Simultaneous reconstruction of an obstacle and its Generalized Impedance Boundary Condition

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The Generalized Impedance Boundary Conditions in acoustic scattering

Context:
- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Advantages:
- Cheaper direct computation (no mesh refinement)
The Generalized Impedance Boundary Conditions in acoustic scattering

Context:
- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...

Advantages:
- Cheaper direct computation (no mesh refinement)
- Inverse problem less unstable

Inverse problem: recover $D$ and $Z$ from the scattered field.
Outline

1. The forward and inverse problems

2. Uniqueness and stability for the inverse problem
   - The case of a single incident wave, known obstacle
   - The case of infinitely many incident plane waves

3. A steepest descent method to solve the inverse problem
   - Presentation of the method
   - Computation of the shape derivative of the scattered field

4. Numerical experiments
Example of generalized impedance boundary condition

Most commonly used impedance operator:

\[ Z = \lambda \]  

a function.

Here, we consider a more general model:

\[ Zu = \text{div}_\Gamma (\eta \nabla_\Gamma u) + \lambda u. \]

For example, this corresponds to the first order approximation of the solution for thin coatings

\[ Zu = \text{div}_\Gamma (\mu^{-1} \delta \nabla_\Gamma u) + \delta k^2 \epsilon u \]

where

- \( \epsilon \) and \( \mu \) are the electromagnetic constants inside the coating,
- \( \delta \) is the width of the coating (non necessarily constant).
Approximate model for a perfect conductor coated with a thin dielectric layer

At least formally: \( \| u_\delta - u_m \| \leq C \delta^{m+1} \)

In dimension 2: (Aslanyüreck, Haddar, Şahintürk [11])

\[
\begin{align*}
Z_1 &= \frac{\partial}{\partial s} \delta \mu^{-1} \frac{\partial}{\partial s} + \delta k^2 \epsilon \\
Z_2 &= \frac{\partial}{\partial s} \left( \delta - \frac{\delta^2 c}{2} \right) \mu^{-1} \frac{\partial}{\partial s} + \left( \delta + \frac{\delta^2 c}{2} \right) k^2 \epsilon.
\end{align*}
\]
The forward problem

Find \( u = u^s + u^i \) such that

\[
\begin{align*}
    u^s &\in \left\{ v \in \mathcal{D}'(\Omega), \ \varphi v \in H^1(\Omega) \ \forall \varphi \in \mathcal{D}(\mathbb{R}^d); \ v|_{\partial D} \in H^1(\partial D) \right\} \\
\end{align*}
\]

and

\[
\begin{align*}
    (\mathcal{P}) \quad &\begin{cases}
    \Delta u + k^2 u = 0 \quad \text{in } \Omega := \mathbb{R}^d \setminus \overline{D} \\
    \frac{\partial u}{\partial \nu} + \text{div}_\Gamma (\eta \nabla u) + \lambda u = 0 \quad \text{on } \partial D \\
    \lim_{R \to \infty} \int_{|x| = R} \left| \frac{\partial u^s}{\partial r} - i k u^s \right|^2 \ ds = 0.
    \end{cases}
\end{align*}
\]

\( u \) exists and is unique if

- \( \Im(m(\lambda)) \geq 0, \ \Im(m(\eta)) \leq 0 \text{ a.e. on } \partial D \) (physical assumption)
- \( \Re(\eta) \geq c \text{ a.e. on } \partial D \text{ for } c > 0. \)
The inverse problem

The far field map
For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define
\[
T : (\lambda, \eta, \partial D, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})
\]
where \( u^\infty \) associated with \( u^s \) is defined in dimension \( d \) by
\[
u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O \left( \frac{1}{r} \right) \right) \quad r \to +\infty.
\]

Given \( N \) far–fields \((u^\infty(\cdot, \hat{\theta}_j))_{j=1,\ldots,N}\), retrieve \( \lambda, \eta \) and the geometry \( \partial D \),
\[
(u^\infty(\cdot, \hat{\theta}_j))_{j=1,\ldots,N} \mapsto (\lambda, \eta, \partial D).
\]
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Uniqueness for $\lambda$

The case $\eta = 0$

Uniqueness for $\lambda$ with a single incident wave (Colton & Kirsch 81)

Let $\partial D$ be a Lipschitz domain. Let $\lambda_1$ and $\lambda_2$ be two continuous functions. If for some incident direction $\hat{\theta}_0$ we have

$$T(\lambda_1, 0, \partial D, \hat{\theta}_0) = T(\lambda_2, 0, \partial D, \hat{\theta}_0)$$

then $\lambda_1 = \lambda_2$.

