Simultaneous reconstruction of an obstacle and its Generalized Impedance Boundary Condition

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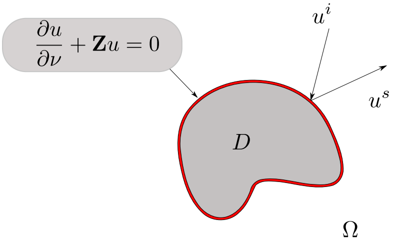
The Generalized Impedance Boundary Conditions in acoustic scattering



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$$u = u^s + u^i$$
$$\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0$$

Context:

- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Advantages:

- Cheaper direct computation (no mesh refinement)

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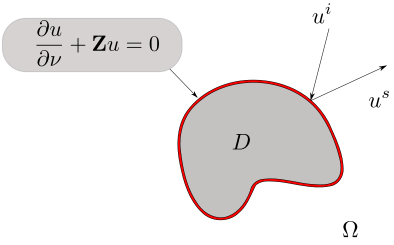

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Advantages:

- Cheaper direct computation (no mesh refinement)
- Inverse problem less unstable

Inverse problem: recover D and $\mathbf{Z}$ from the scattered field.

Outline

Example of generalized impedance boundary condition

Most commonly used impedance operator:

$$\mathbf{Z} = \lambda \quad \text{a function.}$$

Here, we consider a more general model:

$$\boxed{\mathbf{Z}u = \operatorname{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u.}$$

For example, this corresponds to the first order approximation of the solution for thin coatings

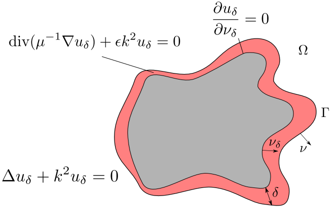
$$\mathbf{Z}u = \operatorname{div}_\Gamma(\mu^{-1}\delta \nabla_\Gamma u) + \delta k^2 \epsilon u$$

where

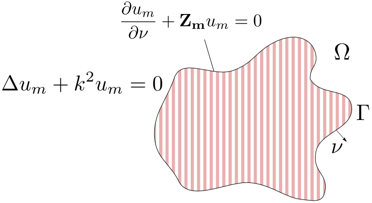
- ϵ and μ are the electromagnetic constants inside the coating,
- δ is the width of the coating (non necessarily constant).

Approximate model for a perfect conductor coated with a thin dielectric layer

$$\boxed{\text{At least formally: } \|u_\delta - u_m\| \le C\delta^{m+1}}$$



Exact model



Approximate model of order m

In dimension 2: *(Aslanyüreck, Haddar, Şahintürk [11])*

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}\delta\mu^{-1}\frac{\partial}{\partial s} + \delta k^2 \epsilon$$

$$\mathbf{Z}_2 = \frac{\partial}{\partial s}\left(\delta - \frac{\delta^2 c}{2}\right)\mu^{-1}\frac{\partial}{\partial s} + \left(\delta + \frac{\delta^2 c}{2}\right)k^2\epsilon.$$

The forward problem

Find $u = u^s + u^i$ such that

$$\boxed{u^s \in \left\{ v \in \mathcal{D}'(\Omega)\,,\ \varphi v \in\ H^1(\Omega)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\partial D}\ \in\ H^1(\partial D) \right\}}$$

and

$$(\mathcal{P}) \quad \left\{ \begin{array}{l} \Delta u + k^2 u = 0 \quad \text{in } \Omega := \mathbb{R}^d \setminus \overline{D} \\ \dfrac{\partial u}{\partial \nu} + \mathrm{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \partial D \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0. \end{array} \right.$$

u exists and is unique if

- $\Im m(\lambda) \geq 0,\ \Im m(\eta) \leq 0$ a.e. on ∂D (physical assumption)
- $\Re e(\eta) \geq c$ a.e. on ∂D for $c > 0$.

