 \end{cases}$$



The data:
For $u^i(x) = e^{ik\hat{\theta}_i \cdot x}$ we know

$$u^\infty_{\text{obs}}(\hat{x}, \hat{\theta}_i)$$

for few $\hat{\theta}_j$ and for $\hat{x} \in S_j$ a portion of the unit circle.

The unknowns:

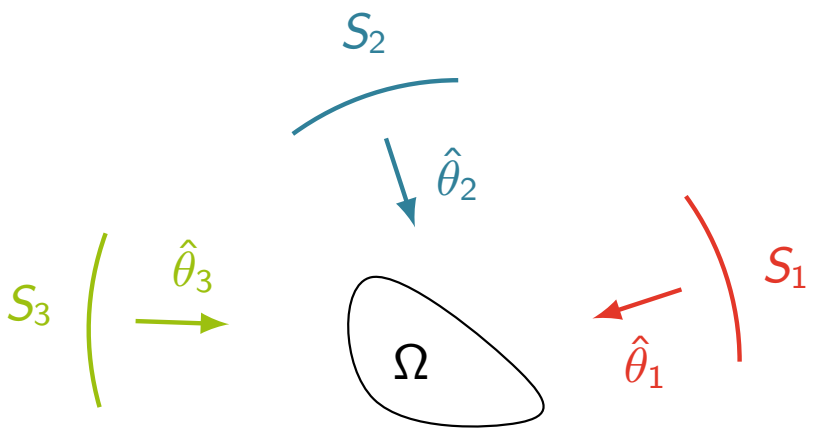
Γ and λ, η.

Require an *a priori* model for $\mathbf{Z}$:

$$\mathbf{Z} = \text{div}_\Gamma \eta \nabla_\Gamma + \lambda$$

Solving the inverse problem with few incident waves

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} \\ u = u^s + u^i(\cdot, \hat{\theta}) \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0 \end{cases}$$



The data:
For $u^i(x) = e^{ik\hat{\theta}_i \cdot x}$ we know

$$u^\infty_{\text{obs}}(\hat{x}, \hat{\theta}_i)$$

for few $\hat{\theta}_j$ and for $\hat{x} \in S_j$ a portion of the unit circle.

The unknowns:

Γ and λ, η.

$$\text{Minimize } F(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u^\infty_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^\infty_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

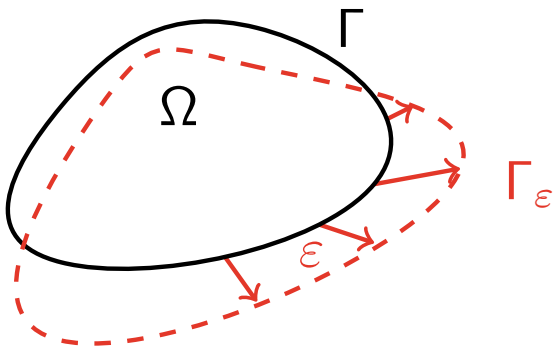
Solving the inverse problem with few incident waves

$$\text{Minimize } F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

1. Reconstruction of parameters with a known shape
   → Given an error on the shape, what is the error on the reconstructed coefficients?
2. Reconstruction of the parameters and the shape
   → How to characterize the shape derivative?

$$\text{Minimize } F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

The inverse coefficient problem on an inexact shape



- The far–field data correspond to (λ, η, Γ) and we reconstruct $(\lambda_\varepsilon, \eta_\varepsilon)$ an approximation of (λ, η) on

$$\Gamma_\varepsilon := (Id + \varepsilon)(\Gamma).$$

Assume that $(\lambda_\varepsilon, \eta_\varepsilon) \in (L^\infty(\Gamma_\varepsilon))^2$ are such that

$$\|u^\infty_{\lambda_\varepsilon,\eta_\varepsilon,\Gamma_\varepsilon} - u^\infty_{\lambda,\eta,\Gamma}\|_{L^2(S^d)} \leq \delta,$$

do we have

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\|_{L^\infty(\Gamma)} + \|\eta_\varepsilon \circ f_\varepsilon - \eta\|_{L^\infty(\Gamma)} \leq G(\delta, \varepsilon)$$

for some function $G(\delta, \varepsilon) \underset{\delta,\varepsilon\to 0}{\longrightarrow} 0$?

The inverse coefficient problem on an inexact shape

Result

Hypothesis: The inverse problem is stable for an exact geometry.
There exists a compact set $K \subset (L^\infty(\Gamma))^2$ and a constant C_K such that for (λ, η) and $(\widetilde{\lambda}, \widetilde{\eta}) \in K$,

$$\boxed{\|\lambda - \widetilde{\lambda}\| + \|\eta - \widetilde{\eta}\| \leq C_K \|u^\infty_{\lambda,\eta,\Gamma} - u^\infty_{\widetilde{\lambda},\widetilde{\eta},\Gamma}\|.}$$

`Allessandrini, Chaabane, Labreuche, Leblond, Rondi, Sincich...` for λ and `Bourgeois-C.-Haddar` for λ and η.

The inverse coefficient problem on an inexact shape

Result

Hypothesis: The inverse problem is stable for an exact geometry.
There exists a compact set $K \subset (L^\infty(\Gamma))^2$ and a constant C_K such that for (λ, η) and $(\widetilde{\lambda}, \widetilde{\eta}) \in K$,

$$\boxed{\|\lambda - \widetilde{\lambda}\| + \|\eta - \widetilde{\eta}\| \leq C_K \|u^\infty_{\lambda,\eta,\Gamma} - u^\infty_{\widetilde{\lambda},\widetilde{\eta},\Gamma}\|.}$$

`Allessandrini, Chaabane, Labreuche, Leblond, Rondi, Sincich...` for λ and `Bourgeois-C.-Haddar` for λ and η.

Theorem

For small ε and for all $(\lambda_\varepsilon \circ f_\varepsilon, \eta_\varepsilon \circ f_\varepsilon) \in K$ that satisfy

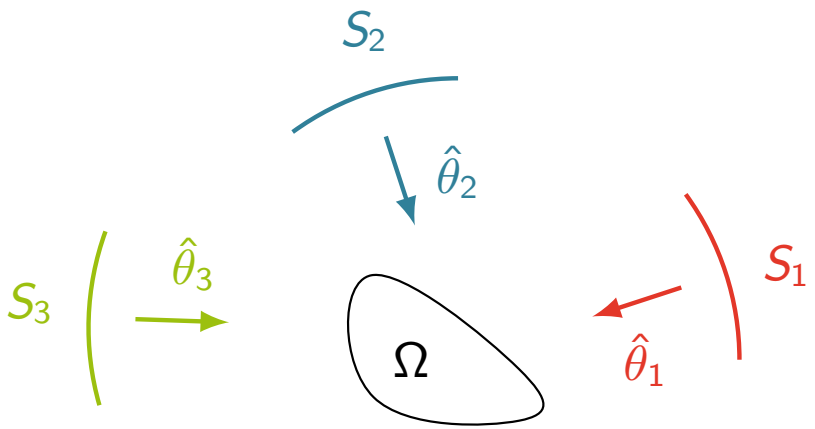
$$\|u^\infty_{\lambda_\varepsilon,\eta_\varepsilon,\Gamma_\varepsilon} - u^\infty_{\lambda,\eta,\Gamma}\| \leq \delta$$

we have

$$\|\lambda_\varepsilon \circ f_\varepsilon - \lambda\| + \|\eta_\varepsilon \circ f_\varepsilon - \eta\| \leq C_K(\delta + \|\varepsilon\|).$$

