Report 5

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Chapter 5
Prime number theorem

This chapter introduced the prime number theorem which states that the relative error in the approximation $\pi(x) \sim Li(x)$ approaches zero as $x \to \infty$[1]

It was well known that the prime number theorem could be deduced from the following theorem that the relative error in the approximation $\psi(x) \sim x$ approaches zero as $x \to \infty$.

There are three main steps to prove the prime number theorem, first step is to prove that there are no roots ρ on the line $Re\ s = 1$. The reason to prove this is to show

$$\lim_{x\to\infty} \sum_{\rho} \frac{x^{\rho-1}}{\rho} = 0$$

the limit of the sum could be taken termwise, so it is suffice to prove $x^{\rho} \to 0$ for all ρ.[2]

The second step is to show that $\psi(x) \sim x$, and the last step is simply duduce that $\pi(x) \sim Li(x)$ from $\psi(x) \sim x$.

Proof of the prime number theorem

1. Hadamard's proof that $Re\ \rho < 1$ for all ρ

   Consider the representation

$$\begin{aligned} log\zeta(s) &= \int_0^{\infty} x^{-s} dJ(x) \\ &= \sum_p \frac{1}{p^s} + \frac{1}{2}\sum_p \frac{1}{p^{2s}} + \frac{1}{3}\sum_p \frac{1}{p^{3s}} + \cdots \end{aligned}$$

   It is valid throughout the halfplane $Re\ s > 1$. The presence of zeros ρ of $\zeta(s)$ on the line $Re\ s = 1$ would imply the presence of points $s = \sigma + it$ slightly to the right of $Re\ s = 1$ where $Re\ log(s)$ is near $-\infty$.

[1] where $Li(x) = \lim_{\epsilon\to 0}\left[\int_0^{1-\epsilon} \frac{dt}{logt} + \int_{1+\epsilon}^{x} \frac{dt}{logt}\right]$
[2] in the other words, $Re\ \rho < 1$ for all ρ

Now look at the series above, it has property that the sum of the terms after the first is bounded by the nubmer

$$B = \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \frac{1}{4}\sum_p \frac{1}{p^4} + \cdots$$

for the entire halfpalne $Re\ s \geq 1$ including $Re\ s = 1$; therefore

$$Re\ log\zeta(\sigma + it) \geq \sum_p \frac{cos(tlogp)}{p^\sigma} - B$$

can approach $-\infty$ as $\sigma \searrow 1$ only if the first term approaches $-\infty$. it would follow that

$$\lim_{\sigma \searrow 1} \sum_p \frac{cos(tlogp)}{p^\sigma} = -\infty$$

. Hence the object is to prove this is impossible. Hadamard proved it by contradiction.

2. Proof of $\psi(x) \sim x$

   Consider von Mangoldt's formula

$$\psi(x) = x - \sum \frac{x^\rho}{\rho} + \sum \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \qquad (x > 1)$$

   is obtained by evaluation the definite integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s ds}{s}\right]$$

   in two different ways,(it was shown in my last week's report), so now it is natural to expect tha thte antiderivative of von Mangoldt's formula,

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum \frac{x^{-2n+1}}{2n(2n-1)} - \frac{\zeta'(s)}{\zeta(s)}x + const$$

   could be obtained by evaluating the antiderivative of this definite integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left[-\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s+1} ds}{s+1}\right]$$

   in two different ways.

   first way is to use the expansion $-\frac{\zeta'(s)}{\zeta(s)} = \sum \Lambda(n)n^{-s}$. This will give

$$\sum_{n=1}^{\infty} \Lambda(n)\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^{s+1}ds}{n^s s(s+1)}$$

   . Now by using the partial fraction

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

this will give that

$$\frac{x^{s+1}}{n^s s(s+1)} = \begin{cases} x-n & \text{if } n \leq x \\ 0 & \text{if } n \geq x \end{cases}$$

therefore the ingetral becomes

$$\sum_{n \leq x} \Lambda(n)(x-n) = \int_0^x (x-t)d\psi(t) = \int_0^x \psi(t)dt$$

Second way to evaluate the intgral is by considering the fact

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s+1}{2(s-1)} - \sum_\rho \frac{s+1}{(\rho+1)(s-\rho)} + \sum_n \frac{s+1}{(2n-1)(s+2n)} - \frac{\zeta'(-1)}{\zeta(-1)}$$

and the expantion

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_\rho \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}$$

So the integrand of the definite integral becomes[3]

$$-\frac{\zeta'(s)}{\zeta(s)} x^{s+1}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

This will prove

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum \frac{x^{-2n+1}}{2n(2n-1)} - \frac{\zeta'(s)}{\zeta(s)} x + const$$

Using the formula for $\int_0^x \psi(t)dt$, it is easy to show that it is approximately $\frac{x^2}{2}$ Notice $\sum x^{\rho-1}/\rho(\rho+1)$ plus terms which goes to zero as x tends to infinity. since $\sum \rho^{-1}(\rho+1)^{-1}$ converges absolutely and since $|x^{\rho-1}| \leq 1$ (because $Re\ \rho \leq 1$).

3. Proof of the prime number theorem

From the my 2nd report there was a relation between $\theta(x)$ and $\psi(x)$ where $\theta(x)$ denote the sum of the logarithms of all primes p less than x. Further the relationship between θ and ψ is similar to the relationship of π and J.

The formula for ψ and θ is,

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \theta(x^{1/4}) + \cdots$$

[3]when multiply out with $\frac{1}{s}$ use the second expansoin for $-\frac{\zeta'(s)}{\zeta(s)}$, whereas $\frac{1}{s+1}$ use the first expansion, in this way the numerator will only be the terms of x^{s+1}

Now let $\epsilon > 0$ be given and let N be such that $(1-\epsilon)x \leq \theta(x) \leq (1+\epsilon)x$ whenever $x \geq N$.Then for $y > x > N$ it follows that

$$\pi(y) - \pi(x) = \int_x^y \frac{d\theta(t)}{logt} = \left[\frac{\theta(t)}{logt}\right]_x^y + \int_x^y \frac{\theta(t)dt}{(logt)^2 t}$$

it at most

$$\begin{aligned}\frac{(1+\epsilon)y}{logy} - \frac{(1-\epsilon)x}{logx} + \int_x^y \frac{(1+\epsilon)tdt}{(logt)^2 t} &= 2\epsilon\frac{x}{logx} + (1+\epsilon)\left\{\left[\frac{t}{logt}\right]_x^y + \int_x^y \frac{tdt}{(logt)^2 t}\right\} \\ &= 2\epsilon\frac{x}{logx} + (1+\epsilon)\left\{\int_x^y \frac{dt}{logt}\right\} \\ &= 2\epsilon\frac{x}{logx} + (1+\epsilon)[Li(y) - Li(x)]\end{aligned}$$

and at least

$$-2\epsilon\frac{x}{logx} + (1-\epsilon)[Li(y) - Li(x)]$$

Thus for fixed x that $\pi(y)/Li(y)$ is at most

$$1 + \epsilon + \frac{\pi(x) - 2\epsilon x(logx)^{-1} - (1+\epsilon)Li(x)}{Li(y)} \leq 1 + 2\epsilon$$

for all sufficently large y and similarly that it at least $1 - 2\epsilon$. Since ϵ is arbitrary, so $Li(y) \sim \pi(y)$