The inverse problem

The far field map

For $u^i(x,\hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$T : (\lambda, \eta, \partial D, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})$$

where u^∞ associated with u^s is defined in dimension d by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

The inverse problem

Given N far–fields $(u^\infty(\cdot, \hat{\theta}_j))_{j=1,\cdots,N}$, retrieve λ, η and the geometry ∂D,

$$(u^\infty(\cdot, \hat{\theta}_j))_{j=1,\cdots,N} \mapsto (\lambda, \eta, \partial D).$$

Outline

1. The forward and inverse problems
2. Uniqueness and stability for the inverse problem
   - The case of a single incident wave, known obstacle
   - The case of infinitely many incident plane waves
3. A steepest descent method to solve the inverse problem
   - Presentation of the method
   - Computation of the shape derivative of the scattered field
4. Numerical experiments

Uniqueness for λ

The case $\eta = 0$

Uniqueness for λ with a single incident wave (Colton & Kirsch 81)

Let ∂D be a Lipschitz domain. Let λ_1 and λ_2 be two continuous functions. If for some incident direction $\hat{\theta}_0$ we have

$$T(\lambda_1, 0, \partial D, \hat{\theta}_0) = T(\lambda_2, 0, \partial D, \hat{\theta}_0)$$

then $\lambda_1 = \lambda_2$.

Proof

If $T(\lambda_1, 0, \partial D, \hat{\theta}_0) = T(\lambda_2, 0, \partial D, \hat{\theta}_0)$ *then* $u_{\lambda_1} = u_{\lambda_2}$ *in* Ω. *For* $u = u_{\lambda_1}$ *we have*

$$\frac{\partial u}{\partial \nu} + \lambda_i u = 0 \quad i = 1, 2 \text{ on } \partial D$$

$$(\lambda_2 - \lambda_1) u = 0 \quad \text{on } \partial D.$$

$\lambda_2 \neq \lambda_1$ *on* $S \subset \partial D \implies u = 0$ *and* $\frac{\partial u}{\partial \nu} = 0$ *on* S.

Then $u = 0$ *in* Ω, *contradiction with the radiation condition.*

What happens if $\eta \neq 0$?

- Let λ_1 and λ_2 be two continuous functions,
- and let η_1 and η_2 be two complex constants

such that

$$\boxed{T(\lambda_1, \eta_1, \partial D, \hat{\theta}) = T(\lambda_2, \eta_2, \partial D, \hat{\theta}).}$$

Denote u_i the total field given by (λ_i, η_i), then $u := u_1 = u_2$ outside D and on ∂D.

$$\frac{\partial u}{\partial \nu} + \eta_i \Delta_\Gamma u + \lambda_i u = 0 \quad i = 1, 2 \text{ on } \partial D$$

$$[(\eta_2 - \eta_1)\Delta_\Gamma \,.\, + (\lambda_2 - \lambda_1)]u = 0 \quad \text{on } \partial D.$$

No conclusion!

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No conclusion!

One can actually find $(\lambda_1, \eta_1) \neq (\lambda_2, \eta_2)$ such that

$$T(\lambda_1, \eta_1, \partial D, \hat{\theta}) = T(\lambda_2, \eta_2, \partial D, \hat{\theta})$$

(Bourgeois & Haddar 10)

Uniqueness and Lipschitz stability

λ and η piecewise constant

- Let $(\partial D_i)_{i=1,\cdots,I}$ be a partition of ∂D,
- let K_I be a compact subset of $L^\infty(\partial D)^2$ such that if $(\lambda, \eta) \in K_I$,

$$\lambda(x) = \sum_{i=1}^{I} \lambda_i \chi_{\partial D_i}(x)\,, \qquad \eta(x) = \sum_{i=1}^{I} \eta_i \chi_{\partial D_i}(x)$$

and assumptions for the forward problem are satisfied.

Global stability for λ (Sincich 07)

There exists $C^\lambda_{K_I} > 0$ such that for all (λ^1, η) and (λ^2, η) in K_I,

$$\|\lambda^1 - \lambda^2\| \leq C^\lambda_{K_I} \|T(\lambda^1, \eta, \partial D) - T(\lambda^2, \eta, \partial D)\|.$$

[Proof: Appropriate Carleman estimates
Continuity of the near field to far field map.]

