Math 2201, Further Linear Algebra: a practical summary. February, 2009

There are just a few themes that were covered in the course.

I. Algebra of integers and polynomials.
II. Structure theory of one endomorphism.
III. Quadratic forms and bilinear forms.
IV. Inner product spaces.

I. The main topics covered were: the Euclidean algorithm, Bezout's Lemma, factorization, and congruences.
You should know:
-How to carry out the Euclidean algorithm for integers and polynomials.
-How to explicitly express GCD's as linear combinations of input integers or polynomials, i.e., Bezout's Lemma.
-How to use the Chinese remainder theorem.

Also,
-Understand basic facts about prime numbers and irreducible polynomials as contained in the lecture notes.
-Understand the proof of the theorem on unique factorization of integers and polynomials.
-Understand the proof of the Chinese remainder theorem.

Some questions to think about:

–How is Bezout's lemma used in the proof of the unique factorization theorem?
–Why is the HCF of two polynomials defined to be a *monic* polynomial?
–What are the invertible elements in the ring of polynomials over a field k?
–What are the invertible elements in $\mathbb{Z}$?
–What are the invertible elements in $\mathbb{Q}$?
–What are the irreducible elements in $\mathbb{C}[x]$?
–Find irreducible elements of degrees 2, 3, and 4 in $\mathbb{F}_2[x]$and $\mathbb{F}_3[x]$.
–What is the basic theorem that helps to compute powers modulo a prime?
–How is the Chinese remainder theorem helpful in computing powers modulo a composite number, in $\mathbb{Z}/39$, for example?
–How would you solve the equation

$$5x \equiv 4 \mod 12 \ ?$$

or other similar equations? What is the role of Bezout's lemma?
–Does

$$45x + 75y = 60$$

have a solution (x, y) in integers?
–Does

$$45x + 75y = 100$$

have a solution (x, y) in integers?
–Can you give a nice description of *all* solutions (in integers, of course) to

$$12x + 18y = 30 \ ?$$

–For what kind of pairs a and b does the equation

$$ax + by = m$$

have solutions no matter what m is?

II. This is a good point at which to review all the basic facts from the previous course, as contained in lectures 8-10 of the notes. Especially:

(0) Notion of basis and dimension.
(1) How to compute the matrix of a linear map with respect to a basis.
(2) Rank-nullity formula.
(3) The characteristic polynomial of a linear map.
After that you should understand the basics of diagonalization, especially the equivalence

$L : V \to V$ is diagonalizable $\Leftrightarrow$ V has a basis consisting entirely of eigenvectors for L $\Leftrightarrow$ the minimal polynomial of L factors entirely into distinct linear factors.

In particular, you need to understand what the minimal polynomial is, how it relates to the characteristic polynomial, and how to compute it.

The next step is to analyze endomorphisms that are *not* diagonalizable. This involves using *generalized eigenspaces* instead of eigenspaces. If $\{\lambda_1, \dots, \lambda_s\}$ are the distinct eigenvalues of L and t_i is the multiplicity of λ_i in the minimal polynomial $m_L(x)$, then

$$V = V_{t_1}(\lambda_1) \oplus V_{t_2}(\lambda_2) \oplus \cdots \oplus V_{t_s}(\lambda_s)$$

where

$$V_{t_i}(\lambda_i)$$

is the t_i-th generalized eigenspace for L. Of course you need to understand the meaning of the various terms, and the proof of this decomposition (Primary Decomposition).
In the process we encountered the important notion of a *Jordan basis* and the associated notion of a Jordan canonical form for an endomorphism.
You should understand how the form of the Jordan basis relates to the generalized eigenspaces. (That is, the meaning of columns and rows when we represent the Jordan basis as a stack.)
Also, make sure you can compute the Jordan canonical form.

Look at the examples in the supplementary material on the course web page concerned with the Jordan canonical form and Jordan bases.