Proof

If $T(\lambda_1, 0, \partial D, \hat{\theta}_0) = T(\lambda_2, 0, \partial D, \hat{\theta}_0)$ then $u_{\lambda_1} = u_{\lambda_2}$ in $\Omega$. For $u = u_{\lambda_1}$ we have

$$\frac{\partial u}{\partial \nu} + \lambda_i u = 0 \quad i = 1, 2 \quad \text{on } \partial D$$

$$(\lambda_2 - \lambda_1)u = 0 \quad \text{on } \partial D.$$  

$\lambda_2 \neq \lambda_1$ on $S \subset \partial D \implies u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $S$.

Then $u = 0$ in $\Omega$, contradiction with the radiation condition.
What happens if $\eta \neq 0$?

- Let $\lambda_1$ and $\lambda_2$ be two continuous functions,
- and let $\eta_1$ and $\eta_2$ be two complex constants
such that

$$T(\lambda_1, \eta_1, \partial D, \hat{\theta}) = T(\lambda_2, \eta_2, \partial D, \hat{\theta}).$$

Denote $u_i$ the total field given by $(\lambda_i, \eta_i)$, then $u := u_1 = u_2$
outside $D$ and on $\partial D$.

$$\frac{\partial u}{\partial \nu} + \eta_i \Delta_{\Gamma} u + \lambda_i u = 0 \quad i = 1, 2 \text{ on } \partial D$$

$$[(\eta_2 - \eta_1) \Delta_{\Gamma} . + (\lambda_2 - \lambda_1)]u = 0 \quad \text{on } \partial D.$$
What happens if $\eta \neq 0$?

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Denote $u_i$ the total field given by $(\lambda_i, \eta_i)$, then $u := u_1 = u_2$ outside $D$ and on $\partial D$.

$$\frac{\partial u}{\partial \nu} + \eta_i \Delta_{\Gamma} u + \lambda_i u = 0 \quad i = 1, 2 \text{ on } \partial D$$

$$[(\eta_2 - \eta_1) \Delta_{\Gamma}. + (\lambda_2 - \lambda_1)]u = 0 \quad \text{on } \partial D.$$ 

No conclusion!

One can actually find $(\lambda_1, \eta_1) \neq (\lambda_2, \eta_2)$ such that

$$T(\lambda_1, \eta_1, \partial D, \hat{\theta}) = T(\lambda_2, \eta_2, \partial D, \hat{\theta})$$

(Bourgeois & Haddar 10)
Uniqueness and Lipschitz stability
\( \lambda \) and \( \eta \) piecewise constant

- Let \((\partial D_i)_{i=1,\ldots,I}\) be a partition of \( \partial D \),
- let \( K_I \) be a compact subset of \( L^\infty(\partial D)^2 \) such that if \((\lambda, \eta) \in K_I\),

\[
\lambda(x) = \sum_{i=1}^{I} \lambda_i \chi_{\partial D_i}(x), \quad \eta(x) = \sum_{i=1}^{I} \eta_i \chi_{\partial D_i}(x)
\]

and assumptions for the forward problem are satisfied.

Global stability for \( \lambda \) (Sincich 07)

There exists \( C_{K_I}^\lambda > 0 \) such that for all \((\lambda^1, \eta)\) and \((\lambda^2, \eta)\) in \( K_I \),

\[
\|\lambda^1 - \lambda^2\| \leq C_{K_I}^\lambda \|T(\lambda^1, \eta, \partial D) - T(\lambda^2, \eta, \partial D)\|.
\]

[Proof: Appropriate Carleman estimates
Continuity of the near field to far field map.]
Uniqueness and Lipschitz stability

$\lambda$ and $\eta$ piecewise constant

- Let $(\partial D_i)_{i=1,\ldots,I}$ be a partition of $\partial D$,
- let $K_I$ be a compact subset of $L^\infty(\partial D)^2$ such that if $(\lambda, \eta) \in K_I$,
  \[
  \lambda(x) = \sum_{i=1}^I \lambda_i \chi_{\partial D_i}(x), \quad \eta(x) = \sum_{i=1}^I \eta_i \chi_{\partial D_i}(x)
  \]
  and assumptions for the forward problem are satisfied.
- $\forall i = 1, \cdots, I$ it exists $S_i \subset \partial D_i$ such that $\forall (\lambda, \eta) \in K_I$
  \[
  (\mathcal{H}) \quad \Delta_\Gamma u_{\lambda, \eta} \neq 0 \quad \text{on } S_i.
  \]