Practical resolution of the inverse problem

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} \\ u = u^s + u^i(\cdot, \hat{\theta}) \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma \\ \lim\limits_{R \to \infty} \displaystyle\int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0 \end{cases}$$



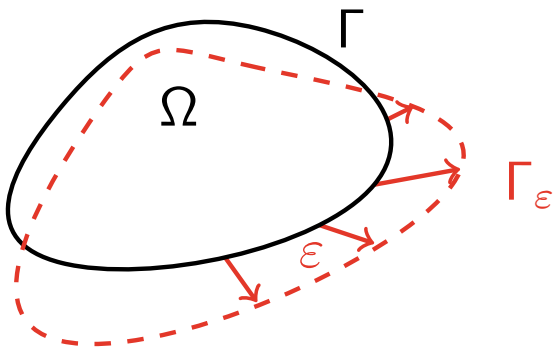
$$F(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u^\infty_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^\infty_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

For minimizing F we use a steepest descent method:
inspired by shape optimization (ex: `Allaire-Jouve...`)

- we need partial derivatives of the far–field with respect to λ and η (quite standard),
- we need an appropriate derivative w.r.t. the obstacle.

Difficulty: the unknown impedances are supported by Γ.

Shape derivative



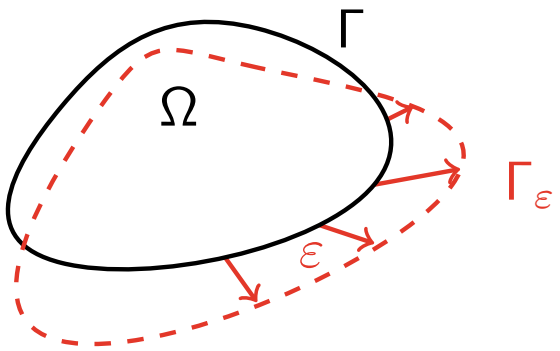
- λ, η and Γ are given
- $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\varepsilon\|_{C^1} < 1$
- $f_\varepsilon := \mathrm{Id} + \varepsilon$
- $\Gamma_\varepsilon := f_\varepsilon(\Gamma)$

Definition 1: constant coefficients

The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$R_0 : \varepsilon \longrightarrow u^s(\lambda, \eta, \Gamma_\varepsilon).$$

Shape derivative



- λ, η and Γ are given
- $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\varepsilon\|_{C^1} < 1$
- $f_\varepsilon := \mathrm{Id} + \varepsilon$
- $\Gamma_\varepsilon := f_\varepsilon(\Gamma)$

Definition 2: non-constant coefficients with intrinsic extension

The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$R_1 : \varepsilon \longrightarrow u^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

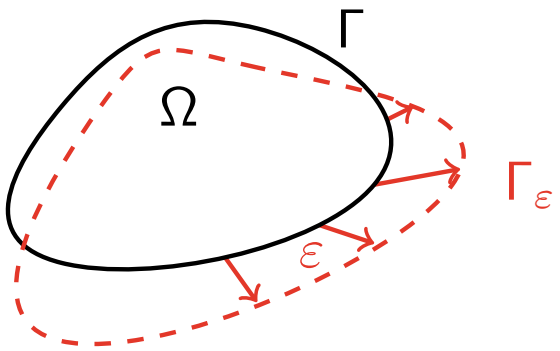
First choice: λ_ε and η_ε: extensions of λ and η in the ν direction

$$\lambda_\varepsilon(x) = \lambda(x_\Gamma)\ , \qquad \eta_\varepsilon(x) = \eta(x_\Gamma)$$

for $x \in \Gamma_\varepsilon$ and x_Γ is the orthogonal projection of x on Γ.

$\longrightarrow$ Same expression for the derivative as in the constant case

Shape derivative



- λ, η and Γ are given
- $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\varepsilon\|_{C^1} < 1$
- $f_\varepsilon := \mathrm{Id} + \varepsilon$
- $\Gamma_\varepsilon := f_\varepsilon(\Gamma)$

Definition 3: non-constant coefficients with extension in the ε direction

The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$R_2 : \varepsilon \longrightarrow u^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

Second choice: $\lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1}, \quad \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}$

→ Different expression and one may find f_ε such that $\Gamma = f_\varepsilon(\Gamma)$ and

$$\boxed{R_2'(0) \cdot \varepsilon \neq 0.}$$

$R_2'(0)$ does not satisfy the classical shape derivative's properties!

Derivative of the scattered field with respect to the obstacle

Let (λ, η, Γ) be given and analytic, for all $\varepsilon \in C^{1,\infty}$ such that $\|\varepsilon\| < 1$ we have

$$\boxed{R_2'(0) \cdot \varepsilon = v_\varepsilon(x),}$$

where $v_\varepsilon(x)$ is the solution of the scattering problem with

$$\frac{\partial v_\varepsilon}{\partial \nu} + \mathbf{Z} v_\varepsilon = B_\varepsilon u \quad \text{on} \quad \Gamma$$

$$\begin{aligned} B_\varepsilon u =& (\varepsilon \cdot \nu)(k^2 - 2H\lambda)u + \mathrm{div}_\Gamma\left((Id + 2\eta(R - H\,Id))(\varepsilon \cdot \nu)\nabla_\Gamma u\right) \\ &+ (\nabla_\Gamma \lambda \cdot \varepsilon)u + \mathrm{div}_\Gamma\left((\nabla_\Gamma \eta \cdot \varepsilon)\nabla_\Gamma u\right) \\ &+ \mathbf{Z}\left((\varepsilon \cdot \nu)\mathbf{Z} u\right), \end{aligned}$$

with

- $2H := \mathrm{div}_\Gamma \nu$, $R := \nabla_\Gamma \nu$, $\mathbf{Z}\cdot = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$,
- u is the total field given by (λ, η, Γ).

Main tools of the proof

- Domain derivative tools: `Murat and Simon 73`, `Kirsch 93`, `Hettlich 94`, `Potthast 94`.
- Green's theorems and integral representation of the scattered field: `Kress and Päivärinta 99`, `Haddar and Kress 04`.