Questions:

–What is the dimension of $\mathbb{C}$ as an $\mathbb{R}$-vector space?
–What is the dimension of $\mathbb{C}$ as a $\mathbb{C}$-vector space?
–What is the dimension of $\mathbb{Q}$ as a $\mathbb{Q}$-vector space?
–What is the dimension of $\mathbb{R}$ as a $\mathbb{Q}$-vector space?
–Consider the space $M_n(\mathbb{C})$ of $n \times n$ complex matrices as a $\mathbb{C}$-vector space. What is its dimension?
–What is a basis for $M_n(\mathbb{C})$?
–In order to find a matrix representation of a linear map, what is the first piece of data you need?
–Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let

$$L_A : M_2(\mathbb{C}) \to M_2(\mathbb{C})$$

be the linear map that multiplies each matrix on the left by A. Find a matrix representation for L_A.
–What is the determinant of L_A?
–Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each vector by $\pi/2$ in the counter clockwise direction. Find a matrix representation.
-What is the determinant of the previous map?
–Having found a matrix representative, regard it now as a complex matrix. Find a diagonalization.
-Can T be diagonalized over $\mathbb{R}$?

–Let $V = \mathbb{C}[x]_{\leq 5}$ the vector space of polynomials with complex coefficients of degree at most 5. Let $D : V \to V$ be the linear map that takes a polynomial f to its derivative f'. Compute the Jordan canonical form and a Jordan basis for D.
–How many linearly independent eigenvectors does D have?
–How do you compute the HCF of a whole collection of polynomials? How does Bezout's lemma generalize to this situation?
–What does the previous question have to do with primary decomposition?
–True or false: The Jordan normal form of a linear map is completely determined by its characteristic polynomial and minimal polynomial.
–True or false: The Jordan normal form of a 6×6 matrix is completely determined by its characteristic polynomial and minimal polynomial.
–Suppose L is a linear map with a single eigenvalue l. What information about the Jordan normal form of L do you get from the dimension of the l-eigenspace?
–Let L be a linear map with a single eigenvalue 1. Suppose there exists a vector v such that $(L-1)^{100}v = 0$, $(L-1)^{99}v \neq 0$. Prove that there exists a vector w such that $(L-1)^2 w = 0$ and $(L-1)w \neq 0$.
–Suppose L is a linear map with a single eigenvalue l. What information about the Jordan normal form of L do you get from the degree of the minimal polynomial?

III. We look at bilinear forms for a few reasons.
(1) A positive-definite symmetric bilinear form on a real vector space allows us to measure length of vectors and gives a notion of orthogonality. This will be discussed below.
(2) The quadratic form associated to a symmetric bilinear form comes up in the study of quadratic equations and normal forms for functions near critical points.
(3) Alternating bilinear forms occur in differential geometry and classical mechanics, but were not discussed in this course.
The main emphasis in (2) was the method of simplifying symmetric bilinear forms by *diagonalization*, using double operations. Make sure you know how to do this efficiently and correctly. It's easy to make mistakes (as I did in class) if you don't write things down, so be careful.
Also, you should know:
-How to compute the matrix of a bilinear form with respect to a basis.
-The proof of the theorem on diagonalization.
-How to find the change of basis matrix that gives you a diagonal form.
-How to recover a symmetric bilinear form from the associated quadratic form. Note that this is necessary in order to move to the diagonalization.
-The real and complex canonical forms, and how to compute them.
Note that the theory of the real and complex canonical forms gives a complete classification of equivalence classes of quadratic forms for real and complex vector spaces.

Questions:

–Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space and the bilinear map given by multiplication

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$(a, b) \mapsto ab$$

Let B be the basis $\{2\}$. What is the matrix of the bilinear form with respect to this basis?
–What is the real canonical form of the previous bilinear form?
–On $\mathbb{C}$ considered as an $\mathbb{R}$-vector space consider the bilinear forms

$$f(z, w) = \Re(zw)$$

and

$$g(z, w) = \mathrm{Im}(zw)$$

Are f or g symmetric? If not, what property does either have?
–Let

$$T : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \rightarrow \mathbb{R}$$

be the bilinear form defined by

$$T(A, B) = Tr(AB)$$

Is T symmetric?
–Find a matrix representative for T.
–What are the real and complex canonical forms for T?
–Consider the bilinear form

$$P : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

that takes (v, w) to $t \cdot (v \times w)$, where $\times$ is the cross-product of vectors, $\cdot$ is the dot product, and t is the vector $(1, 1, 1)^T$. Check that P is bilinear.
–Is P symmetric?
–Find a matrix representative for P.
–Is the geometry of the curve

$$x^2 + 2xy + 2y^2 = 1$$

in $\mathbb{R}^2$ more similar to a circle or a hyperbola?
–Is the geometry of the curve

$$x^2 + 4xy + 3y^2 = 1$$

more similar to a circle or a hyperbola?
–What is a positive definite blinear form?
–What is the real canonical form of a positive definite bilinear form?

