1 Problem Class exercise 2, 17 Oct

Consider a flow in a circular tank of radius π specified by a streamfunction

$$\psi = A\sin^2 r \sin 2\theta, \tag{1}$$

where $A > 0$. Show that the circular tank and the streamfunction are compatible.

(a) Show that there are 5 fixed points in the system and that there is no flow through the x- and the y-axis.

(b) The circular tank is now rotating with angular velocity $\Omega > 0$ and the streamfunction is given by

$$\psi_2 = \Omega\frac{r^2}{2} + A\sin^2 r \sin 2\theta. \tag{2}$$

Assume that $1 < \frac{\pi\Omega}{A} < 2$. Find the number of fixed points of the system and write down an equation they satisfy. Sketch the streamline through the origin and show that particles either travel around the origin obtaining all possible values of θ or they oscillate around one of the fixed points and are restricted to a quarterplane.

Solution: The circle of radius π is a streamline, since $A\sin^2 r\sin 2\theta = A\sin^2\pi\sin 2\theta = 0$, constant. Therefore the circular tank is compatible with the flow.

(a) At fixed points, $\boldsymbol{u} = \mathbf{0}$. In polars, $\partial_\theta\psi = ru_r = 2A\sin^2 r\cos 2\theta$ and $\partial_r\psi = -u_\theta = A\sin 2r\sin 2\theta$. If $\sin r = 0$, then both velocity components are zero. The flow is restricted to $r < \pi$, meaning that $r = 0$ and the origin is a fixed point. If $\sin r \neq 0$, then we need $\cos 2\theta = 0$ and $\cos r\sin 2\theta = 0$. Since $\cos 2\theta = 0$ and $\sin 2\theta = 0$ cannot be satisfied simultaneously, $\cos r$ must be zero, i.e. $r = \pi/2$. As $\cos 2\theta = 0$, the five fixed points are $\mathbf{0}$ and $(r,\theta) = (\pi/2, \pi/4 + k\pi/2)$ for $k = 0, 1, 2, 3$.

To show that there is no flow through the x- and y-axis, we need to show that they are streamlines: on the positive x-axis $\sin 2\theta = \sin 0 = 0$, on the negative x-axis $\sin 2\theta = \sin 2\pi = 0$, on the positive y-axis $\sin 2\theta = \sin\pi = 0$, on the negative y-axis $\sin 2\theta = \sin 3\pi = 0$, both of the axes are streamlines with $\psi = 0$. See Fig. 1 for streamlines.

(b) In this case, $\boldsymbol{u} = \mathbf{0}$ requires $0 = \partial_\theta\psi_2 = 2A\sin^2 r\cos 2\theta$ and $0 = \partial_r\psi_2 = A\sin 2r\sin 2\theta + \Omega r$. Similarly to part (a), either $\sin r = 0$ and in that case $\Omega r = 0$ resulting in the origin as fixed point or $\sin r \neq 0$ and in that case $\cos 2\theta = 0$. The four possible values for θ are $\pi/4 + k\pi/2$, $k = 0, 1, 2, 3$.

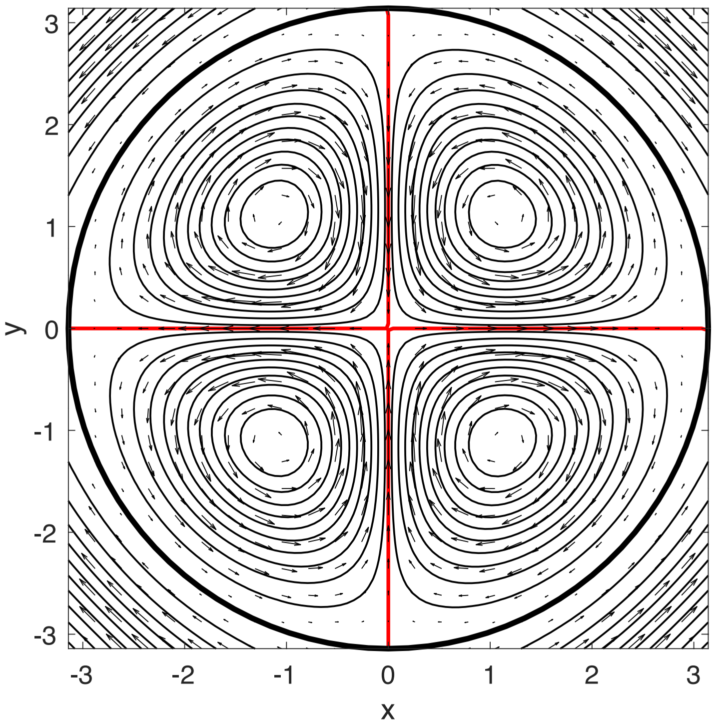

Figure 1: Streamlines for Problem Class 2, Question 2/a. Thick black line indicates the location of the solid solid boundary, red lines correspond to $\psi = 0$. Here, the value $A = 2\pi/3$ was chosen.