Main tools of the proof

- Domain derivative tools: `Murat and Simon 73, Kirsch 93, Hettlich 94, Potthast 94.`
- Green's theorems and integral representation of the scattered field: `Kress and Päivärinta 99, Haddar and Kress 04.`

Green's theorems and integral representation: prove that

$$u_\varepsilon^s - u^s = -\int_\Gamma (B_\varepsilon u) w(\cdot, y) ds(y) + \mathrm{o}(\|\varepsilon\|)$$

where for $y \in \Omega_{\text{ext}}$ $w(\cdot, y) = w^s(\cdot, y) + \Phi(\cdot, y)$ is the Green function associated with the GIBC scattering problem

$$\begin{cases} \Delta w(\cdot, y) + k^2 w(\cdot, y) = \delta_y \quad \text{in } \Omega_{\text{ext}} \\ \dfrac{\partial w}{\partial \nu} + \mathbf{Z} w = 0 \quad \text{on } \Gamma \\ +\text{radiation condition.} \end{cases}$$

A steepest descent algorithm to solve the inverse problem

$$F(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u^{\infty}_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

A steepest descent algorithm to solve the inverse problem

$$F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

Numerical procedure for minimizing w.r.t. λ:

1. Take and initial guess λ_{init}
2. Solve the forward problem for I incidents plane waves to compute $u_{\lambda,\eta,\Gamma}$
3. Solve the forward problem with I adjoint incident fields

$$G^i(y,\hat{\theta}_j) = \int_{S_j} e^{-ik\hat{x}\cdot y}\overline{(u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\text{obs}})}(\hat{x},\hat{\theta}_j)d\hat{x}$$

4. Deduce $F'_{\eta,\Gamma}(\lambda)\cdot h = \sum_{j=1}^{I} \Re e\left(\int_\Gamma G(y,\hat{\theta}_j)u(y,\hat{\theta}_j)h(y)dy\right)$ and update λ
5. Return to 2. until convergence

A steepest descent algorithm to solve the inverse problem

$$F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

Numerical procedure for minimizing w.r.t. λ:

1. Take and initial guess λ_{init}
2. Solve the forward problem for I incidents plane waves to compute $u_{\lambda,\eta,\Gamma}$
3. Solve the forward problem with I adjoint incident fields

$$G^i(y,\hat{\theta}_j) = \int_{S_j} e^{-ik\hat{x}\cdot y}\overline{(u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\text{obs}})}(\hat{x},\hat{\theta}_j)d\hat{x}$$

4. Deduce $F'_{\eta,\Gamma}(\lambda)\cdot h = \sum_{j=1}^{I} \Re e\left(\int_\Gamma G(y,\hat{\theta}_j)u(y,\hat{\theta}_j)h(y)dy\right)$ and update λ
5. Return to 2. until convergence

A steepest descent algorithm to solve the inverse problem

$$F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

Numerical procedure for minimizing w.r.t. λ:

1. Take and initial guess λ_{init}
2. Solve the forward problem for I incidents plane waves to compute $u_{\lambda,\eta,\Gamma}$
3. Solve the forward problem with I adjoint incident fields

$$G^i(y,\hat{\theta}_j) = \int_{S_j} e^{-ik\hat{x}\cdot y}\overline{(u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\text{obs}})}(\hat{x},\hat{\theta}_j)d\hat{x}$$

4. Deduce $F'_{\eta,\Gamma}(\lambda)\cdot h = \sum_{j=1}^{I} \Re\text{e}\left(\int_{\Gamma} G(y,\hat{\theta}_j)u(y,\hat{\theta}_j)h(y)dy\right)$ and update λ
5. Return to 2. until convergence

A steepest descent algorithm to solve the inverse problem

$$F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

Numerical procedure for minimizing w.r.t. λ:

1. Take and initial guess λ_{init}
2. Solve the forward problem for I incidents plane waves to compute $u_{\lambda,\eta,\Gamma}$
3. Solve the forward problem with I adjoint incident fields

$$G^i(y,\hat{\theta}_j) = \int_{S_j} e^{-ik\hat{x}\cdot y}\overline{(u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\text{obs}})}(\hat{x},\hat{\theta}_j)d\hat{x}$$

4. Deduce $F'_{\eta,\Gamma}(\lambda)\cdot h = \sum_{j=1}^{I} \Re e\left(\int_\Gamma G(y,\hat{\theta}_j)u(y,\hat{\theta}_j)h(y)dy\right)$ and update λ
5. Return to 2. until convergence

A steepest descent algorithm to solve the inverse problem

$$F(\lambda,\eta,\Gamma) := \frac{1}{2}\sum_{j=1}^{I} \|u^{\infty}_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot,\hat{\theta}_j)\|^2_{L^2(S_j)}$$

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function,

Numerical procedure for minimizing w.r.t. λ:

1. Take and initial guess λ_{init}
2. Solve the forward problem for I incidents plane waves to compute $u_{\lambda,\eta,\Gamma}$
3. Solve the forward problem with I adjoint incident fields

$$G^i(y,\hat{\theta}_j) = \int_{S_j} e^{-ik\hat{x}\cdot y}\overline{(u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\text{obs}})}(\hat{x},\hat{\theta}_j)d\hat{x}$$

4. Deduce $F'_{\eta,\Gamma}(\lambda)\cdot h = \sum_{j=1}^{I} \Re e\left(\int_\Gamma G(y,\hat{\theta}_j)u(y,\hat{\theta}_j)h(y)dy\right)$ and update λ
5. Return to 2. until convergence

The regularization procedure

$$F(\lambda, \eta, \Gamma) = \frac{1}{2} \sum_{j=1}^{I} \| u^{\infty}_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

We regularize the gradient, NOT the cost function, using a $H^1(\Gamma)$ gradient (inspired by the shape optimization techniques: `Allaire...`)

- Descent direction for λ: $\delta\lambda$ that solves for every ϕ in some finite dimensional space:

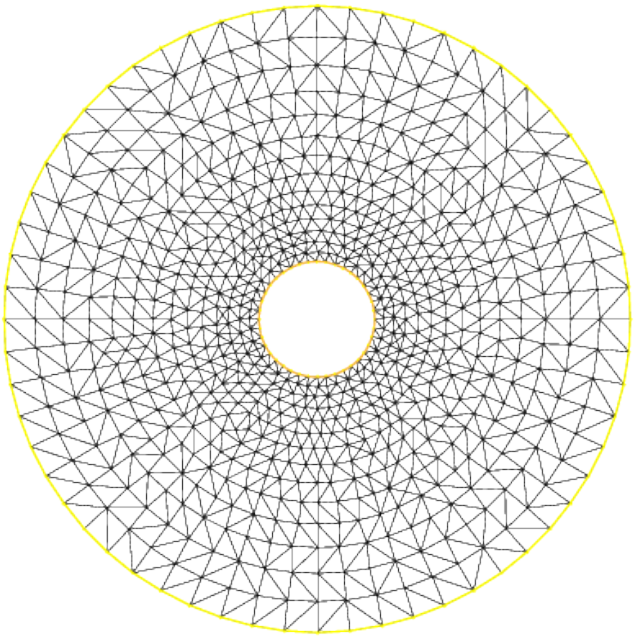
$$\beta_\lambda \int_\Gamma \nabla_\Gamma(\delta\lambda) \cdot \nabla_\Gamma \phi \, ds + \int_\Gamma \delta\lambda\phi \, ds = -\alpha_\lambda \, F'_{\eta,\Gamma}(\lambda) \cdot \phi$$