Global stability for $\eta$ (Bourgeois, C. & Haddar 11)

There exists $C^m_{K_I} > 0$ such that for all $(\lambda, \eta^1)$ and $(\lambda, \eta^2)$ in $K_I$,
\[
\|\eta^1 - \eta^2\| \leq C^m_{K_I} \|T(\lambda, \eta^1, \partial D) - T(\lambda, \eta^2, \partial D)\|.
\]

[Proof: Appropriate Carleman estimates
Continuity of the near field to far field map.]
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Uniqueness of the identification of $\lambda$, $\eta$ and $\partial D$

Uniqueness

Let $D_1$, $D_2$ be two $C^2$ open bounded sets and take $(\lambda_1, \eta_1)$ and $(\lambda_2, \eta_2) \in L^\infty \times W^{1,\infty}$.

$$u_1^\infty = T(\lambda_1, \eta_1, \partial D_1) \text{ and } u_2^\infty = T(\lambda_2, \eta_2, \partial D_2)$$

if $u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta}) \forall (\hat{x}, \hat{\theta}) \in (S^{d-1})^2$ then

$$D_1 = D_2 \text{ and } (\lambda_1, \eta_1) = (\lambda_2, \eta_2).$$

Main tools

- The mixed reciprocity principle: for $z$ outside $D_1$ and $D_2$

$$v^\infty(-\hat{x}, z) = u^s(z, \hat{x}),$$

leads to $D_1 = D_2$.

- Density of $\{u(\cdot, \hat{\theta}), \hat{\theta} \in S^{d-1}\}$ in $H^1(\partial D)$ gives $(\lambda_1, \eta_1) = (\lambda_2, \eta_2)$.

Uniqueness still holds for a general symmetric surface operator $Z$. 
Proof of uniqueness

\[ (Z_1, D_1) \rightarrow u_1^\infty, (Z_2, D_2) \rightarrow u_2^\infty \text{ and } u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta}) \]

Assume that \( D_1 \neq D_2 \).

Mixed reciprocity principle:

\[ 4\pi v^\infty(\hat{x}, z) = u^s(z, \hat{x}) \quad \text{for} \quad z \in \mathbb{R}^3 \setminus \overline{D_1 \cup D_2}. \]

Far-field given by \( \Phi_z(x) := \frac{e^{ik|x-z|}}{4\pi|x-z|} \)

Scattered field given by \( e^{ik\hat{x} \cdot y} \)
Proof of uniqueness

\[(Z_1, D_1) \rightarrow u_1^\infty, (Z_2, D_2) \rightarrow u_2^\infty \text{ and } u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})\]

Assume that \(D_1 \neq D_2\).

Mixed reciprocity principle: \(4\pi v^\infty(-\hat{x}, z) = u^s(z, \hat{x})\) for \(z \in \mathbb{R}^3 \setminus \overline{D_1 \cup D_2}\).

\[u^s_1(x, x_0) = u^s_2(x, x_0) \quad \forall (x, x_0) \in (\mathbb{R}^N \setminus (D_1 \cup D_2))^2\]
Proof of uniqueness

\[(Z_1, D_1) \rightarrow u_1^\infty, (Z_2, D_2) \rightarrow u_2^\infty \text{ and } u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})\]

Assume that \(D_1 \neq D_2\).

Mixed reciprocity principle: \(4\pi v^\infty(-\hat{x}, z) = u^s(z, \hat{x})\) for \(z \in \mathbb{R}^3 \setminus \overline{D_1 \cup D_2}\).

\[
\downarrow
\]

\[
\forall (x, x_0) \in (\mathbb{R}^N \setminus (D_1 \cup D_2))^2
u_1^s(x, x_0) = u_2^s(x, x_0)
\]

\[
\downarrow
\]

\[
\begin{align*}
\frac{\partial u_2^s}{\partial \nu}(x_1, x_n) + Z_1 u_2^s(x_1, x_n) &= \frac{\partial u_1^s}{\partial \nu}(x_1, x_n) + Z_1 u_1^s(x_1, x_n) \\
&= - \left( \frac{\partial \Phi_{x_n}}{\partial \nu}(x_1) + Z_1 \Phi_{x_n}(x_1) \right) \\
&\xrightarrow{x_n \rightarrow x_1} \infty
\end{align*}
\]
Proof of uniqueness

$$(Z_1, D_1) \rightarrow u_1^\infty, \ (Z_2, D_2) \rightarrow u_2^\infty \ \text{and} \ u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})$$