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  and assumptions for the forward problem are satisfied.
- $\forall i = 1,\cdots,I$ it exists $S_i \subset \partial D_i$ such that $\forall(\lambda,\eta) \in K_I$

$$(\mathcal{H}) \qquad \Delta_\Gamma u_{\lambda,\eta} \neq 0 \quad \text{on } S_i.$$

Global stability for η (Bourgeois, C. & Haddar 11)

There exists $C^\eta_{K_I} > 0$ such that for all (λ,η^1) and (λ,η^2) in K_I,

$$\|\eta^1 - \eta^2\| \leq C^\eta_{K_I} \|T(\lambda,\eta^1,\partial D) - T(\lambda,\eta^2,\partial D)\|.$$

[Proof: Appropriate Carleman estimates
Continuity of the near field to far field map.]

Outline

Uniqueness of the identification of λ, η and ∂D

Uniqueness

Let D_1, D_2 be two C^2 open bounded sets and take (λ_1, η_1) and $(\lambda_2, \eta_2) \in L^\infty \times W^{1,\infty}$.

$$u_1^\infty = T(\lambda_1, \eta_1, \partial D_1) \text{ and } u_2^\infty = T(\lambda_2, \eta_2, \partial D_2)$$

if $u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta}) \; \forall(\hat{x}, \hat{\theta}) \in (S^{d-1})^2$ then

$$D_1 = D_2 \quad \text{and} \quad (\lambda_1, \eta_1) = (\lambda_2, \eta_2).$$

<u>Main tools</u>

– The mixed reciprocity principle: for z outside D_1 and D_2

$$v^\infty(-\hat{x}, z) = u^s(z, \hat{x}),$$

leads to $D_1 = D_2$.

– Density of $\{u(\cdot, \hat{\theta}), \; \hat{\theta} \in S^{d-1}\}$ in $H^1(\partial D)$ gives $(\lambda_1, \eta_1) = (\lambda_2, \eta_2)$.

Uniqueness still holds for a general symmetric surface operator $\mathbf{Z}$.

Proof of uniqueness

$$\boxed{(\mathbf{Z}_1, D_1) \longrightarrow u_1^\infty,\ (\mathbf{Z}_2, D_2) \longrightarrow u_2^\infty \text{ and } u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})}$$

- Assume that $D_1 \neq D_2$.

  Mixed reciprocity principle: $\boxed{4\pi v^\infty(-\hat{x}, z)} = \boxed{u^s(z, \hat{x})}$ for $z \in \mathbb{R}^3 \setminus \overline{D_1 \cup D_2}$.

Far–field given by $\Phi_z(x) := \frac{e^{ik|x-z|}}{4\pi|x-z|}$

Scattered field given by $e^{ik\hat{x}\cdot y}$

Proof of uniqueness

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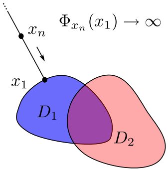
$$\Downarrow$$

$$u_1^s(x, x_0) = u_2^s(x, x_0) \quad \forall (x, x_0) \in (\mathbb{R}^N \setminus (D_1 \cup D_2))^2$$



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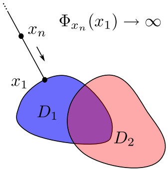

$$\begin{aligned}
\frac{\partial u_2^s}{\partial \nu}(x_1, x_n) + \mathbf{Z}_1 u_2^s(x_1, x_n) &= \frac{\partial u_1^s}{\partial \nu}(x_1, x_n) + \mathbf{Z}_1 u_1^s(x_1, x_n) \\
&= -\left(\frac{\partial \Phi_{x_n}}{\partial \nu}(x_1) + \mathbf{Z}_1 \Phi_{x_n}(x_1)\right) \\
&\underset{x_n \longrightarrow x_1}{\longrightarrow} \infty
\end{aligned}$$

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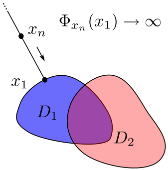

but $\partial_\nu u_2^s(x_1, x_1) + \mathbf{Z}_1 u_2^s(x_1, x_1)$ remains bounded!

