Non-abelian cohomology varieties in Diophantine geometry

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X: smooth variety over a field F.

$Cov(X)$: category of finite étale coverings of X.

$b : \text{Spec}(K) \to X$, a geometric point of X.

$$f_b : Cov(X) \to \text{finite sets}$$

$$\begin{array}{ccc} Y & & Y_b \\ \downarrow & \mapsto & \downarrow \\ X & & b \end{array}$$

Pro-finite fundamental group:

$$\widehat{\pi}_1(X, b) := \mathrm{Aut}(f_b)$$

Pro-finite torsor of paths:

$$\widehat{\pi}_1(X; b, a) := \mathrm{Isom}(f_b, f_a)$$

Flexible variation of points is a crucial advantage of this definition.

F: a number field.

$\bar{X} := X \otimes_F \bar{F}$

$a, b \in X(F)$: rational points, regarded as geometric points.

$\Gamma :=$ Gal$(\bar{F}/F)$

Then Γ acts on $Cov(\bar{X})$ preserving the fiber functors f_a, f_b, so

$$\widehat{\pi}_1(\bar{X}; a, b)$$

has a Γ-action.

Natural maps

$$\hat{\pi}_1(\bar{X}; b, c) \times \hat{\pi}_1(\bar{X}; a, b) \rightarrow \hat{\pi}_1(\bar{X}; a, c)$$

compatible with Γ-action.

Thus, $\hat{\pi}_1(\bar{X}; b, c)$ becomes a $\Gamma-$equivariant right-torsor for $\hat{\pi}_1(\bar{X}, b)$.

Classified by

$$H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

Main object of study

$$\kappa : X(F) \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

$$x \mapsto [\hat{\pi}_1(\bar{X}; b, x)]$$

What information about $X(F)$ is encoded in this map?

Grothendieck's section conjecture:

When X is a compact hyperbolic curve, κ is a bijection.

Grothendieck and Deligne expected:

Section conjecture $\Rightarrow$ Faltings' theorem

κ is very much studied already in *abelian* contexts.

(X, b) an elliptic curve. Then $\hat{\pi}_1(\bar{X}, b)$ is abelian, and

$$X(F) \to \widehat{X(F)} \to H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is the usual connecting homomorphism of Kummer theory.

In this case, the image lies inside a subspace

$$H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b)) \subset H^1(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

satisfying local Selmer conditions.

The bijectivity of

$$\widehat{X(F)} \to H^1_f(\Gamma, \hat{\pi}_1(\bar{X}, b))$$

is an important part of the Birch and Swinnerton-Dyer conjecture.

Even for general X, κ^{ab}, obtained by replacing $\widehat{\pi}_1$ with its abelianization $\widehat{\pi}^{ab}(\bar{X}, b) = H_1(\bar{X}, \widehat{\mathbb{Z}})$ is commonly studied.

$$k^{ab} : X(F) \rightarrow H^1(\Gamma, \widehat{\pi}^{ab}(\bar{X}, b)),$$

$$x \mapsto [\widehat{\pi}^{ab}(\bar{X}; b, x)]$$

X smooth projective over a finite field $\mathbf{F}_q$.

$$\kappa^{ab} : X(\mathbf{F}_q) \rightarrow H^1(\mathrm{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q), H_1(\bar{X}, \hat{\mathbb{Z}}))$$

But the target group in this case is

$$H_1(\bar{X}, \hat{\mathbb{Z}})/Im[(Fr_q - I)] = (\hat{\pi}_1^{ab}(X))^0,$$

which fits into the exact sequence

$$0 \rightarrow (\hat{\pi}_1^{ab}(X))^0 \rightarrow \hat{\pi}_1^{ab}(X) \rightarrow \mathrm{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q) \rightarrow 0$$

κ^{ab} becomes identified with the reciprocity map

$$rec : CH_0^0(X) \to (\widehat{\pi}_1^{ab}(X))^0$$

restricted to the cycles $(x) - (b)$.