- Assume that $D_1 \neq D_2$.

$$\frac{\partial u_2^s}{\partial \nu}(x_1, x_n) + Z_1 u_2^s(x_1, x_n) \xrightarrow{x_n \rightarrow x_1} \infty$$

but $\partial_\nu u_2^s(x_1, x_1) + Z_1 u_2^s(x_1, x_1)$ remains bounded!

Conclusion: $D_1 = D_2$. 
Proof of uniqueness

\[(Z_1, D_1) \rightarrow u_1^\infty, (Z_2, D_2) \rightarrow u_2^\infty \text{ and } u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})\]

- Assume that \(D_1 \neq D_2\).

\[
\frac{\partial u_2^s}{\partial \nu}(x_1, x_n) + Z_1 u_2^s(x_1, x_n) \xrightarrow{x_n \rightarrow x_1} \infty
\]

but \(\partial \nu u_2^s(x_1, x_1) + Z_1 u_2^s(x_1, x_1)\) remains bounded!

Conclusion: \(D_1 = D_2\).

- For every \(\hat{\theta}\), \(u_1(\cdot, \hat{\theta})\) satisfies

\[
Z_1 u_1(\cdot, \hat{\theta}) = Z_2 u_1(\cdot, \hat{\theta}).
\]

The density of \(\{u_1(\cdot, \hat{\theta}), \hat{\theta} \in S^{d-1}\}\) in \(H^1(\partial D)\) gives

\[
Z_1 \varphi = Z_2 \varphi \quad \forall \varphi \in H^1(\partial D).
\]

Test with well chosen functions to have \((\lambda_1, \eta_1) = (\lambda_2, \eta_2)\).
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Solving the inverse problem with a finite number of incident waves

\[ F(\lambda, \eta, \partial D) := \frac{1}{2} \sum_{j=1}^{I} \| T(\lambda, \eta, \partial D, \hat{\theta}_j) - u_{\text{obs}}^\infty(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2 \]

For minimizing \( F \):

- we need partial derivatives of the far-field with respect to \( \lambda \) and \( \eta \) (quite standard),
- we need an appropriate derivative w.r.t. the obstacle.

Difficulty: the unknown impedances are supported by \( \partial D \).
Derivative of the cost function with respect to the obstacle

\( \varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \) is “small”

\[
\begin{align*}
    f_\varepsilon & := \text{Id} + \varepsilon \\
    \partial D_\varepsilon & := f_\varepsilon(\partial D) \\
    \lambda_\varepsilon & := \lambda \circ f_\varepsilon^{-1}, \quad \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}
\end{align*}
\]

We define the derivative \( v_\varepsilon \) of the scattered field with respect to the geometry at point \((\lambda, \eta, \partial D)\) by

\[
u^s(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon) - u^s(\lambda, \eta, \partial D) = v_\varepsilon + o(||\varepsilon||)
\]

where \( \varepsilon \mapsto v_\varepsilon \) is linear.
Derivative of the cost function with respect to the obstacle

\[ \varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ is “small”} \]

\[ f_\varepsilon := \text{Id} + \varepsilon \]

\[ \partial D_\varepsilon := f_\varepsilon(\partial D) \]

\[ \lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1}, \quad \eta_\varepsilon := \eta \circ f_\varepsilon^{-1} \]

We define the derivative \( v_\varepsilon \) of the scattered field with respect to the geometry at point \((\lambda, \eta, \partial D)\) by

\[ u^s(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon) - u^s(\lambda, \eta, \partial D) = v_\varepsilon + o(\|\varepsilon\|) \]

where \( \varepsilon \mapsto v_\varepsilon \) is linear.