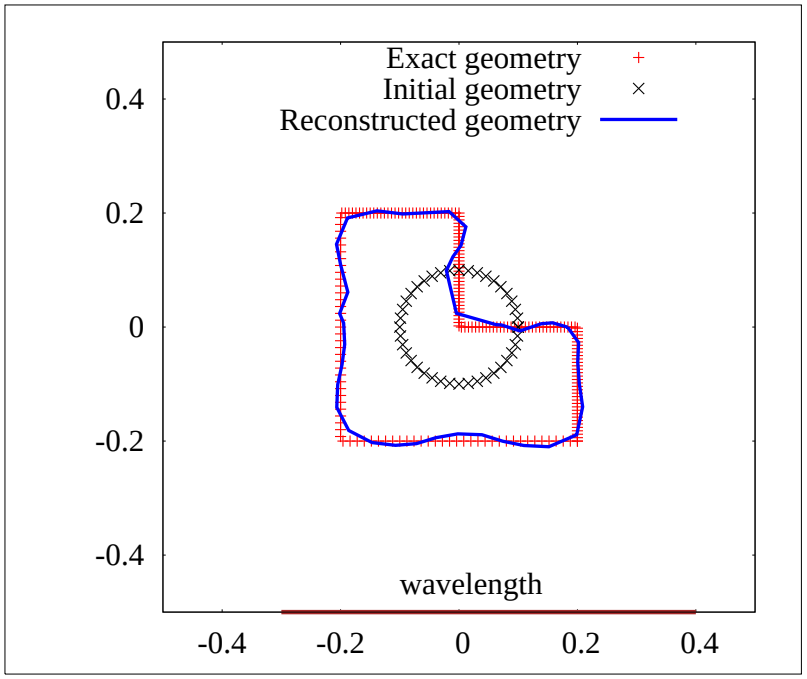
where β_λ is the regularization coefficient and α_λ is the descent coefficient.

- Do the same for $\delta\eta$ and $\delta\Gamma$.

Numerical reconstruction

Finite elements method and remeshing procedure using FreeFem++

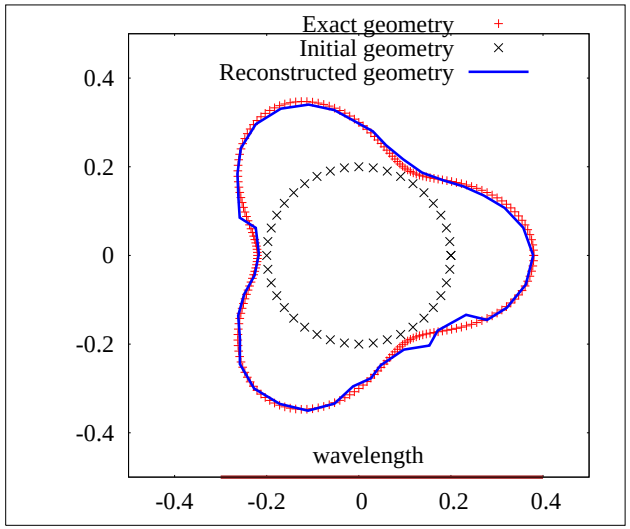




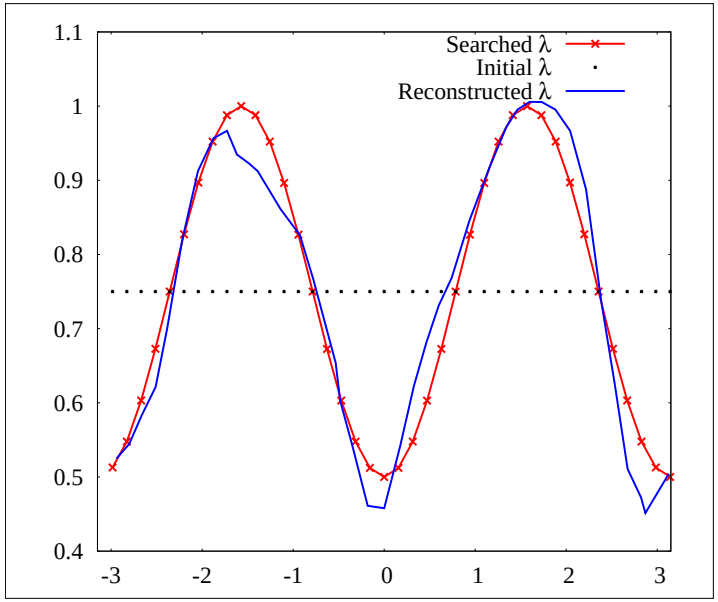
Reconstruction of the geometry with 2 incident waves and 1% noise on the far–field, $\lambda = ik/2$ and $\eta = 2/k$ being known

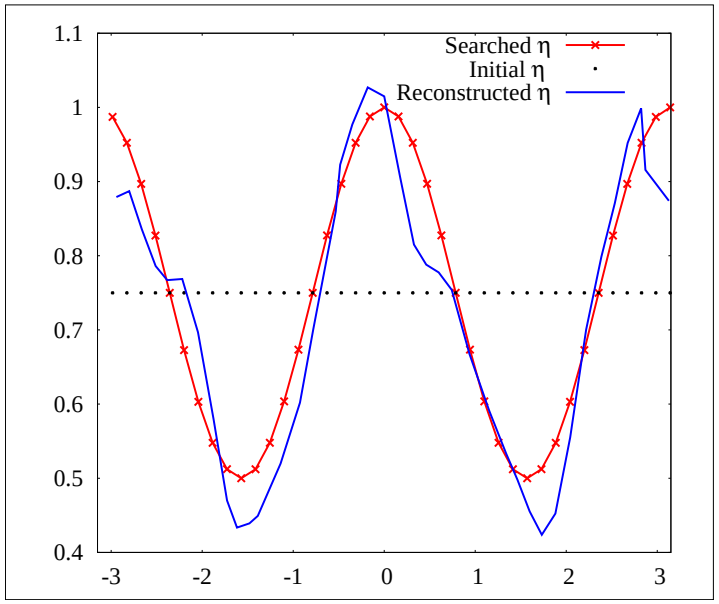
Numerical reconstruction

Simultaneous reconstruction of λ, η and Γ



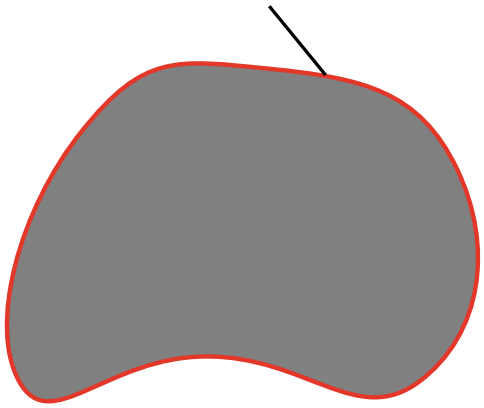
8 incident waves, 5% of noise on far–field data.