For k=1,3, $0 = A \sin 2r \sin 2\theta + \Omega r$ results in $0 = A \sin 2r - \Omega r$. Note that $r < \pi$ because of the presence of a solid boundary.The graphs of $\sin 2r$ and $\Omega r/A$ can therefore meet only at most one point (apart from 0). Since at $r = \pi/4$ we have $\sin 2r - \Omega r/A = 1 - \Omega\pi/(4A) > 0$, since $4 > \frac{\pi\Omega}{A}$. This means that the graph of $\sin 2r$ will be above the line $\Omega r/A$ at $r = \pi/4$. However, the former will decrease to 0 when $r = \pi/2$ and the latter is still positive. Therefore $\sin 2r - \Omega r/A = 0$ have a (unique) solution for some $\pi/4 < r_c < \pi/2$. The solution is unique as the graph of $\sin 2r$ is concave. The fixed points are then given by $(r, \theta) = (r_c, \pi/4)$ and $(r, \theta) = (r_c, 5\pi/4)$

For k=0,2, $0 = A \sin 2r \sin 2\theta + \Omega r$ results in $0 = A \sin 2r + \Omega r$, however $A, \Omega, r > 0$ meaning that $\sin 2r$ has to be negative otherwise the right hand side is positive, i.e. possible solutions are restricted to $\pi/2 < r < \pi$. The graphs of $\sin 2r$ and $-\Omega r/A$ can therefore meet either 0,1 or 2 times, depending on the value of Ω/A. To find a critical value where the line $-\Omega r/A$ is tangent to $\sin 2r$, they derivatives also need to match, i.e. $-\Omega/A = 2\cos 2r$. Therefore, $\left(-\frac{\Omega}{2A}\right)^2 + \frac{\Omega^2 r^2}{A^2} = \cos^2 2r + \sin^2 2r = 1$ and $-\frac{1}{\sqrt{1/4 + r^2}} = 2\cos 2r$. In $\pi/2 < r < \pi$ this has a unique solution (think about it why!) r_s which is approximately 2.2467. The corresponding value of $\Omega\pi/A \approx 1.36492 = \beta$ which is indeed in the range (1,2). If we increase the steepness Ω/A, we will have no solution, while if we decrease the steepness we have 2 solutions for r (and 4 fixed points as there are two values for θ).

To sum up, if $1 < \Omega\pi/A < \beta$, there are 7 fixed point in total (see Fig. 3 for streamlines). If $\Omega\pi/A = \beta$, there are 5 fixed points (not shown). If $\beta < \Omega\pi/A < 2$, there are 3 fixed points (see Fig. 2 for streamlines).

The streamfunction through the origin takes the value of 0, therefore $\sin 2\theta$ must be non-positive on a streamline through the origin, restricting the dynamics to the 2nd and 4th quarter-planes for particles inside the lobes specified by this streamline (see figures).

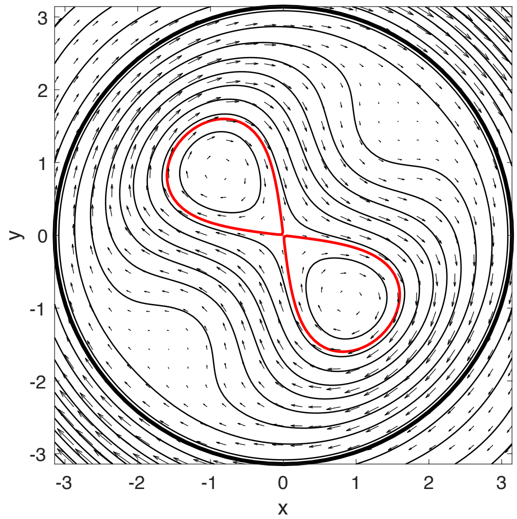

Figure 2: Streamlines for Problem Class 2, Question 2/b. Thick black line indicates the location of the solid solid boundary, red lines correspond to $\psi = 0$. Here, the values $\Omega = 1$ and $A = 2\pi/3$ were chosen.

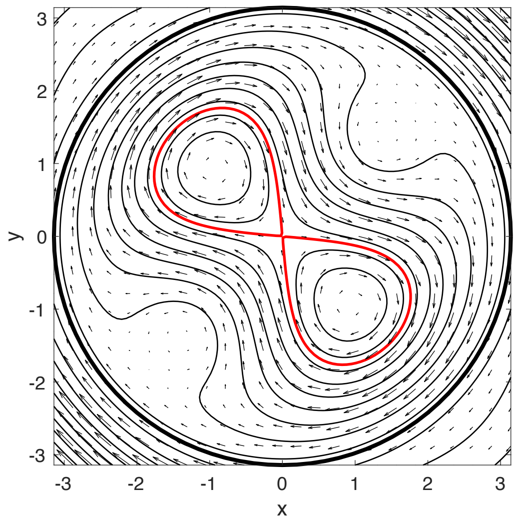

Figure 3: Streamlines for Problem Class 2, Question 2/b. Thick black line indicates the location of the solid solid boundary, red lines correspond to $\psi = 0$. Here, the values $\Omega = 1$ and $A = \pi/1.2$ were chosen.