One may find \( f_\varepsilon \) such that \( \partial D = f_\varepsilon(\partial D) \) and

\[ F'_{\lambda,\eta}(\partial D) \cdot \varepsilon \neq 0. \]

\( F'_{\lambda,\eta}(\partial D) \) does not satisfy the classical shape derivative’s properties!
Derivative of the scattered field with respect to the obstacle

Let $(\lambda, \eta, \partial D)$ be given and analytics, for some small $\varepsilon \in C^{1,\infty}$ define

$$\partial D_\varepsilon = f_\varepsilon(\partial D), \quad \lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1} \text{ and } \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}.$$ 

Let $u^s_\varepsilon \ [u^s]$ be scattered field associated with $(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon) \ [(\lambda, \eta, \partial D)]$.

Let $u^s_\varepsilon (u^s) be scattered field associated with $(\lambda_\varepsilon, \eta_\varepsilon, \partial D_\varepsilon) \ [(\lambda, \eta, \partial D)]$.

$$u^s_\varepsilon(x) - u^s(x) = v_\varepsilon(x) + o(||\varepsilon||),$$

where $v_\varepsilon(x)$ is the solution of the scattering problem with

$$\frac{\partial v_\varepsilon}{\partial \nu} + Zv_\varepsilon = B_\varepsilon u \quad \text{on} \quad \partial D$$

$$B_\varepsilon u = (\varepsilon \cdot \nu)(k^2 - 2H\lambda)u + \text{div}_\Gamma ((Id + 2\eta(R - H Id))(\varepsilon \cdot \nu)\nabla_\Gamma u)$$
$$+ (\nabla_\Gamma \lambda \cdot \varepsilon)u + \text{div}_\Gamma ((\nabla_\Gamma \eta \cdot \varepsilon)\nabla_\Gamma u)$$
$$+ Z((\varepsilon \cdot \nu)Zu),$$

with $2H := \text{div}_\Gamma \nu$, $R := \nabla_\Gamma \nu$ and $Z \cdot = \text{div}_\Gamma (\eta \nabla_\Gamma \cdot) + \lambda$. 

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Main tools of the proof

- Domain derivative tools: Murat and Simon [73], Kirsch [93], Hettlich [94], Potthast [94].
- Green’s theorems and integral representation of the scattered field: Kress and Päivärinta [99], Haddar and Kress [04].

Green’s theorems and integral representation: write

\[ u_\varepsilon^s - u^s = -\int_{\partial D} (B_\varepsilon u) w(\cdot, y) ds(y) + o(\|\varepsilon\|) \]

where for \( y \in \Omega \) \( w(\cdot, y) = w^s(\cdot, y) + \Phi(\cdot, y) \) is the Green function associated with the GIBC scattering problem

\[
\begin{cases}
\Delta w(\cdot, y) + k^2 w(\cdot, y) = \delta_y & \text{in } \Omega \\
\frac{\partial w}{\partial \nu} + Z w = 0 & \text{on } \partial D \\
\text{+radiation condition.}
\end{cases}
\]
Sketch of the proof (1/2)

Volume extension of the surface objects between $\partial D$ and $\partial D_\varepsilon$

$D_\varepsilon$ is outside $D$

\[ D^* = D_\varepsilon \setminus \overline{D} \]

- We parametrize $D^*$ with $f_\varepsilon^t := Id + t\varepsilon$ for $t \in [0, 1],$

\[ D^* \ni x_t = x_0 + t\varepsilon(x_0) \]

- $\lambda_t = \lambda \circ (f_\varepsilon^t)^{-1}$, $\eta_t = \eta \circ (f_\varepsilon^t)^{-1}$

- for a given $t$, $\partial D_t := f_\varepsilon^t(\partial D)$,

- $\nu_t$: outward unit normal of $\partial D_t$, the direction of $\nu_t$ depends on $t!$

- $\nabla \Gamma_t \cdot := \left( \nabla \cdot - \frac{\partial}{\partial \nu_t} \nu_t \right) |_{\partial D_t}$
Sketch of the proof (2/2)

Write an integral representation on $\partial D$

Objective: write the $u^s_\varepsilon - u^s$ as

$$u^s_\varepsilon(x) - u^s(x) = -\int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + o(\|\varepsilon\|).$$

Integral representation formula for $x$ outside $D_\varepsilon$:

$$u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u_\varepsilon \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \text{div}_\varepsilon (\eta_\varepsilon \nabla_\varepsilon w) + \lambda_\varepsilon w \right\} ds_\varepsilon,$$
Sketch of the proof (2/2)

Write an integral representation on \(\partial D\)