Bijectivity of rec is known through Lang, Bloch, Kato and Saito.

Section conjecture and its relation to Diophantine geometry appear to be quite deep.

Considerably easier to study

$$\kappa^u : X(F) \rightarrow H^1(\Gamma_T, U_n^{et})$$

a unipotent $p-$adic version of κ.

Un(X): category of unipotent $\mathbb{Q}_p$-local systems on $\bar{X}$.

$$f_b : \mathrm{Un}(X) \to \mathrm{Vect}_{\mathbb{Q}_p},$$

$$\mathcal{F} \mapsto \mathcal{F}_b$$

$U^{et} := \mathrm{Aut}^{\otimes}(f_b)$, pro-unipotent, pro-algebraic group over $\mathbb{Q}_p$ equipped with Γ-action.

Given $x \in X(F)$,

$P^{et}(x) := \mathrm{Isom}^{\otimes}(f_b, f_x)$, pro-algebraic variety over $\mathbb{Q}_p$ with structure of Γ-equivariant U^{et}-torsor.

$Z^{\cdot}$ descending central series of U^{et}.

$Z^1 = U^{et}$, $Z^{i+1} = [U^{et}, Z^i]$.

$U_n^{et} := Z^n \backslash U^{et}$, unipotent algebraic group over $\mathbb{Q}_p$.

Also have U_n^{et}-torsors $P_n^{et}(x)$, algebraic varieties over $\mathbb{Q}_p$.

To construct a good classifying space, need integral models.

S: finite set of places of F.

R: ring of S-integers.

Assume given a model with good compactification over R:

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookrightarrow & \bar{\mathcal{X}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \hookrightarrow & \mathrm{Spec}(R) & = & \mathrm{Spec}(R) \end{array}$$

that is, $\bar{\mathcal{X}}$ is a proper and smooth, relative normal crossing compactification of $\mathcal{X}$ over R. Choose p not divisible by the primes in S and put

$T := S \cup \{v|p\}$.

$b, x \in \mathcal{X}(R)$.

Then the Γ-action on all P^{et} factors through $\Gamma_T = \mathrm{Gal}(F_T/F)$, the Galois group of the maximal extension F_T of F unramified above T.

Consider functor on $\mathbb{Q}_p$-algebras E:

$E \mapsto$ isomorphism classes of U_n^{et}-torsors over E equipped with continuous G_T-action on E-points.

Representable by an affine $\mathbb{Q}_p$-variety

$$H^1(\Gamma_T, U_n^{et})$$

Proof: follows by induction from the vector group case.

These fit together over n to form a pro-algebraic variety.

Main point: We have more structure to work with than $H^1(\Gamma, \hat{\pi}_1)$.

We again have a natural map

$$k^u : \mathcal{X}(R) \to H^1(\Gamma_T, U_n^{et})$$

$$x \mapsto [P^{et}(x)]$$

This map is definitely not bijective in general. Image is still hard to understand.

Much easier local version: $v|p$, R_v completion of R at v, $G_v = \mathrm{Gal}(\bar{F}_v/F_v)$.

$$k_v^u : \mathcal{X}(R_v) \to H^1(G_v, U_n^{et})$$

Easier because of (non-abelian) p-adic Hodge theory:

Image of k_v^u lies inside the subspace $H^1_f(G_v, U_n^{et})$ classifying torsors that trivialize over B_{cr}, ring of crystalline periods, and

$$H^1_f(G_v, U_n^{et}) \simeq U_n^{dr}/F^0$$

U^{dr}: De Rham fundamental group of $X_v = X \otimes_F F_v$.

$F^{\cdot}$: Hodge filtration on U^{dr}.

$\mathsf{Un}^{dr}(X_v)$: Category of unipotent vector bundles with flat connection on X_v.