Application to the reconstruction of a coated obstacle

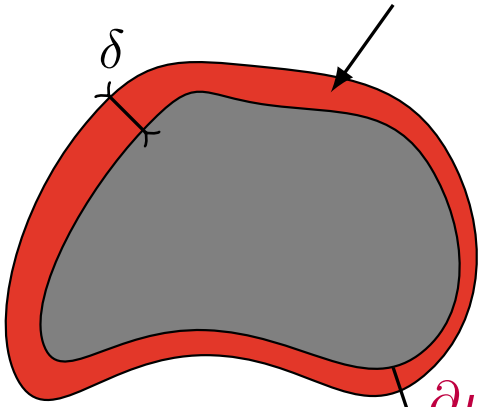
$$\frac{\partial u_{\text{app}}}{\partial \nu} + \text{div}_\Gamma(\epsilon^{-1}\delta\nabla_\Gamma u_{\text{app}}) + k^2\mu\delta u_{\text{app}} = 0$$



$$\Delta u_{\text{app}} + k^2 u_{\text{app}} = 0$$

(TE mode)

$$\text{div}(\epsilon^{-1}\nabla u_\delta) + k^2\mu u_\delta = 0$$

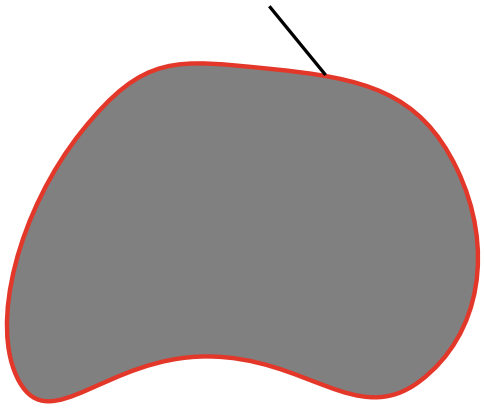


$$\Delta u_\delta + k^2 u_\delta = 0$$

$$\frac{\partial u_\delta}{\partial \nu} = 0$$

Application to the reconstruction of a coated obstacle

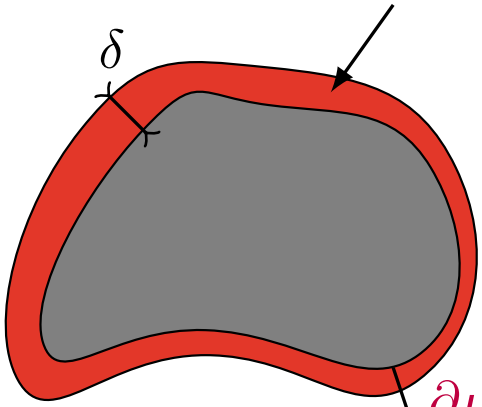
$$\frac{\partial u_{\text{app}}}{\partial \nu} + \text{div}_\Gamma(\epsilon^{-1}\delta\nabla_\Gamma u_{\text{app}}) + k^2\mu\delta u_{\text{app}} = 0$$



$$\Delta u_{\text{app}} + k^2 u_{\text{app}} = 0$$

(TE mode)

$$\text{div}(\epsilon^{-1}\nabla u_\delta) + k^2\mu u_\delta = 0$$



$$\Delta u_\delta + k^2 u_\delta = 0 \qquad \frac{\partial u_\delta}{\partial \nu} = 0$$

Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

$$F(\epsilon, \delta, \Gamma) := \frac{1}{2}\sum_{j=1}^{I} \| u^\infty_{\text{app}}(\epsilon, \delta, \Gamma, \hat{\theta}_j) - u^\infty_{\delta,\text{obs}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

with $\mu = 0.1$ known.

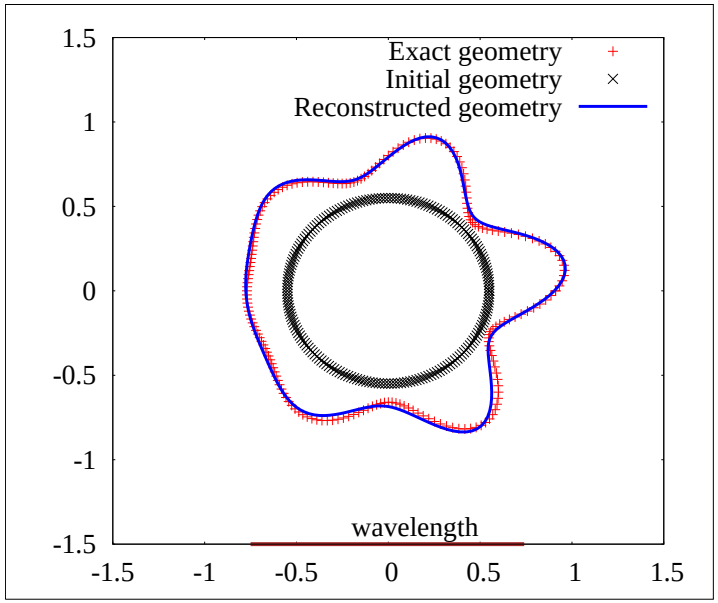
Application to the reconstruction of a coated obstacle

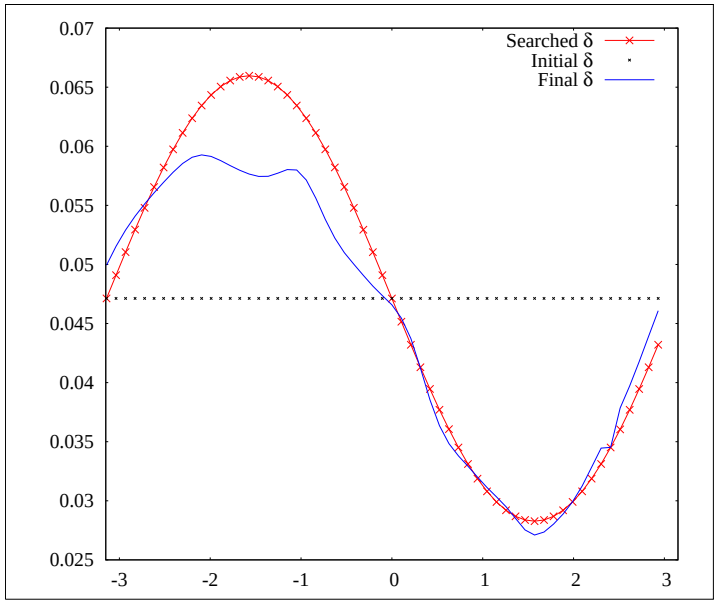
Numerical results

Synthetic data created with

- $\mu = 0.1$ is known,
- $\delta = 0.04/(1 - 0.4\sin(\theta))$ is unknown; l being the wavelength,
- $\epsilon = 2.5$ is unknown.

Reconstructed ϵ: 2.3.





Fails with a classical impedance boundary condition model!