$\boxed{\text{Conclusion: } D_1 = D_2.}$

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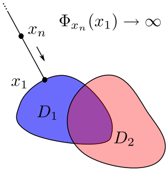

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$$\boxed{\text{Conclusion: } D_1 = D_2.}$$

- For every $\hat{\theta}$, $u_1(\cdot, \hat{\theta})$ satisfies

$$\boxed{\mathbf{Z}_1 u_1(\cdot, \hat{\theta}) = \mathbf{Z}_2 u_1(\cdot, \hat{\theta}).}$$

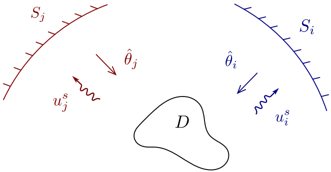
The density of $\{u_1(\cdot, \hat{\theta})\,,\ \hat{\theta} \in S^{d-1}\}$ in $H^1(\partial D)$ gives

$$\mathbf{Z}_1 \varphi = \mathbf{Z}_2 \varphi \quad \forall \varphi \in H^1(\partial D).$$

Test with well chosen functions to have $(\lambda_1, \eta_1) = (\lambda_2, \eta_2)$.

Outline

Solving the inverse problem with a finite number of incident waves



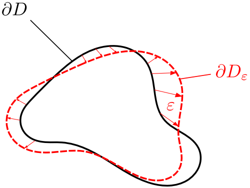
$$F(\lambda, \eta, \partial D) := \frac{1}{2} \sum_{j=1}^{I} \| T(\lambda, \eta, \partial D, \hat{\theta}_j) - u^\infty_{\mathrm{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

For minimizing F:

- we need partial derivatives of the far–field with respect to λ and η (quite standard),
- we need an appropriate derivative w.r.t. the obstacle.

Difficulty: the unknown impedances are supported by ∂D.

Derivative of the cost function with respect to the obstacle



$$\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ is “small”}$$

$$f_\varepsilon := \mathrm{Id} + \varepsilon$$

$$\partial D_\varepsilon := f_\varepsilon(\partial D)$$

$$\lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1}, \quad \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}$$

We define the derivative v_ε of the scatered field with respect to the geometry at point $(\lambda, \eta, \partial D)$ by

$$u^s(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon) - u^s(\lambda, \eta, \partial D) = v_\varepsilon + o(||\varepsilon||)$$

where $\varepsilon \mapsto v_\varepsilon$ is linear.

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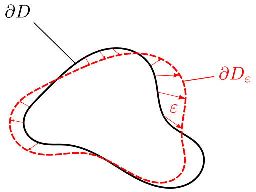

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where $\varepsilon \mapsto v_\varepsilon$ is linear.

One may find f_ε such that $\partial D = f_\varepsilon(\partial D)$ and

$$\boxed{F'_{\lambda,\eta}(\partial D) \cdot \varepsilon \neq 0.}$$

$\hookrightarrow$ $F'_{\lambda,\eta}(\partial D)$ does not satisfy the classical shape derivative's properties!

Derivative of the scattered field with respect to the obstacle

Let $(\lambda, \eta, \partial D)$ be given and analytics, for some small $\varepsilon \in C^{1,\infty}$ define

$$\partial D_\varepsilon = f_\varepsilon(\partial D)\,, \quad \lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1} \text{ and } \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}.$$

Let u^s_ε $[u^s]$ be scattered field associated with $(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon)$ $[(\lambda, \eta, \partial D)]$.

$$\boxed{u^s_\varepsilon(x) - u^s(x) = v_\varepsilon(x) + o(||\varepsilon||),}$$

where $v_\varepsilon(x)$ is the solution of the scattering problem with

$$\frac{\partial v_\varepsilon}{\partial \nu} + \mathbf{Z} v_\varepsilon = B_\varepsilon u \quad \text{on} \quad \partial D$$

$$\begin{aligned} B_\varepsilon u =& (\varepsilon \cdot \nu)(k^2 - 2H\lambda)u + \mathrm{div}_\Gamma\left((Id + 2\eta(R - H\,Id))(\varepsilon \cdot \nu)\nabla_\Gamma u\right) \\ &+ (\nabla_\Gamma \lambda \cdot \varepsilon)u + \mathrm{div}_\Gamma\left((\nabla_\Gamma \eta \cdot \varepsilon)\nabla_\Gamma u\right) \\ &+ \mathbf{Z}\left((\varepsilon \cdot \nu)\mathbf{Z}u\right), \end{aligned}$$

with $2H := \mathrm{div}_\Gamma \nu$, $R := \nabla_\Gamma \nu$ and $\mathbf{Z}\cdot = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$