$f_b : \mathsf{Un}^{dr}(X_v) \rightarrow \mathsf{Vect}_{F_v}$

$U^{dr} = \mathsf{Aut}^{\otimes}(f_b)$, pro-unipotent, pro-algebraic group over F_v.

$P^{dr}(x) = \mathsf{Isom}^{\otimes}(f_b, f_x)$, pro-variety over F_v with structure of U^{dr}-torsor.

Hodge filtration on these objects defined by Morgan and Hain over $\mathbb{C}$ and descended to arbitrary fields of characteristic zero by Wojtkowiak.

P^{dr} also has a Frobenius action ϕ obtained from an isomorphism

$$P^{dr}(X_v) \simeq P^{cr}(Y_v),$$

where Y_v is the reduction of $\mathcal{X}_v$ and P^{cr} is defined using unipotent overconvergent isocrystals on Y_v.

U^{dr}/F^0 classifies torsors for U^{dr} with compatible Hodge filtrations and ϕ-action, and hence, becomes the target of a map

$$k^u_{dr} : \mathcal{X}(R_v) \to U^{dr}/F^0$$

$$x \mapsto [P^{dr}(x)]$$

Actually a compatible tower:

$$
\begin{array}{ccc}
\| & & \downarrow \\
\mathcal{X}(R_v) & \rightarrow & U^{dr}_{n+1}/F^0 \\
\| & & \downarrow \\
\mathcal{X}(R_v) & \rightarrow & U^{dr}_{n}/F^0 \\
\| & & \downarrow \\
\vdots & \vdots & \vdots \\
\| & & \downarrow \\
\mathcal{X}(R_v) & \rightarrow & U^{dr}_{2}/F^0
\end{array}
$$

Bottom arrow is the log of the Albanese map to

$$H^{dr}_1(X_v)/F^0 = T_e J_X,$$

the tangent space to the Jacobian of X. Over $\mathbb{C}$, these maps were defined by Hain and called the higher Albanese maps. $\mathbb{Q}_p$-version appears to be more useful so far.

Can sometimes be described rather explicitly.

Example: $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

$$U^{dr} = \mathrm{Spec}(F_v < a_w >)$$

w runs over words on $\{A, B\}$.

When b is taken to be tangential (pointing towards 1) at 0,

$$a_w \circ k^u_{dr} = Pol_w,$$

a p-adic multiple polylogarithm.

For words of the form

$$w = A^{k_1-1}BA^{k_2-1}B \cdots A^{k_m-1}B,$$

$k_m > 1$, there is an expansion near 0

$$Pol_w = \Sigma_{n_1<n_2<\cdots<n_m}[\frac{z^{n_1}}{n_1^{k_1}n_2^{k_2}\cdots n_m^{k_m}}]$$

The fundamental groups and torsor spaces introduced so far can be viewed as components of a motivic fundamental group

$$U^M$$

and motivic torsors of paths

$$P^M(x)$$

Thus, we are given a motivic higher Albanese map

$$x \in \mathcal{X}(R) \mapsto [P^M(x)]$$

Here, 'motivic' in the sense of Deligne, 1987: compatible systems of realizations.

Commutative diagram

$$\begin{array}{ccc} \mathcal{X}(R_v) & \rightarrow & U_n^{DR}/F^0 \\ & \searrow & \uparrow\simeq \\ & & H^1_f(G_v, U_n^{et}). \end{array}$$

coming from p-adic non-abelian comparison isomorphism

$$D(\pi_1^{et}(\bar{X}_v; b, x)) \simeq \pi_{1,DR}(X_v; b, x)$$

(Shiho, Vologodsky, Olsson, Faltings)

Big commutative diagram.

$$\begin{array}{ccccc} \mathcal{X}(R) & \rightarrow & \mathcal{X}(R_v) & \rightarrow & U_n^{DR}/F^0 \\ \downarrow & & \downarrow & \nearrow & \\ H^1_f(\Gamma_T, U_n^{et}) & \stackrel{loc_v}{\rightarrow} & H^1_f(G_v, U_n^{et}) & & \end{array}$$

$H^1_f(\Gamma_T, U_n^{et}) \subset H^1(\Gamma_T, U_n^{et})$: subspace of elements that locally lie inside $H^1_f(G_v, U_n^{et})$.