Extension to the Maxwell case

$$\nu \times \mathbf{E} + \mathbf{rot}_\Gamma(\eta \mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T = 0 \text{ on } \Gamma$$

The data:

For incident waves

$$\mathbf{E}^i(z, \hat{\theta}, \mathbf{p}) = ik[(\hat{\theta} \times \mathbf{p}) \times \hat{\theta}]e^{ik\hat{\theta}\cdot z}$$

$$\mathbf{H}^i(z, \hat{\theta}, \mathbf{p})) = ik(\hat{\theta} \times \mathbf{p})e^{ik\hat{\theta}\cdot z}$$

we know

$$\mathbf{E}^\infty_{\text{obs}}(\hat{x}, \hat{\theta}_j, \mathbf{p}_j)$$

for few $\hat{\theta}_j$ and $\mathbf{p}_j$ and for $\hat{x} \in S_j$ a portion of the unit circle.

The unknowns:

Γ, λ and η.

$$\text{Minimize } F(\lambda, \eta, \Gamma) := \frac{1}{2}\sum_{j=1}^{I} \left\| \mathbf{E}^\infty_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j, \mathbf{p}_j) - \mathbf{E}^\infty_{\text{obs}}(\cdot, \hat{\theta}_j, \mathbf{p}_j) \right\|^2_{\mathbf{L}^2_t(S_j)}$$

Shape derivative for Maxwell

Definition

The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$R : \varepsilon \longrightarrow \mathbf{E}^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

Notations:

$$\Gamma_\varepsilon := f_\varepsilon(\Gamma)\ , \quad \lambda_\varepsilon := \lambda \circ f_\varepsilon\ , \quad \eta_\varepsilon := \eta \circ f_\varepsilon$$

Result:

$$dR(0) \cdot \varepsilon = \mathbf{v}^s_\varepsilon$$

where $(\mathbf{v}^s_\varepsilon, \mathbf{w}^s_\varepsilon)$ is an outgoing solution to the Maxwell equations outside Ω and

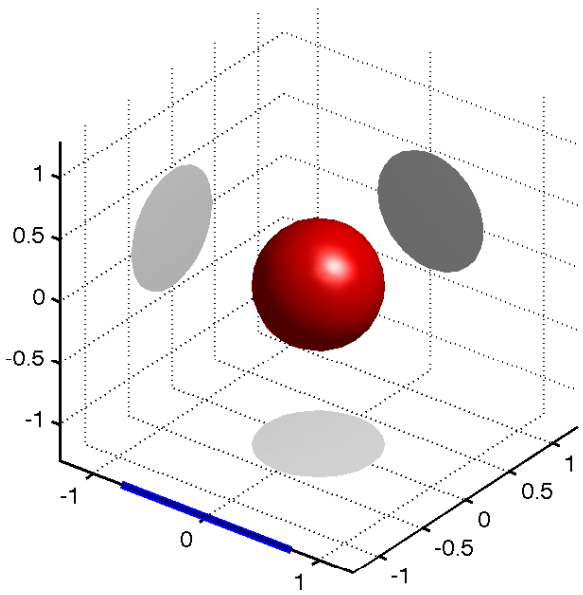
$$\nu \times \mathbf{v}^s_\varepsilon + \mathbf{Z}\mathbf{w}^s_{T,\varepsilon} = B_\varepsilon(\mathbf{E}, \mathbf{H}) \quad \text{on } \Gamma$$

$$\begin{aligned} B_\varepsilon(\mathbf{E}, \mathbf{H}) := &- ik(\nu \cdot \varepsilon)\mathbf{H}_T + \mathbf{rot}_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathbf{E})] + \lambda(\nu \cdot \varepsilon)\,(2R - 2H)\,\mathbf{H}_T \\ &- \lambda \nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathbf{H})] + 2\mathbf{rot}_\Gamma[H(\nu \cdot \varepsilon)\eta \mathrm{rot}_\Gamma(\mathbf{H}_T)] \\ &+ (\nabla_\Gamma \lambda \cdot \varepsilon)\mathbf{H}_T + \mathbf{rot}_\Gamma[(\nabla_\Gamma \eta \cdot \varepsilon)\mathrm{rot}_\Gamma(\mathbf{H}_T) \\ &+ ik\mathbf{Z}[(\nu \cdot \varepsilon)\mathbf{Z}\mathbf{H}_T] \end{aligned}$$

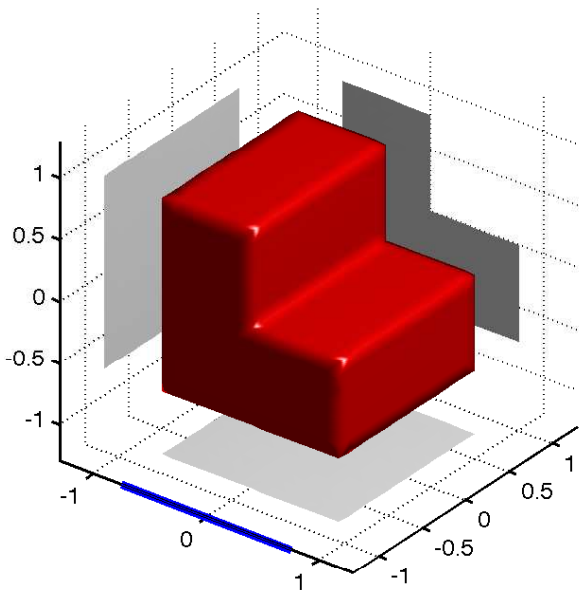
where $\mathbf{E}$ and $\mathbf{H}$ are the total fields for the reference shape Γ.

Numerical results

- $\lambda = 0$, $\eta = -0.25i$, $k = 4$, $\delta = 2\%$
- 4 incident plane waves



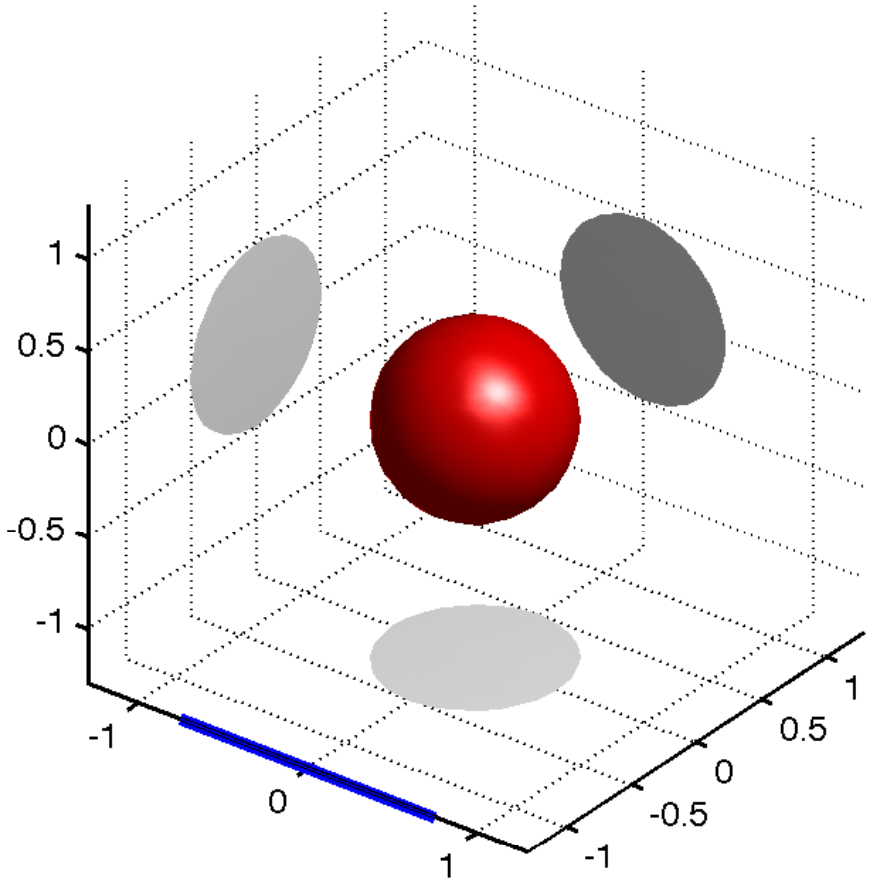
(a) Initial shape



(b) Target

Numerical results

- $\lambda = 0$, $\eta = -0.25i$, $k = 4$, $\delta = 2\%$
- 4 incident plane waves