IV. Inner product spaces

An inner product $< \cdot, \cdot >$ on a vector space V is a special case of a quadratic form in the case of real vector spaces. But for complex vector spaces, the inner product is not linear in the second variable. Rather, it is *conjugate linear* i.e.,

$$< v, aw >= \bar{a} < v, w >$$

for all $v, w \in V$, and $a \in \mathbb{C}$. The motivation for this is that the natural inner product on $\mathbb{C}^n$ that is used to measure lengths is given by

$$< z, w >= \Sigma z_i \bar{w}_i$$

For example, on $\mathbb{C}$-itself, since it is $\mathbb{R}^2$ as a real vector space, we know that the natural length-squared is $< a + ib, a + ib >= a^2 + b^2$. But this is exactly $(a + ib)\overline{(a + ib)}$. What are some examples of other inner products on $\mathbb{R}^n$ or $\mathbb{C}^n$?
The most important property to remember in both the real and complex case is that the product is *positive definite*: $< v, v >> 0$ for all $v \neq 0$ in V.
The inner product gives us the extra structure necessary to isolate a special class of bases, namely, *the orthonormal bases*.
You should know the definition of an orthonormal basis, and some examples. Any inner product space has an orthonormal basis that you can construct starting from any basis using the Gram-Schmidt orthonormalization process. Be sure to review this procedure.
The inner product allows us to carry over to an inner product space notions of Euclidean geometry, and many theorems also carry over, such as the Cauchy-Schwartz inequality and the triangle inequality. Review the proof of these inequalities.
One can then investigate the interaction between the structure of an endomorphism and the inner product. In particular, we studied at a superficial level the notions of a *self-adjoint map* and an *isometry*.

You should know the proof of the fact that given a self-adjoint map $L : V \rightarrow V$, V admits an orthonormal basis of eigenvectors for L.

Questions:

–What are some examples of self-adjoint maps? On $\mathbb{C}^n$? On $\mathbb{R}^n$?
–What are some examples of isometries on $\mathbb{R}^2$, $\mathbb{R}^3$ and, $\mathbb{R}^4$?
–Can you describe all isometries from $\mathbb{C}$ to $\mathbb{C}$ (as a $\mathbb{C}$ vector space with the standard inner product)?
–Can you describe all isometries from $\mathbb{R}$ to $\mathbb{R}$ (as an $\mathbb{R}$ vector space with the standard inner product)?
–Let $L : V \to V$ be an isometry. Show that V admits an orthonormal basis of eigenvectors for L.
–What is special about the eigenvalues of a self-adjoint map? Can you prove this?
–True or false: a matrix in $M_n(\mathbb{C})$ can be diagonalized if and only if it is self-adjoint.
–True or false: a symmetric matrix in $M_n(\mathbb{R})$ is self-adjoint with respect to any inner product on $\mathbb{R}^n$.
–Illustrate your answer to the previous question with some examples.
–Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry. True or false: If A is a matrix representative for L with respect to any basis, then

$$A^T A = I$$

–True or false: If A is any diagonalizable $n \times n$ real matrix, then there is an orthogonal matrix P such that $P^{-1}AP$ is diagonal.
–What is special about the eigenvalues of an isometry on a real inner product space?
–What is special about the eigenvalues of an isometry on a complex inner product space?

Some important theorems, propositions, corollaries, . . .

1.1.4, 1.1.8, 1.2.12, 1.2.14, 1.2.15, 1.3.21, 1.3.22, 2.1.29, 2.1.36, 2.2.42, 2.2.47, 2.2.51, 3.1.56, 3.2.57, 3.2.60, 3.2.61, 3.2.64, 3.3.67, 3.3.69, 3.3.72, 3.3.73, 3.4.79, 3.4.82, 3.5.87, 3.6.92, 3.9.95, 4.1.101, 4.2.105, 4.3.110, 4.3.112, 4.4.114, 4.6.127, 5.1.137, 5.1.138, 5.2.145, 5.2.147, 5.3.149, 5.4.150, 5.4.152, 5.5.157, 5.5.159, 5.5.160.

Needless to say, it is assumed that you will have a full understanding of the material surrounding the results listed above, especially the definitions and the examples.