Also, maps between local classifying spaces is actually

$$H^1_f(G_v, U_n^{et}) \simeq Res^{F_v}_{\mathbb{Q}_p}(U_n^{dr}/F^0)$$

All maps between the classifying spaces are algebraic. However, maps from points are highly transcendental: image of

$$k^u_{dr} : \mathcal{X}(R_v) \to U_n^{dr}/F^0,$$

described using p-adic iterated integrals, is Zariski dense.

Let X be a hyperbolic curve.

Conjecture: For n sufficiently large, the image of

$$H^1_f(\Gamma_T, U_n^{et}) \rightarrow U_n^{dr}/F^0$$

is inside a proper, Zariski-closed subset.

Conjecture implies the theorems of Faltings and Siegel: The image of $\mathcal{X}(R)$ lies inside that of $H^1_f(\Gamma_T, U_n^{et})$. So there exists a non-zero algebraic function on U_n^{dr}/F^0 vanishing on this image. However, when considered on $\mathcal{X}(R_v)$, this function is non-zero and belongs to the ring of Coleman functions. In particular, has finitely many zeros on compact sets.

When $n = 2$, reduces to the method of Chabauty. Recall that Chabauty's method requires the hypothesis

$$\text{rank} J_X(R) < \dim_{F_v} T_e J_X$$

But we have lifted Chabauty's map using the tower:

$$\begin{array}{ccccc}
\| & & \| & & \downarrow \\
\mathcal{X}(R) & \hookrightarrow & \mathcal{X}(R_v) & \rightarrow & U_{n+1}^{dr}/F^0 \\
\| & & \| & & \downarrow \\
\mathcal{X}(R) & \hookrightarrow & \mathcal{X}(R_v) & \rightarrow & U_n^{dr}/F^0 \\
\| & & \| & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\| & & \| & & \downarrow \\
\mathcal{X}(R) & \hookrightarrow & \mathcal{X}(R_v) & \rightarrow & U_2^{dr}/F^0
\end{array}$$

Hypothesis should be unnecessary if we go high enough.

The conjecture over $\mathbb{Q}$ is implied by many standard conjectures in the structure theory of mixed motives.

-Bloch-Kato conjecture (on the image of the p-adic Chern class map from motivic cohomology).

-Fontaine-Mazur conjecture on Galois representations of geometric origin.

-Jannsen's conjecture on vanishing of second Galois cohomology (when X is affine).

-Probably some strong conjecture in non-abelian Iwasawa theory.

Can prove the conjecture for:

-$X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ and $F = \mathbb{Q}$.

-X: CM elliptic curve of rank 1 minus origin over an imaginary quadratic field and S empty, with minor modification of definitions involving local conditions at other primes.

Use the inductive structure:

$$0 \to Z^{n+1} \backslash Z^n \to U_{n+1} \to U_n \to 0$$

in étale and De Rham settings. Can keep track of the jump in dimensions

$$d_n := \mathrm{dim} Z^{n+1} \backslash Z^n$$

As a group, U is a unipotent completion of a free group on $m = 2g + s - 1$ generators, where s is the number of points at infinity, in the affine case. In the compact case, it is he completion of a free group on $2g$ generators modulo a single relation of degree 2.

Recursive formula

$$\Sigma_{k|n} k d_k = m^n$$

in the open case and

$$\Sigma_{k|n} k d_k = (g+\sqrt{g^2-1})^n + (g-\sqrt{g^2-1})^n$$

in the compact case.