Conclusions

- The forward problem.
  - ✓ It is well posed for Helmholtz equation for general Fredholm type impedance operator
  - ✓ It is also well posed in the Maxwell case with more restrictive assumptions on the impedance operator

Conclusions

- The forward problem.
  - ✓ It is well posed for Helmholtz equation for general Fredholm type impedance operator
  - ✓ It is also well posed in the Maxwell case with more restrictive assumptions on the impedance operator

- Qualitative methods.
  - ✓ Fast and require very few *a priori* assumptions
  - ✗ The reconstructions are a bit blurry

Conclusions

- The forward problem.
  - ✓ It is well posed for Helmholtz equation for general Fredholm type impedance operator
  - ✓ It is also well posed in the Maxwell case with more restrictive assumptions on the impedance operator

- Qualitative methods.
  - ✓ Fast and require very few *a priori* assumptions
  - ✗ The reconstructions are a bit blurry

- Quantitative methods.
  - ✓ Provide accurate and robust results and allow thin coating reconstructions
  - ✗ Difficult to extend to more general operators
    ⚠ We only considered second order surface operators
  - ✗ Difficult to implement in $3D$
    ⚠ Possible problems with the quality of the successive meshes

Open questions and future work

- The forward problem in the Maxwell case.
- ? How can we numerically solve a $\nabla_\Gamma \eta \mathrm{div}_\Gamma$ problem?
- ? Implementation of boundary integral methods? ($\rightarrow$ `Pernet & al.`)

- The Qualitative methods.
- ? Extension to the case of $\mathbf{Z} : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$?
- ? Extension to the Maxwell case?
- ? Many open questions related to the interior transmission eigenvalues

- The Quantitative methods.
- ? Use of Newton type methods? ($\rightarrow$ `Farhat-Tezaur-Djellouli 02`)
- ? Can we consider more general operators?
- ? Can boundary integral equation simplify the numerical resolution in $3D$?

Open questions and future work

- The forward problem in the Maxwell case.
- ? How can we numerically solve a $\nabla_\Gamma \eta \mathrm{div}_\Gamma$ problem?
- ? Implementation of boundary integral methods? ($\rightarrow$ `Pernet & al.`)

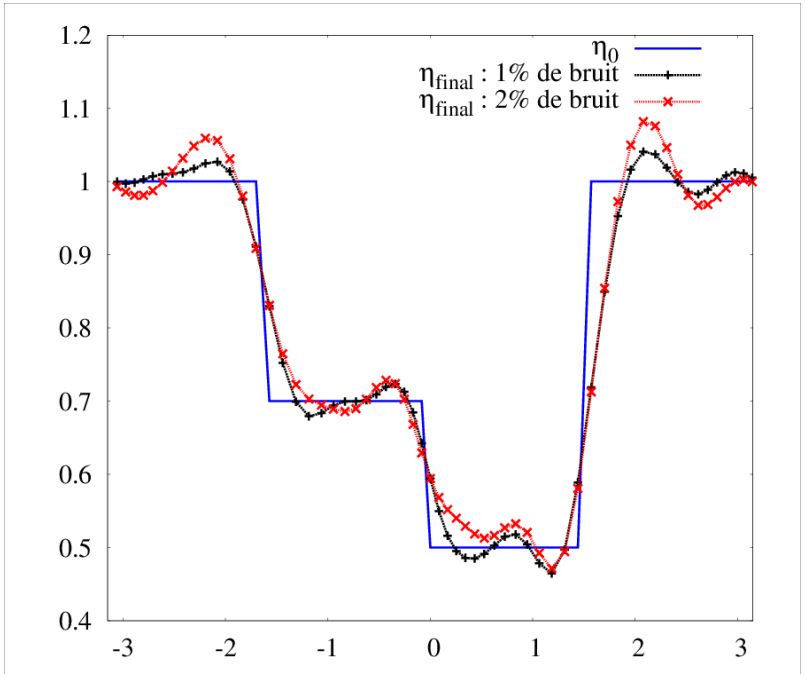
- The Qualitative methods.
- ? Extension to the case of $\mathbf{Z} : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$?
- ? Extension to the Maxwell case?
- ? Many open questions related to the interior transmission eigenvalues

- The Quantitative methods.
- ? Use of Newton type methods? ($\rightarrow$ `Farhat-Tezaur-Djellouli 02`)
- ? Can we consider more general operators?
- ? Can boundary integral equation simplify the numerical resolution in $3D$?

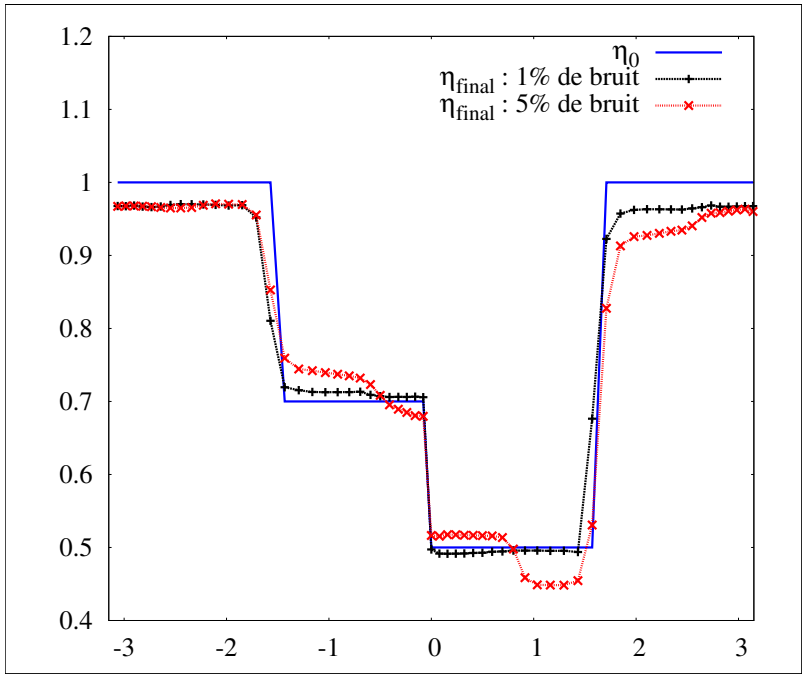
Thank You!