Main tools of the proof

- Domain derivative tools: Murat and Simon [73], Kirsch [93], Hettlich [94], Potthast [94].
- Green's theorems and integral representation of the scattered field: Kress and Päivärinta [99], Haddar and Kress [04].

Green's theorems and integral representation: write

$$u_\varepsilon^s - u^s = -\int_{\partial D} (B_\varepsilon u) w(\cdot, y) ds(y) + \mathrm{o}(\|\varepsilon\|)$$

where for $y \in \Omega$ $w(\cdot, y) = w^s(\cdot, y) + \Phi(\cdot, y)$ is the Green function associated with the GIBC scattering problem

$$\left\{ \begin{array}{l} \Delta w(\cdot, y) + k^2 w(\cdot, y) = \delta_y \quad \text{in } \Omega \\ \dfrac{\partial w}{\partial \nu} + \mathbf{Z} w = 0 \quad \text{on } \partial D \\ +\text{radiation condition.} \end{array} \right.$$

Sketch of the proof (1/2)

Volume extension of the surface objects between ∂D and ∂D_ε

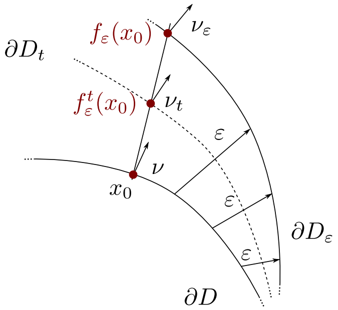
D_ε is outside D

$$D^\star = D_\varepsilon \setminus \overline{D}$$

- We parametrize $D^\star$ with $f_\varepsilon^t := Id + t\varepsilon$ for $t \in [0,1]$,

$$D^\star \ni x_t = x_0 + t\varepsilon(x_0)$$

- $\lambda_t = \lambda \circ (f_\varepsilon^t)^{-1}\,, \quad \eta_t = \eta \circ (f_\varepsilon^t)^{-1}$
- for a given t, $\partial D_t := f_\varepsilon^t(\partial D)$,
- ν_t: outward unit normal of ∂D_t, the direction of ν_t depends on t!
- $\nabla_{\Gamma_t} \cdot := \left(\nabla \cdot - \dfrac{\partial \cdot}{\partial \nu_t} \nu_t \right) |_{\partial D_t}$



Sketch of the proof (2/2)

Write an integral representation on ∂D

Objective: write the $u^s_\varepsilon - u^s$ as

$$u^s_\varepsilon(x) - u^s(x) = -\int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + \mathrm{o}(\|\varepsilon\|).$$

Integral representation formula for x outside D_ε:

$$u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \mathrm{div}_{\Gamma_\varepsilon}(\eta_\varepsilon \nabla_{\Gamma_\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon,$$

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$$u^s_\varepsilon(x) - u^s(x) = -\int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + \mathrm{o}(\|\varepsilon\|).$$

Integral representation formula for x outside D_ε:

$$u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \mathrm{div}_{\Gamma_\varepsilon}(\eta_\varepsilon \nabla_{\Gamma_\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon + o(\|\varepsilon\|),$$

Gauss divergence theorem

$$= \int_{D_\varepsilon \setminus \overline{D}} \mathrm{div} \left\{ u \nabla w - \eta_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \, \nu_t + \lambda_t u w \, \nu_t \right\} dx + o(\|\varepsilon\|)$$

Sketch of the proof (2/2)

Write an integral representation on ∂D

Objective: write the $u_\varepsilon^s - u^s$ as

$$u_\varepsilon^s(x) - u^s(x) = -\int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + \mathrm{o}(\|\varepsilon\|).$$

