1 Problem Class exercise 2, 24 Oct

A stationary, circular cylinder with boundary $x^2 + y^2 = a^2$ is put in a 2D irrotational flow field that has velocity component (u, v) such that

$$\begin{aligned} &u - 4Mxy(y^2 - x^2) \to 0, \\ &v - M(-x^4 + 6x^2y^2 - y^4) \to 0, \end{aligned} \tag{1}$$

as $|\boldsymbol{x}| \to \infty$, $M > 0$, and the circulation around the cylinder is 0.

(a) Find a streamfunction.

(b) Now fluid is blowing steadily outwards across the cylinder with speed $W + U \cos 2\theta$ at (a, θ), where $0 < V < W$. What is the streamfunction now, if the large-distance behaviour of the flow is unchanged?

Solution:

(a) At large distance,

$$\begin{aligned} \partial_y \psi' &= u = 4Mxy(y^2 - x^2), \\ -\partial_x \psi' &= v = M(-x^4 + 6x^2y^2 - y^4), \end{aligned} \tag{2}$$

therefore, by integration, we recover $\psi' = M(x^5/5 - 2x^3y^2 + xy^4)$. Now, we aim to express ψ' in polars. The x^5 terms motivates the inclusion of a $r^5 \cos 5\theta$ term. By using additional formulae, one should derive that $r^5 \cos 5\theta = x^5 - 10x^3y^2 + 5xy^4$, therefore $\psi' = \frac{M}{5} r^5 \cos 5\theta$. In order to satisfy the 'no normal flow' condition on the cylinder, we need to include a $r^{-5} \cos 5\theta$ term, $\psi = \frac{M}{5} r^5 \cos 5\theta + Ar^{-5} \cos 5\theta$. The streamfunction is constant on streamlines (and compatible solid bodies), i.e. $(\frac{M}{5} a^5 + Aa^{-5}) \cos 5\theta$ is constant. As θ can take any value, $(\frac{M}{5} a^5 + Aa^{-5}) = 0$, therefore $\psi = \frac{M}{5} r^5 \cos 5\theta (1 - \frac{a^{10}}{r^{10}})$.

(b) The large-scale behaviour of the flow is unchanged, therefore $\psi' = \frac{M}{5} r^5 \cos 5\theta$ as in part (a). The condition on the cylinder has changed, now it is $\frac{1}{r} \partial_\theta \psi = u_r = W + V \cos 2\theta$ on $r = a$. To satisfy this, we need to include a source term and a $\sin 2\theta$ (as we need to take a derivative to get a cosine) term, i.e. we seek a solution in form $\psi = \frac{M}{5} r^5 \cos 5\theta + Ar^{-5} \cos 5\theta + Br^{-2} \sin 2\theta + C\theta$. Now, $\frac{1}{a}(Ma^5 + 5Aa^{-5}) \sin 5\theta + 2\frac{B}{a^3} \cos 2\theta + \frac{C}{a} = \frac{1}{r} \partial_\theta \psi = W + V \cos 2\theta$. Equating terms and substituting back to the streamfunction we find $\psi = \frac{M}{5} r^5 \cos 5\theta (1 - \frac{a^{10}}{r^{10}}) + \frac{a^3 V}{2} r^{-2} \sin 2\theta + aW\theta$.