Hence,

$$d_n \approx m^n/n$$

in the non-compact case and

$$d_n \approx (g+\sqrt{g^2-1})^n/n$$

in the compact case. For the Hodge filtration, an elementary argument shows

$$F^0(Z^{n+1}\backslash Z^n) \leq g^n.$$

So the jump in dimensions as one goes from U_n^{dr}/F^0 to U_{n+1}^{dr}/F^0 is determined by the asymptotics of d_n.

In the étale realization, we have

$$0 \to H^1(\Gamma_T, Z^{n+1}\backslash Z^n) \to H^1(\Gamma_T, U^{et}_{n+1}) \to$$

$$\to H^1(\Gamma_T, U^{et}_n) \xrightarrow{\delta} H^2(H^1(\Gamma_T, Z^{n+1}\backslash Z^n)$$

which is exact in the sense that

$$H^1(\Gamma_T, U^{et}_{n+1})$$

is a torsor for the vector group

$$H^1(\Gamma_T, Z^{n+1}\backslash Z^n)$$

over the kernel of δ.

So we can control the change in dimensions using the Euler characteristic formula

$$\dim H^1(\Gamma_T, Z^{n+1}\backslash Z^n) - \dim H^2(\Gamma_T, Z^{n+1}\backslash Z^n)$$

$$= \dim(Z^{n+1}\backslash Z^n)^-$$

where the negative superscript refers to the (-1) eigenspace of complex conjugation. By comparison with complex Hodge theory, we see that the right hand side is

$$d_n/2$$

for n odd.

When the genus is zero,

$$H^2(\Gamma_T, Z^{n+1} \backslash Z^n) = 0$$

for $n \geq 2$ (Soulé).

Therefore, eventually,

$$\text{dim} H^1(\Gamma_T, U_n^{et}) < \text{dim} U_n^{dr}/F^0$$

The general implications over $\mathbb{Q}$ rely on bounds of the form

$$\dim H^2(\Gamma_T, Z^{n+1}\backslash Z^n) \leq P(n)g^n$$

implied by Bloch-Kato or Fontaine-Mazur.

We have a surjection

$$H^2(\Gamma_T, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \rightarrow H^2(\Gamma_T, Z^{n+1}\backslash Z^n) \rightarrow 0$$

and an exact sequence

$$0 \rightarrow Sh^2_n \rightarrow H^2(\Gamma_T, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n}) \rightarrow$$

$$\oplus_{w\in T} H^2(G_w, H_1(\bar{X}, \mathbb{Q}_p)^{\otimes n})$$

The local groups are bounded by a quantity of the form $P(n)g^n$ using Hodge-Tate decomposition and the monodromy-weight filtration.

Furthermore, $Sh_n^2 \simeq (Sh_n^1)^*$ with the latter group defined by

$$0 \rightarrow Sh_n^1 \rightarrow H^1(\Gamma_T, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1)) \rightarrow$$

$$\oplus_{w \in T} H^1(G_w, H^1(\bar{X}, \mathbb{Q}_p)^{\otimes n}(1)))$$

But either Bloch-Kato or Fontaine-Mazur implies that $Sh_n^1 = 0$ for $n \geq 2$.

One possible viewpoint:

Bloch-Kato $\Rightarrow$ Faltings' theorem

is a 'linearized' version of

Section conjecture $\Rightarrow$ Faltings' theorem

X: CM elliptic curve minus the origin over F an imaginary quadratic field.

$$\begin{array}{ccc} & & Z^3_{dr}\backslash Z^2_{dr} \\ & \nearrow & \uparrow \\ H^1_f(\Gamma_T, U^{et}_3) & \rightarrow & U^{dr}_3/F^0 \\ \simeq\downarrow & & \downarrow \\ H^1_f(\Gamma_T, U^{et}_2) & \rightarrow & U^{dr}_2/F^0 \end{array}$$

Speculation:

-Non-abelian Kolyvagin systems?

-Archimedean theory?

-Use of reductive completions?

Last suggestion due to Weil in 1930's.