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Gauss divergence theorem

$$= \int_{\partial D} (\varepsilon \cdot \nu) \int_0^1 \mathrm{div}\left(-\eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t u w \nu_t\right) dt\, ds + o(||\varepsilon||)$$

Sketch of the proof (2/2)

Write an integral representation on ∂D

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Gauss divergence theorem and Taylor expansion:

$$= \int_{\partial D} (\varepsilon \cdot \nu) \,\mathrm{div}\left(-\eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t u w \nu_t\right)|_{t=0} ds + o(||\varepsilon||)$$

Sketch of the proof (2/2)

Write an integral representation on ∂D

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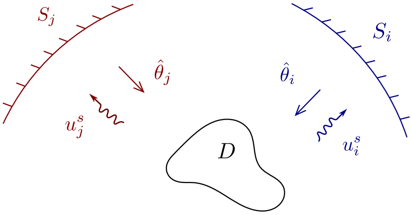
$$= \int_{\partial D} (\varepsilon \cdot \nu) \,\mathrm{div}\left(-\eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t u w \nu_t\right)|_{t=0} ds + o(||\varepsilon||)$$

$$\mathrm{div}\left(\lambda_t u w \nu_t\right)|_{t=0} = uw\, (\nabla \lambda_t)|_{t=0} \cdot \nu + \lambda \frac{\partial (uw)}{\partial \nu} + \lambda uw \,\mathrm{div}(\nu_t)|_{t=0}$$

$$\boxed{(\nu \cdot \varepsilon)(\nabla \lambda_t)|_{t=0} \cdot \nu = -\nabla_\Gamma \lambda \cdot \varepsilon}$$

Outline

A steepest descent method to solve the inverse coefficient problem



$$F(\lambda, \eta, \partial D) := \frac{1}{2} \sum_{j=1}^{I} \| T(\lambda, \eta, \partial D, \hat{\theta}_j) - u^\infty_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

Numerical procedure:

- update alternatively λ, η and ∂D with a direction given by the partial derivative of the cost function,
- when we update the geometry we also transport the impedance coefficients to the new boundary.

The regularization procedure

$$F(\lambda, \eta, \partial D) = \frac{1}{2} \sum_{j=1}^{I} \|T(\lambda, \eta, \partial D, \hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

We regularize the gradient, NOT the cost function, using a $H^1(\partial D)$ regularization.

- Descent direction for λ: $\delta\lambda$ that solves for every ϕ in some finite dimensional space:

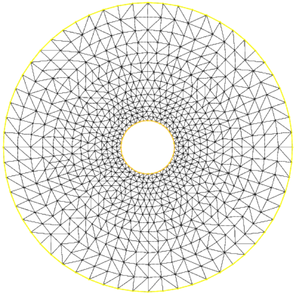
$$\beta_\lambda \int_{\partial D} \nabla_\Gamma(\delta\lambda) \cdot \nabla_\Gamma \phi \, ds + \int_{\partial D} \delta\lambda \phi \, ds = -\alpha_\lambda \, F'_{\eta,\partial D}(\lambda) \cdot \phi$$

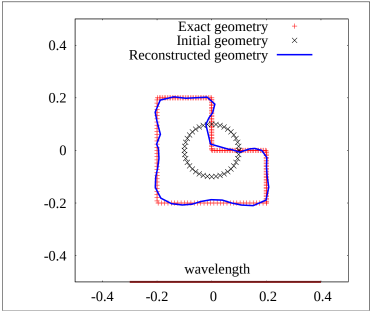
where β_λ is the regularization coefficient and α_λ is the descent coefficient.

- Do the same for $\delta\eta$ and $\delta(\partial D)$.

