3rd Report

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3rd July 2008

Chapter 2
The Product Formula for ξ

In this chapter it gives the proof of the product formula

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)$$

The idea is the following,

1. Jensen's theorem

   Let $f(z)$ be a function which defined and analytic on disk $|z| < R$. Suppose $f(z)$ has no zeros on the bounding circle $|z| = R$, and inside the disk it has zeros $z_1, z_2, z_3, \cdots, z_n$, suppose finally $f(0) \neq 0$. Then

   $$log\left|f(0)\frac{R}{z_1}\cdot\frac{R}{z_2}\cdot\frac{R}{z_3}\cdots\frac{R}{z_n}\right| = \frac{1}{2\pi}\int_0^{2\pi} log|f(Re^{i\theta})|d\theta$$

2. A simple estimate of $|\xi(s)|$

   Theorem: For all suffices large values of R, the estimate $|\xi(s)| \leq R^R$ holds on the disk $|s - \frac{1}{2}| \leq R$ [1]

3. The resulting estimateof the root ρ

   Theorem: Let $n(R)$ denote the number of roots ρ of $\xi(\rho) = 0$ which lie inside or on the circle $|s - \frac{1}{2}| = R$ (counted with multiplicities). Then $n(R) \leq 2RlogR$ for all sufficiently large R.

4. Convergentce of the product

   In order to prove the convergence of the product

   $$\prod\left(1 - \frac{s}{\rho}\right) = \prod_{Im\rho>0}\left[1 - \frac{s(1-s)}{\rho(1-\rho)}\right]$$

   for all s, it suffices to prove the convergence of the sum $\sum|\rho - \frac{1}{2}|^{-1}$. However for finite number of roots ρ inequality

   $$\frac{1}{|\rho(1-\rho)|} = \frac{1}{(|\rho - \frac{1}{2})^2 - \frac{1}{4}|} < \frac{1}{|\rho - \frac{1}{2}|^2}$$

[1] The purpose of first 2 steps is that they will be needed in 3rd which proves the esitimate of the root ρ

holds. So now it is suffices to prove the convergence of the sum $\sum |\rho - \frac{1}{2}|^{-2}$. Now consider the folowing theorem

Theorem: For any given $\epsilon > 0$ the series $\sum \frac{1}{|\rho - \frac{1}{2}|^{1+\epsilon}}$ converges,where ρ ranges over all roots ρ of $\xi(\rho) = 0$. [2]

5. Rate of growth of the quotient

This section contains two parts, 1st part is that the growth of real part of $log\xi(s) - \sum log[1 - (s/\rho)]$ is no faster than $|s|^{1+\epsilon}$, 2nd partis to show that since it is an even function, so it actually is a constant.The following theorems proves two parts respectively.

Theorem (1st part): Let $\epsilon > 0$ be given, Then

$$Re\ log \frac{\xi(s)}{\prod_\rho \left(1 - \frac{s-\frac{1}{2}}{\rho - \frac{1}{2}}\right)} \le \left|s - \frac{1}{2}\right|^{1+\epsilon}$$

for all suffices large $|s - \frac{1}{2}|$

Theorem (2nd part): Let $f(s)$ be ananalytic function, defined in the entire s-plane, which is even [3] and grows more slowly than $|s|^2$ in the sense of $\forall \epsilon > 0$ there is an R such that $Ref(s) < \epsilon |s|^2$ at all points s satisfying $|s| \ge R$. Then f must be constant. [4]

6. The product formula for ξ

Consider the function $F(s) = \xi(s) \prod_\rho [1 - (s - \frac{1}{2})/(\rho - \frac{1}{2})]$ is analytic in the entire s-plane and it is even funtion of $s - \frac{1}{2}$. Moveover it has no zeros, by the results of the last two parts then combined to get $F(s) = constant$.

$$\xi(s) = c \prod \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right)$$

, where c is a constant. Dividing this by

$$\xi(0) = c \prod \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)$$

gives

$$\frac{\xi(s)}{\xi(0)} = \prod \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right) \left(1 - \frac{-\frac{1}{2}}{\rho - \frac{1}{2}}\right)^{-1}$$

[2] in the case of $\epsilon = 1$ is what we want to prove in the first place

[3] i.e $f(x) = f(-x)$

[4] the following lemma is used for the proof of this theorem, Lemma: Let $f(s)$ be an analytic function on the disk $|s| \le r$, let $f(0) = 0$, and let M be the maximum value of the $Ref(s)$ on the bounding circle $|s| = r$. Then for $r_1 < r$ the modulus of f on the smaller disk $|s| \le r_1$ is bounded by $|f(s)| \le 2r_1 M/(r - r_1)$

Simplify the right hand side will give,

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$