**Objective:** write the \(u^s_\varepsilon - u^s\) as

\[
u^s_\varepsilon(x) - u^s(x) = -\int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + o(\|\varepsilon\|).
\]

Integral representation formula for \(x\) outside \(D_\varepsilon\):

\[
u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \text{div}_{\Gamma_\varepsilon} (\eta_\varepsilon \nabla_{\Gamma_\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon + o(\|\varepsilon\|),
\]
Sketch of the proof (2/2)
Write an integral representation on $\partial D$

**Objective:** write the $u_{\varepsilon}^s - u^s$ as

$$u_{\varepsilon}^s(x) - u^s(x) = -\int_{\partial D} (B_{\varepsilon} u) w(\cdot, x) ds + o(||\varepsilon||).$$

Integral representation formula for $x$ outside $D_{\varepsilon}$:

$$u_{\varepsilon}^s(x) - u^s(x) = \int_{\partial D_{\varepsilon}} u \left\{ \frac{\partial w}{\partial \nu_{\varepsilon}} + \text{div}_{\Gamma_{\varepsilon}} (\eta_{\varepsilon} \nabla_{\Gamma_{\varepsilon}} w) + \lambda_{\varepsilon} w \right\} ds_{\varepsilon} + o(||\varepsilon||),$$

Gauss divergence theorem

$$= \int_{D_{\varepsilon} \setminus \overline{D}} \text{div} \{u \nabla w - \eta_t (\nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w) \nu_t + \lambda_t u w \nu_t \} dx + o(||\varepsilon||)$$
Sketch of the proof (2/2)

Write an integral representation on $\partial D$

**Objective:** write the $u_\varepsilon^s - u^s$ as

$$u_\varepsilon^s(x) - u^s(x) = - \int_{\partial D} (B_\varepsilon u) w(\cdot, x) ds + o(\|\varepsilon\|).$$

Integral representation formula for $x$ outside $D_\varepsilon$:

$$u_\varepsilon^s(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \text{div}_{\Gamma_\varepsilon} (\eta_\varepsilon \nabla_{\Gamma_\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon + o(\|\varepsilon\|),$$

Gauss divergence theorem

$$= \int_{\partial D} (\varepsilon \cdot \nu) \int_0^1 \text{div} \left( -\eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w \nu_t + u \nabla w + \lambda_t uw \nu_t \right) dt ds + o(\|\varepsilon\|)$$
Sketch of the proof (2/2)

Write an integral representation on \( \partial D \)

Objective: write the \( u^s_\varepsilon - u^s \) as

\[
u^s_\varepsilon(x) - u^s(x) = - \int_{\partial D} (B_\varepsilon u)w(\cdot, x)ds + o(\|\varepsilon\|).
\]

Integral representation formula for \( x \) outside \( D_\varepsilon \):

\[
u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \text{div}_{\varepsilon} (\eta_\varepsilon \nabla_{\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon + o(\|\varepsilon\|),
\]

Gauss divergence theorem and Taylor expansion:

\[
= \int_{\partial D} (\varepsilon \cdot \nu) \text{div} ( - \eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} \nu_t u + u\nabla w + \lambda_t u w \nu_t ) \big|_{t=0} ds + o(\|\varepsilon\|)
\]
Sketch of the proof (2/2)

Write an integral representation on $\partial D$

Objective: write the $u^s_\varepsilon - u^s$ as

$$u^s_\varepsilon(x) - u^s(x) = - \int_{\partial D} (B_\varepsilon u)w(\cdot, x)ds + o(\|\varepsilon\|).$$

Integral representation formula for $x$ outside $D_\varepsilon$:

$$u^s_\varepsilon(x) - u^s(x) = \int_{\partial D_\varepsilon} u \left\{ \frac{\partial w}{\partial \nu_\varepsilon} + \text{div}_{\varepsilon} (\eta_\varepsilon \nabla_{\varepsilon} w) + \lambda_\varepsilon w \right\} ds_\varepsilon + o(\|\varepsilon\|),$$

