

Report 4

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Chapter 3

Riemann's main formula

This chapter shows how von Mangoldt proved Riemann's main formula

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2 - 1)\log t} \quad (x > 1)$$

1. Von Mangoldt recast this formula in a simpler form which has virtually replaced Riemann's original statement. The derivation of the simpler form are follows.

Consider the relationship

$$\log \zeta(s) = \int_0^{\infty} x^{-s} dJ(x)$$

and then the use of

$$\prod \left(\frac{s}{2}\right)^{\pi^{-s/2}} (s-1) \zeta(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

The function $\log \zeta(s)$ has singularities at all the roots, however it's derivative $\zeta'(s)/\zeta(s)$ is analytic in the entire plane except for the poles at the roots, the pole 1, and the zeros $-2n$. Hence

$$\frac{\zeta'(s)}{\zeta(s)} = - \int_0^{\infty} x^{-s} (\log x) dJ(x)$$

Since the measure $(\log x) dJ(x)$ is a point measure which assigns the weight $\log(p^n(1/n))$ to prime powers p^n and weight 0 to all other point. Thus it can be written as a Stieltjes measure $d\psi(x)$ where

$$\psi(x) = \sum_{p^n < x} \log p$$

So

$$\frac{\zeta'(s)}{\zeta(s)} = \int_0^{\infty} x^{-s} d\psi(x)$$

The simplest formulation of Riemann's result is his formula

$$dJ = \left(\frac{1}{\log x} - \sum_{\rho} \frac{x^{\rho-1}}{\log x} - \frac{1}{x(x^2 - 1)\log x} \right) dx \quad (x > 1)$$

which gives

$$\begin{aligned} d\psi &= (\log x)dJ \\ &= \left(1 - \sum_{\rho} x^{\rho-1} - \sum_{\rho} x^{-2n-1}\right) dx \end{aligned}$$

this leads to

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_n \frac{x^{-2n}}{2n} + \text{const} \quad (x > 1)$$

2. Derivation of von Mangoldt's formula for $\psi(x)$

Consider $-\zeta'(s)/\zeta(s) = s \int_0^{\infty} \psi(x)x^{-s-1}dx$, Then put $\psi(x)$ in the form of a definite integral

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left[\frac{-\zeta'(s)}{\zeta(s)} \right] x^s \frac{ds}{s}$$

von Mangoldt proves the formula for $\psi(x)$ by using two different ways to evaluate this definite integral. One gives $\psi(x)$, another one gives $x - \sum(x^{\rho}/\rho) + \sum(x^{-2n}/2n) + \text{const}$

The first method begins with

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^{\infty} x^{-s} d\psi(x)$$

and let $\Lambda(n)$ denote the weight assigned to the integer n by the measure $d\psi$, i.e. $\Lambda(n)$ is zero unless n is a prime power, therefore the integral can be replaced by

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \Lambda(n)n^{-s} \quad (\text{Re } s > 1)$$

substitute this back to the definite integral and assume termwise integration is valid, then

$$\sum_{n=2}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s}$$

by the formula $\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{s-\beta} y^s \frac{ds}{s} = y^{\beta}$ if $y > 1$ (with $\beta = 0$), the integral corresponding to n in the sum is 1 if $x/n > 1$ and 0 if $x/n < 1$. therefore the sum is just

$$\sum_{n < x} \Lambda(n) = \psi(x)$$

¹to get this equation first consider the integral $-\zeta'(s)/\zeta(s) = \int_0^{\infty} x^{-s} d\psi(x)$ and then integrate it by parts

The second way to evaluate the integral is the following, consider the the formula

$$\Pi\left(\frac{s}{2}\right)\pi^{-s/2}(s-1)\zeta(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

take log of it and differetiate to get,

$$\frac{d}{ds} \log \Pi\left(\frac{s}{2}\right) - \frac{1}{2} \log \pi + \frac{1}{(s-1)} + \frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \frac{1}{1 - (s/\rho)} \left(-\frac{1}{\rho}\right)$$

using the formula $\Pi(s) = \prod_{n=1}^{\infty} (1 + s/n)^{-1} (1 + 1/n)^s$ and differentiate termwise then gives,

$$\frac{-\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + \sum_{n=1}^{\infty} \left[\frac{-1}{s+2n} + \frac{1}{2} \log \left(1 + \frac{1}{n}\right) \right] - \frac{1}{2} \log \pi$$