Numerical reconstruction

Finite elements method and remeshing procedure using FreeFem++

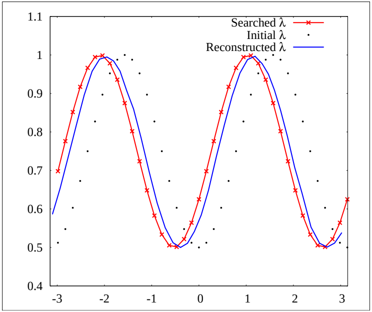


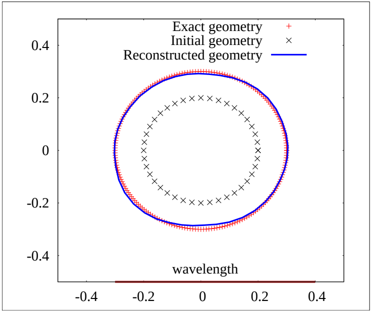


Reconstruction of the geometry with 2 incident waves and 1% noise on the far–field, $\lambda = ik/2$ and $\eta = 2/k$ being known

Numerical reconstruction

Simultaneous reconstruction of λ, ∂D with $\eta = 0$



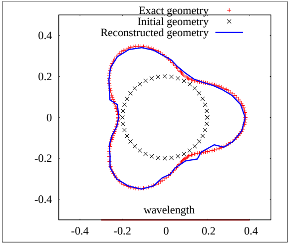


8 incident waves, 5% of noise on far–field data.
We iterate only on the geometry.

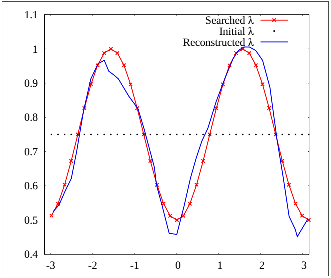
$$B_\varepsilon u = (\nabla_\Gamma \lambda \cdot \varepsilon) u + \cdots$$

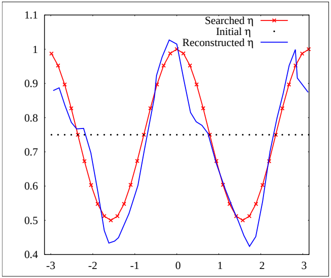
Numerical reconstruction

Simultaneous reconstruction of λ, η and ∂D

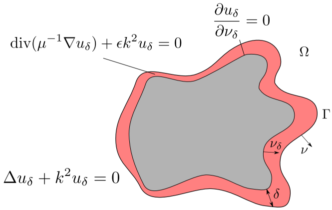


8 incident waves, 5% of noise on far–field data.

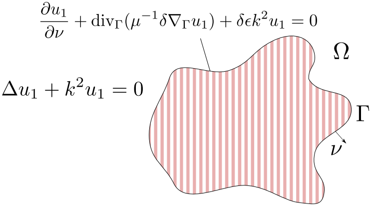




Application to the reconstruction of a coated obstacle



Exact model



Approximate model of order 1

Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

$$F(\epsilon, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \|T(\epsilon, \delta, \Gamma, \hat{\theta}_j) - u^{\infty}_{\mathrm{obs,mince}}(\cdot, \hat{\theta}_j)\|^2_{L^2(S_j)}$$

with $\mu = 0.1$ known.

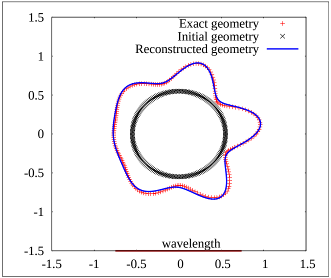
Application to the reconstruction of a coated obstacle

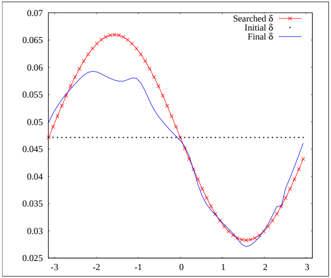
Numerical results

Artificial data created with

- $\mu = 0.1$ is known,
- $\delta = 0.04l(1 - 0.4\sin(\theta))$ is unknown; l being the wavelength,
- $\epsilon = 2.5$ is unknown.

Reconstructed ϵ: 2.3.





Fails with a classical impedance boundary condition model!

Conclusion

- The inverse problem is ill-posed but not too much.
- It is solvable using a steepest descent method with regularization.
- Possible reconstruction of coated obstacles.

- Extension to the $3D$ Maxwell equations (ongoing work).
- The case of a general symmetric operator on the boundary?