Gauss divergence theorem and Taylor expansion:

$$= \int_{\partial D} (\varepsilon \cdot \nu) \text{div} (-\eta_t \nabla_{\Gamma_t} u \cdot \nabla_{\Gamma_t} w\nu_t + u \nabla w + \lambda_t uw\nu_t) \big|_{t=0} ds + o(\|\varepsilon\|)$$

$$\text{div} (\lambda_t uw\nu_t) \big|_{t=0} = uw (\nabla\lambda_t) \big|_{t=0} \cdot \nu + \lambda \frac{\partial (uw)}{\partial \nu} + \lambda uw \text{div}(\nu_t) \big|_{t=0}$$

$$(\nu \cdot \varepsilon) (\nabla\lambda_t) \big|_{t=0} \cdot \nu = -\nabla_{\Gamma} \lambda \cdot \varepsilon$$
Outline

1. The forward and inverse problems

2. Uniqueness and stability for the inverse problem
   - The case of a single incident wave, known obstacle
   - The case of infinitely many incident plane waves

3. A steepest descent method to solve the inverse problem
   - Presentation of the method
   - Computation of the shape derivative of the scattered field

4. Numerical experiments
A steepest descent method to solve the inverse coefficient problem

\[ F(\lambda, \eta, \partial D) := \frac{1}{2} \sum_{j=1}^{I} \| T(\lambda, \eta, \partial D, \hat{\theta}_j) - u_{\text{obs}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2 \]

**Numerical procedure:**

- update alternatively \( \lambda, \eta \) and \( \partial D \) with a direction given by the partial derivative of the cost function,
- when we update the geometry we also transport the impedance coefficients to the new boundary.
The regularization procedure

\[ F(\lambda, \eta, \partial D) = \frac{1}{2} \sum_{j=1}^{I} \| T(\lambda, \eta, \partial D, \hat{\theta}_j) - u_{\text{obs}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2 \]

We regularize the gradient, NOT the cost function, using a \( H^1(\partial D) \) regularization.

- Descent direction for \( \lambda \): \( \delta \lambda \) that solves for every \( \phi \) in some finite dimensional space:

\[ \beta_\lambda \int_{\partial D} \nabla \Gamma(\delta \lambda) \cdot \nabla \Gamma \phi \, ds + \int_{\partial D} \delta \lambda \phi \, ds = -\alpha_\lambda F'_{\eta, \partial D}(\lambda) \cdot \phi \]

where \( \beta_\lambda \) is the regularization coefficient and \( \alpha_\lambda \) is the descent coefficient.

- Do the same for \( \delta \eta \) and \( \delta(\partial D) \).
Numerical reconstruction

Finite elements method and remeshing procedure

using FreeFem++

Reconstruction of the geometry with 2 incident waves and 1% noise on the far-field, $\lambda = ik/2$ and $\eta = 2/k$ being known
Numerical reconstruction
Simultaneous reconstruction of $\lambda$, $\partial D$ with $\eta = 0$

8 incident waves, 5% of noise on far-field data.
We iterate only on the geometry.

$$B_\varepsilon u = (\nabla \Gamma \lambda \cdot \varepsilon)u + \cdots$$
Numerical reconstruction

Simultaneous reconstruction of $\lambda$, $\eta$ and $\partial D$

8 incident waves, 5% of noise on far-field data.
Application to the reconstruction of a coated obstacle

**Exact model**

\[
\begin{align*}
\text{div}(\mu^{-1}\nabla u_\delta) + \epsilon k^2 u_\delta &= 0 \\
\Delta u_\delta + k^2 u_\delta &= 0
\end{align*}
\]

**Approximate model of order 1**

\[
\begin{align*}
\frac{\partial u_1}{\partial \nu} + \text{div}_\Gamma(\mu^{-1}\delta \nabla u_1) + \delta k^2 u_1 &= 0 \\
\Delta u_1 + k^2 u_1 &= 0
\end{align*}
\]

Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

\[
F(\epsilon, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| T(\epsilon, \delta, \Gamma, \hat{\theta}_j) - u_{\text{obs}, \text{mince}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2
\]

with \( \mu = 0.1 \) known.
Application to the reconstruction of a coated obstacle

Numerical results

Artificial data created with

- $\mu = 0.1$ is known,
- $\delta = 0.04l(1 - 0.4\sin(\theta))$ is unknown; $l$ being the wavelength,
- $\epsilon = 2.5$ is unknown.

Reconstructed $\epsilon$: 2.3.

Fails with a classical impedance boundary condition model!
Conclusion

- The inverse problem is ill-posed but not too much.
- It is solvable using a steepest descent method with regularization.
- Possible reconstruction of coated obstacles.

- Extension to the 3D Maxwell equations (ongoing work).
- The case of a general symmetric operator on the boundary?