Numerical reconstruction

TV regularization for piecewise constant coefficients



$H^1(\Gamma)$ regularization



TV regularization

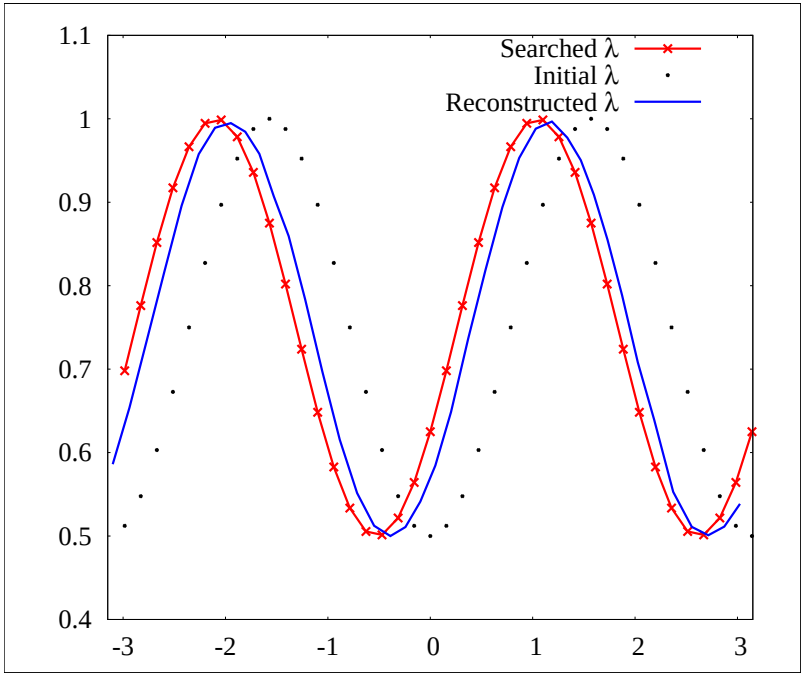
Reconstruction of a piecewise constant η on an ellipse

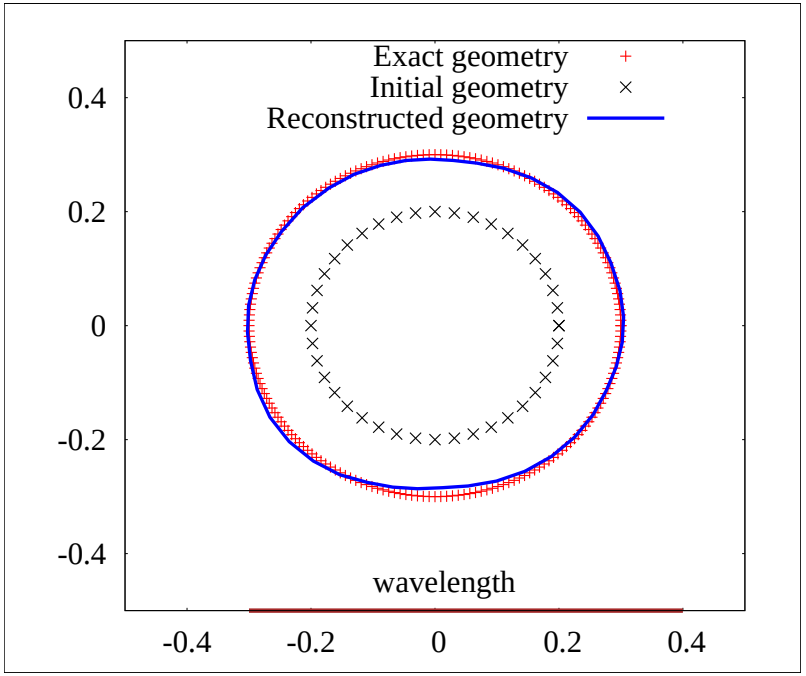
TV cost functional:

$$F_{\mathrm{TV}}(\eta) = \frac{1}{2}\|u^{\infty}_{\lambda,\eta,\Gamma} - u^{\infty}_{\mathrm{obs}}\|_{L^2(S^d)} + \gamma|\nabla_\Gamma \eta|_{L^1(\Gamma)}$$

Numerical reconstruction

Simultaneous reconstruction of λ, Γ with $\eta = 0$





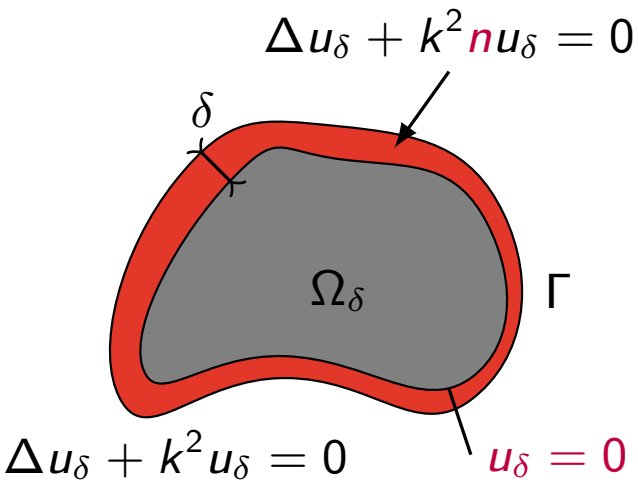
8 *incident waves*, 5% *of noise on far–field data.*

We iterate only on the geometry.

$$B_\varepsilon u = (\nabla_\Gamma \lambda \cdot \varepsilon) u + \cdots$$

The interior transmission eigenvalues for coatings

(TM mode)



Def: Interior transmission eigenvalue problem
Find $(v_\delta, w_\delta) \in L^2(\Omega) \times L^2(\Omega \setminus \overline{\Omega_\delta})$ and $k_\delta^2 > 0$ such that

$$\begin{cases} \Delta v_\delta + k_\delta^2 v_\delta = 0 \text{ in } \Omega, \\ \Delta w_\delta + k_\delta^2 n w_\delta = 0 \text{ in } \Omega \setminus \overline{\Omega_\delta} \\ \frac{\partial v_\delta}{\partial \nu} = \frac{\partial w_\delta}{\partial \nu}, \quad v_\delta = w_\delta \text{ on } \Gamma \\ w_\delta = 0 \text{ on } \Gamma_\delta \end{cases}$$

Prop: [Cakoni-Cossonière-Haddar 13]
If $0 < n < 1$ then the interior transmission eigenvalues exist and form a discrete set of $\mathbb{R}$.

Theorem [Cakoni-C.-Haddar]

The first eigenvalue k_δ^2 expands as

$$k_\delta^2 = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + \mathcal{O}(\delta^3).$$

Notation: λ_0 = first Laplacien-Dirichlet eigenvalue inside Ω

$\lambda_1 = \int_\Gamma \left|\frac{\partial v_0}{\partial \nu}\right| ds$ where v_0 is the first Dirichlet eigenvector.