with $s = 0$ it shows

$$\frac{-\zeta'(0)}{\zeta(0)} = -1 - \sum_{\rho} \frac{1}{-\rho} + \sum_{n=1}^{\infty} \left[\frac{-1}{2n} + \frac{1}{2} \log \left(1 + \frac{1}{n}\right) \right] - \frac{1}{2} \log \pi$$

so subtraction gives

$$\frac{-\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}$$

substitute this back to the definite integral and assume that termwise integration is valid, this gives the value

$$\begin{aligned} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{s-1} - \sum_{\rho} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{\rho(s-\rho)} + \sum_n \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{2n(s+2n)} \\ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \left[-\frac{\zeta'(0)}{\zeta(0)} \right] \frac{ds}{s} \end{aligned}$$

Now consider²

$$\begin{aligned} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{s-\beta} &= \frac{1}{2\pi i} \int_{a-\beta-i\infty}^{a-\beta+i\infty} x^{\beta} x^t \frac{dt}{t} \\ &= x^{\beta} \frac{1}{2\pi i} \int_{\operatorname{Re}(a-\beta)-i\infty}^{\operatorname{Re}(a-\beta)+i\infty} x^t \frac{dt}{t} \\ &= x^{\beta} \end{aligned}$$

Therefore the value can be reduced when $x > 1$, to

$$x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_n \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}$$

²using change of variable $t = s - \beta$

3. Prove of von Mangoldt's formula

To prove the formula, need first to prove that the termwise integration is valid, furthermore the series converge uniformly on any finite segment $[a - ih, a + ih]$ ($a > 0$), so that the integral can be evaluate termwise on finite segment. Lastly need to show that the limit of their sum as $h \rightarrow \infty$ is equal to the sum of their limits.

4. Numerical evaluation of the constant

Consider the function

$$\frac{\zeta(s)}{\Gamma(-s)} = \frac{1}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

where the path are the same as in the 1st chapter. it will be shown that

$$\frac{1}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)(\log x - i\pi)}{e^x - 1} \frac{dx}{x} = 0$$

Let the path of integration be written as a sum of three parts as in chapter 1, so it becomes

$$\begin{aligned} \frac{1}{2\pi i} \int_{+\infty}^{\epsilon} \frac{(-x)(\log x - i\pi)}{e^x - 1} \frac{dx}{x} + \frac{1}{2\pi i} \int_{|x|=\epsilon} \frac{(-x)(\log x + i\theta - i\pi)}{e^x - 1} \frac{dx}{x} + \\ \frac{1}{2\pi i} \int_{\epsilon}^{\infty} \frac{(-x)(\log x + i\pi)}{e^x - 1} \frac{dx}{x} = - \int_{\epsilon}^{\infty} \frac{dx}{e^x - 1} \\ - \frac{\log \epsilon}{2\pi i} \int_{|x|=\epsilon} \frac{x}{e^x - 1} \frac{dx}{x} - \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{x}{e^x - 1} \phi d\phi \end{aligned}$$

where $x = \epsilon e^{i(\phi+\pi)}$ in the last integral. the middle integral by the Cauchy integral formula and the last integral approaches zero as $\epsilon \rightarrow 0$.

Left with only the first integral, it can be evaluated directly,

$$\begin{aligned} - \int_{\epsilon}^{\infty} \frac{dx}{e^x - 1} &= - \int_{\epsilon}^{\infty} \left(\sum_{n=1}^{\infty} e^{-nx} \right) dx = - \left[\sum_{n=1}^{\infty} \frac{e^{-nx}}{-n} \right]_{\epsilon}^{\infty} = - \sum \frac{(e^{-\epsilon})^n}{n} \\ &= \log(1 - e^{-\epsilon}) = \log\left(\epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6}\right) \\ &= \log \epsilon + \log\left(1 - \frac{\epsilon}{2}\right) \end{aligned}$$

In this way $\log \epsilon$'s cancel and the limit as $\epsilon \rightarrow 0$ of the remaining terms is zero. which proves

$$\frac{1}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)(\log x - i\pi)}{e^x - 1} \frac{dx}{x} = 0$$

Then by taking the log of functional equation

$$\zeta(s)/\Pi(-s) = (2\pi)^{s-1}\zeta(1-s)2\sin(\pi s/2)$$

Since its derivative is zero at $s=1$, its logarithmic derivative

$$\log(2\pi) - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\pi \cos(\pi s/2)}{2 \sin(\pi s/2)}$$

is zero at $s=1$, which gives

$$\zeta'(0)/\zeta(0) = \log 2$$